Université Libre de Bruxelles

Faculté des Sciences

Département de Mathématique

ON TWO UNSOLVED PROBLEMS IN PROBABILITY

Thèse présentée en vue de l'obtention
du grade de Docteur en Sciences, orientation mathématiques

Promoteur: F. Thomas BRUSS

Année académique 2006-2007

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Remerciements

Ce travail a été réalisé sous la direction de Monsieur le Professeur F. Thomas Bruss. Je tiens à le remercier de m'avoir permis de réaliser cette thèse en me proposant de travailler sur des problèmes passionnants et ouverts. Je lui suis reconnaissant du temps qu'il m'a consacré. Nos nombreuses discussions mathématiques au sujet des différents problèmes que j'ai été amené à étudier ont été pour moi une grande source d'inspiration. Je le remercie également pour le soutien constant qu'il m'a apporté durant toutes ces années, ainsi que pour sa relecture patiente et attentive de mes différents manuscrits.

Je remercie le Département de Mathématique de m'avoir proposé le poste d'assistant qui m'a permis de découvrir un autre aspect de la vie universitaire. Je remercie également les titulaires des différents cours pour lesquels j'ai été amené à diriger les séances d'exercice. Ces charges d'enseignement m'ont apporté de nombreuses joies et resteront parmi les meilleurs souvenirs que je garderai de ces premières années de travail à l'Université.

Je voudrais également exprimer ma gratitude à tous les membres du jury, Madame le professeur Deelstra, Messieurs les professeurs Bruss, Grübel, Hallin, Latouche, Lefèvre, et Paindaveine.

Je remercie mes amis, pour les rires, les joies, les maths et la musique qui ont rythmé ma vie ces dernières années. Je les remercie non seulement pour leur présence et disponibilité sans faille, mais également pour leur absence compréhensive ces derniers mois. Cela m'a permis de me concentrer totalement sur la rédaction de ce travail.

Je remercie ma famille. Merci à mes parents, pour leur joie de vivre et pour le soutien qu'ils m'apportent à tant de niveaux depuis le début. Merci également à Marguerite et Bruno, et à toute la tribu Deke, pour tout et le reste. Et, par dessus tout, merci à Isabelle, pour son amour, sa patience et sa confiance qui nous ont permis de traverser des épreuves difficiles et d'en sortir renforcés.

Enfin, merci à Simon d'être là, tout simplement.

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Foreword

In the course of this thesis, we have worked on two specific problems in Probability. Both problems are well-known and unsolved so far. The aim of this dissertation is to contribute to these.

The first problem is a generalization of the classical gambler's ruin problem for two players. We begin with a study of this problem in discrete time for an arbitrary number N of players. We then present an asymptotic resolution in continuous time which is valid when $N \leq 3$.

The second of these problems belongs to the domain of optimal stopping. It is known as Robbins' problem. We motivate why we study a version of this problem in continuous time and contribute several parts of the solution.

These two problems require a different background and different mathematical tools. For this reason, the presentation of this work is divided into two distinct parts, each of which is concluded with a list of the relevant references.

A complete bibliography is given at the end of this dissertation.

Part I

On the N-Player ruin problem

Introduction

The first part of this thesis originated in a generalization of a problem known as the *gambler's ruin problem*, in which two players with finite initial capital play a sequence of games until one of them is ruined. Our research is therefore carried out under the auspices of applied probability, even though - as is typical for applied mathematics in general - our main interest focuses on the theoretical aspects and implications of this problem. We aim to obtain exact and asymptotic solutions, and for this we will use tools as varied as Markov processes, Matrix-Analytic theory, Brownian motions and conformal transformations. The results included in the next few pages are intended to be self-contained, and thus we have tried to give a concise theoretical background to each of the mathematical tools we have applied in our research.

1 The gambler's ruin problem and related problems

The gamblers' ruin problem for two players is often stated as follows. Two players, owning respectively a and b euros, play a sequence of games in each of which they either win or lose one euro. If the games are fair, the expected time E until the ruin of one of the two players occurs and the probability P that the player with a euros is ruined first are given, respectively, by

$$E = ab \text{ and } P = \frac{b}{a+b}.$$

This solution can be obtained for example through martingale arguments or random walk analysis, and it is standard and well known (see Feller (1950) or Steele (2001)).

As soon as this problem is posed, one is almost forced to inquire whether this kind of simple solution still holds when the games are not fair, or when the number of players is greater than 2. The answer to this question is affirmative, but only in certain specific cases. Let us consider a few examples of ruin problems for more than 2 players.

Example 1

Initially three players own a, b and c euros, respectively. They play a sequence of fair games during each of which one player is selected at random and receives one unit from the other two players. The ruin problem associated to these games is known as the symmetric 3-player ruin problem and has been studied, among others, by Engel (1993), Stirzaker (1994), Ferguson (1995) or Bruss et al. (2002).

We will see that it is useful to view the flows between capitals as a random walk on a triangle in the plane with transitions defined by the structure of the games, as illustrated in Figure 1.

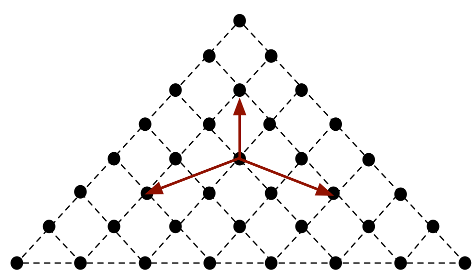

Figure 1: *Admissible transitions for the symmetric three player problem.*

Now let $\Sigma = a + b + c$ and define $T(a, b, c)$ as the expected number of games until one of the players is ruined. Clearly we have

$$T(a, b, c) = 0 \quad \textit{whenever} \quad \min\{a, b, c\} = 0. \tag{1}$$

Moreover for $a, b, c > 0$, a conditioning argument yields

$$T(a, b, c) = 1 + \frac{1}{3}(T(a+2, b-1, c-1) + T(a-1, b+2, c-1) + T(a-1, b-2, c+2)). \tag{2}$$

From computer simulations, Engel (1993) surmised that

$$T(a, b, c) = \frac{abc}{\Sigma - 2},$$

and showed that this function satisfies (1) and (2). Consequently, by uniqueness, it is the exact solution. Stirzaker (1994) obtains this result by using a more general martingale argument. Ferguson (1995) studies an asymptotic version of this problem.

Example 2

Initially three players own a, b and c euros, respectively. At each game, two players are chosen at random. Then one of these is again chosen at random as the winner, who receives one euro from the other one (see e.g. Engel (1993), Stirzaker (1994), Bruss et al. (2002) or Alabert et al. (2003)). This problem is better known under the name of 3-tower problem (or Hanoi tower problem), and was invented by the French mathematician Edouard Lucas in 1883 (see Beck et al. (2000)). Figure 2 illustrates the transitions associated to this problem.

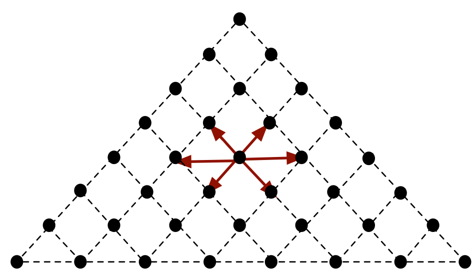

Figure 2: *Admissible transitions for the 3-tower problem.*

Engel and Stirzaker obtain, with the same methods as in Example 1, that the expected time until ruin of a player is given by

$$T(a, b, c) = \frac{3abc}{\Sigma}.$$

Moreover, Bruss et al. (2003) obtain the complete probability distribution of the time until absorption.

Example 3

A generalization of the last example for N players is given by the N-tower problem: N towers contain initially n_i $(i = 1, 2, \ldots, N)$ counters. At each game a tower X is chosen at random, then another tower Y is also chosen at random and a counter is moved from X to Y. Engel (1993) attributes this problem to Lennart Råde from Gotheburg University, Sweden. No simple formulas are known so far for the distribution of T in the case $N > 3$ (see Bruss et al. (2002)).

Now let (X_i) be a sequence of i.i.d. random variables with probability distribution given by

$$\mathrm{P}[X_i = 1] = \mathrm{P}[X_i = -1] = \frac{1}{2},$$

and define the random walk S_n by $S_0 = 0$, $S_n = \sum_{i=1}^{n} X_i$. Then the 2-player ruin problem with fair games is clearly equivalent to the problem of determining the distribution of the first time S_n reaches level a or $-b$. Moreover, an appropriate scale change of S_n will converge in law to a one dimensional Brownian motion B_t in the interval $[-b, a]$ starting at 0. Thus if we let $\tau = \min\{t : B_t = a \text{ or } B_t = -b\}$, one can show (see e.g. Steele (2001)) that

$$\mathrm{P}[\tau < \infty] = 1, \ \mathrm{P}[B_\tau = a] = b/(a+b) \text{ and } \mathrm{E}[\tau] = ab,$$

and we see that the asymptotic solution obtained through approximation by a Brownian motion is the same as the exact solution of the original problem. Again a natural generalization of this situation is to look at similar Brownian motion approximations for ruin problems with more than two players.

Example 4

Let us consider the case $N = 3$ with fair games as a random walk in a triangle as illustrated in Figures 1 and 2. As above one can show that an appropriate scale change will converge in law, this time to a two-dimensional Brownian motion in a triangle. Hence an asymptotic version of the 3-Player ruin problem is given by the distribution of the first exit times of a two dimensional Brownian motion starting at a point inside a triangle, through a given edge of this triangle.

We could give here several more examples, since any integer sequence of identical, independent games of chance define a problem which falls under the very general heading of what we call N-player ruin problems. Several such problems (with different objectives and measures of interest) have been studied in the literature, see e.g. Beyer et al. (1977), Amano et al. (2001) or Kmet and Petrovsek (2002) for different definitions and approaches. See also Asmussen (2000) for a survey of the applications of these results in insurance theory.

2 Preview of Part One

We now give a brief outline of the research we have done on the subject of N-player ruin problems. The presentation we shall give is counter-chronological, since our research originated in a note by Professor Ferguson (see Ferguson (1995)) on a Brownian motion approximation of the symmetric ruin problem. We extended the results of this note into a study of the interplay between Brownian motions, conformal transformations and their applications to hitting time problems. We present these results in Chapter 2. It is only after this work was completed that we turned our attention to the discrete problem. Our first steps in this direction consisted in trying to adapt the martingale arguments of Stirzaker (1994) to other more general ruin problems. However, these attempts turned out fruitless because it appeared that the simplifications needed for the martingale arguments to apply, as far as we understand, only occur for certain very specific problems. We modeled the ruin problems as Markov processes on a finite state space with a finite number of absorbing states. This led to the results that are presented in Chapter 1.

Chapter 1.

For any number N of players, one can see that the flows between capitals define a random walk J_t on a simplex $\Delta \subset \mathbb{Z}^N$. The ruin of a player then corresponds to J_t reaching a point which has at least one of its coordinate negative or nil. For example, if $N = 3$ and $a, b, c > 0$, then any sequence of games will define a random walk on a triangle of discrete points given by $\Delta = \{(X, Y, Z) \in \mathbb{N}^3 \mid X + Y + Z = a + b + c\}$, as illustrated in Figures 1 and 2. If we suppose that the sequence of games stops as soon as a player is ruined, then we see that the points of Δ which correspond to the ruin of a player are absorbing for J_t. Hence the time until the ruin of a player occurs is given by the time until absorption of a random walk on a state space which has a finite number of transient points. The probability that a given player is ruined first is given by that of the random walk reaching specific points of Δ. PH distributions and Matrix Analytic methods are the natural setting for such problems. The methods and terminology we use are borrowed from Matrix Analytic Theory. These methods were developed over the years following the impetus of Marcel Neuts, who, to our knowledge, coined the terminology of 'matrix-geometric distributions' and 'phase-type processes' (see for example the books by Neuts (1975), (1978), (1981)). More specifically we use different results from Latouche and Ramaswami (1999) in which the authors describe applications and algorithms derived from and for Matrix Analytic Theory.

PH distributions arise as generalizations of the exponential and Erlang distributions. In the discrete case they are defined by considering the time until absorption into state 0 of a Markov process on a finite state space $\{0, 1, \ldots, n\}$ with transition probability matrix

$$\boldsymbol{P} = \begin{pmatrix} 1 & \boldsymbol{0} \\ \boldsymbol{t} & \boldsymbol{T} \end{pmatrix}.$$

In Section 1.2 we modify this definition to include discrete distributions with an arbitrary number of absorbing states. In Section 1.3 we set up the N-player ruin problem as a multivariate absorbing Markov chain with absorbing states corresponding to the ruin of a player. We discuss this in the context of the modified PH distributions and derive an explicit solution to the N-player problem. The main results of Chapter 1 are given in Sections 1.4 and 1.5 in which we define a partition of the set of transient states into different levels on which we give an extension of the folding algorithm[1]. This achieves an

[1]This material has been published in the paper "A matrix-analytic approach to the N-player ruin problem", see Swan and Bruss (2006).

efficient computational procedure for calculating some of the key measures.

Chapter 2.

In this chapter we study exit time problems for 2 dimensional Brownian motions. To start with a specific example, suppose that we have a Brownian motion $(B_t)_{t\geq o}$ without drift, starting at the origin ($B_0 = (0,0)$). Let Γ be the disk centered at $(0,0)$ with radius r, and let L be a fixed arc segment of length l on the circumference of the circle. The probability that the first exit of (B_t) will occur through L is then simply the angular measure of L with respect to the origin, that is $l/2\pi r$. Clearly, in this specific case, the symmetry argument is not only sufficient for the answer to be intuitively clear, but also to prove its correctness. Moving the starting point B_0 away from the origin changes the situation completely, and alternative arguments must be found to obtain the exit probabilities. The problem is even more complex if the Brownian motion takes place in a non symmetrical domain $\Gamma \subset \mathbb{R}^2$ (such as a polygon, an infinite strip, etc.), although it seems intuitively clear that the exit probabilities through a segment L of the boundary of any domain Γ must depend somehow on the length of L (or at least the proportionality of L with respect to the border of Γ) and on the starting point of the Brownian motion.

The fundamental idea behind our approach lies in the fact that Brownian motions are invariant under conformal transformations. Hence, if we construct a conformal transformation f which maps Γ conformally into the unit disk and which sends the starting point of the Brownian motion onto the origin, then we will be able to compute the exit probability of (B_t) through L by use of the same symmetry argument as that which we gave above, i.e. the probability that the Brownian motion first exits Γ through a segment L of the border of Γ will be given by the angular measure of $f(L)$.

In Sections 2.2 and 2.3 we outline the construction of a family of conformal transformations known as *Schwarz-Christoffel transformations* which can be used efficiently in order to obtain conformal transformations of a closed convex polygon into the unit disk. From the principle outlined above, this also yields a solution to the exit time problem for 2 dimensional Brownian motions. In Section 2.4 we use these transformations to obtain a solution to the asymptotic versions of the two and three player problems for symmetric games. Moreover, we solve a modification of the 3-player problem which we call the *three player ruin problem with capital constraints*. Some numerical results are given at the end of this section[2].

[2]This material has been published in the paper "The Schwarz-Christoffel transformation as a tool in applied probability", see Swan and Bruss (2004).

Chapter 1

A Matrix-Analytic Approach to the N-Player Ruin Problem

1.1 The N-Player ruin problem

In the spirit of the examples we gave in the Introduction, one could say that any sequence of games between N players will implicitly define a N-player ruin problem. However, to avoid ambiguities we need to be more specific about the kind of games with which we are going to work, and from now on we restrict our attention to ruin problems which satisfy the following definition.

N people play a sequence of identical and independent games during each of which a certain number of players either win or lose a determined amount of money. We suppose that the probabilities of winning and losing in each game are known and that they remain identical throughout the sequence of games. We also suppose that no exterior input of money is allowed, and that the sequence of games stops as soon as a player has no money left on the table. Finally we suppose that the games are defined in such a way that ties are excluded i.e. that there is no initial distribution of fortunes and no rule of the game such that it is possible that the sequence of games never ends. Our first objective consists in determining the probability that a given player is ruined first. Secondly we also want to determine the expected number of games before the ruin of a player occurs. Both answers should be given as a function of the initial state only, since clearly, once the games are fixed, these measures only depend on the initial distribution of fortunes. These two objectives together constitute the core of what we call the *N-player ruin problem.*

Remark 1 *It should be noted that we do not suppose that the games are fair, nor do we restrict our attention to games in which ruin occurs only at zero.*

1.2 PH^m random variables

Consider a Markov process J_t on a set with m transient states (numbered 1 to m) and n absorbing states $O_1, \dots, O_n$ whose transition matrix is given by

$$\boldsymbol{P} = \begin{pmatrix} \boldsymbol{T} & \boldsymbol{r}^1 & \dots & \boldsymbol{r}^n \\ \mathbf{0} & & \boldsymbol{I} & \end{pmatrix}, \tag{1.1}$$

where $\boldsymbol{r}^1, \dots, \boldsymbol{r}^n$ are n vectors of $\mathbb{R}^m$, $\boldsymbol{T}$ is a $m \times m$ matrix and $\boldsymbol{I}$ is the $n \times n$ identity matrix. Let $\boldsymbol{\tau}$ be the initial distribution of this Markov process [1]. Since $\mathbf{P}$ is stochastic, we clearly have the relationship

$$\boldsymbol{r^1} + \dots + \boldsymbol{r^n} + \boldsymbol{T}\mathbf{1} = \mathbf{1}, \tag{1.2}$$

where $\mathbf{1} = (1, 1, \dots, 1)'$. Also from the structure of $\boldsymbol{P}$ we see immediately, that for $k \geq 1$,

$$\boldsymbol{P}^k = \begin{pmatrix} \boldsymbol{T}^k & (\boldsymbol{I} + \dots + \boldsymbol{T}^{k-1})\boldsymbol{r}^1 & \dots & (\boldsymbol{I} + \dots + \boldsymbol{T}^{k-1})\boldsymbol{r}^n \\ \mathbf{0} & & \boldsymbol{I} & \end{pmatrix}.$$

Definition 1.1 *The distribution of the time X till absorption of a Markov process with transition matrix $\boldsymbol{P}$ and initial distribution $\boldsymbol{\tau}$ is a PH^m distribution with representation $(\boldsymbol{\tau}, \boldsymbol{r}^1, \dots, \boldsymbol{r}^n, \boldsymbol{T})$. We will write*

$$X \sim PH^m(\boldsymbol{\tau}, \boldsymbol{r}^1, \dots, \boldsymbol{r}^n, \boldsymbol{T}).$$

Remark 2 *For $m = 1$, PH^1 and PH distributions are equivalent. Moreover, we could group all the absorbing states into one single absorbing state $O = O_1 \cup \dots \cup O_n$. The distribution of the time until absorption into O is a PH^1 distribution.*

It should be noted that for a given random variable X following a PH^m distribution, there are in fact infinitely many choices of matrices $(\boldsymbol{\tau}, \boldsymbol{r}^1, \dots, \boldsymbol{r}^n, \boldsymbol{T})$ which describe X. This is the reason for which $(\boldsymbol{\tau}, \boldsymbol{r}^1, \dots, \boldsymbol{r}^n, \boldsymbol{T})$ is known as a representation of X.

Because of the structure of the matrix $\boldsymbol{P}$ we easily obtain the distribution of PH^m random variables.

Proposition 1.2 *Let X follow a $PH^m(\boldsymbol{\tau}, \boldsymbol{r}^1, \dots, \boldsymbol{r}^n, \boldsymbol{T})$ distribution starting in a state $1 \leq \alpha \leq m$ (which is not an edge point). Then the distribution of X conditional on this initial state is given by:*

$$\mathrm{P}_\alpha[X = 0] = 0, \tag{1.3}$$

$$\mathrm{P}_\alpha[X = k] = \boldsymbol{\tau}\boldsymbol{T}^{k-1}\mathbf{1} - \boldsymbol{\tau}\boldsymbol{T}^k\mathbf{1}, \tag{1.4}$$

and

$$F_X(k) = \mathrm{P}_\alpha[X \leq k] = \boldsymbol{\tau}(\boldsymbol{I} - \boldsymbol{T}^k)\mathbf{1}. \tag{1.5}$$

[1] We use results from Latouche and Ramaswami (1999). These authors use the symbol $\boldsymbol{\tau}$ for the initial probability distribution.

Proof: Equation (1.3) is obvious. To see (1.4), let J_t be a random walk with transition matrix $\boldsymbol{P}$ and let $O_\zeta, \zeta = 1, \ldots, n$ be the absorbing states of J_t. Define

$$\mathrm{P}_\alpha[J_k \in O_\zeta],\ \zeta = 1, \ldots, n$$

as the probability that, starting from α, the kth step is the first transition of J_t into the absorbing state O_ζ. Then we see that

$$\mathrm{P}_\alpha[X = k] = \sum_{\zeta=1}^{n} \mathrm{P}_\alpha[J_k \in O_\zeta].$$

For each $\zeta = 1, \ldots, n$ we condition on the state of the second last step. This yields

$$\begin{aligned}\mathrm{P}_\alpha[J_k \in O_\zeta] &= \sum_{\beta=1}^{m} \mathrm{P}_\alpha[J_{k-1} = \beta]\mathrm{P}[J_k \in O_\zeta | J_{k-1} = \beta] \\ &= \sum_{\beta=1}^{m} T_{\alpha\beta}^{k-1} r_\beta^\zeta \\ &= \boldsymbol{\tau}\boldsymbol{T}^{k-1}\boldsymbol{r}^\zeta.\end{aligned}$$

Hence,

$$\mathrm{P}_\alpha[X = k] = \sum_{\zeta=1}^{n} \boldsymbol{\tau}\boldsymbol{T}^{k-1}\boldsymbol{r}^{\boldsymbol{\zeta}} = \boldsymbol{\tau}\boldsymbol{T}^{k-1}\sum_{\zeta=1}^{n}\boldsymbol{r}^{\boldsymbol{\zeta}}.$$

From (1.2) we know that $\sum_{\zeta=1}^{n} \boldsymbol{r}^{\boldsymbol{\zeta}} = \mathbf{1} - \boldsymbol{T}\mathbf{1}$. This yields (1.4) and (1.5). ∎

Although our definition of the transition matrix $\boldsymbol{P}$ is not the same as that which defines a standard PH-distribution, we see, as expected from Remark 2, that the distribution of X is that of a standard PH-distribution (see Latouche and Ramaswami (1999), pp. 49).

Proposition 1.3 *Absorption of a $PH^m(\boldsymbol{\tau}, \boldsymbol{r}^1, \ldots, \boldsymbol{r}^n, \boldsymbol{T})$ random variable occurs almost surely if and only if the matrix $(\boldsymbol{I} - \boldsymbol{T})$ is regular.*

Moreover, $(\boldsymbol{I} - \boldsymbol{T})^{-1}_{\alpha\beta}$ is the expected number of steps in phase β, given that the initial phase is α.

Proof: Assume that absorption occurs almost surely. We denote by $E_{\alpha\beta}$ the expected sojourn time in state β starting from α, and we see that almost sure absorption implies that $E_{\alpha\beta}$ must be finite for all transient states $\alpha, \beta \in \{1, \ldots, m\}$. By conditioning on the first state visited after α, we can write:

$$\begin{aligned}E_{\alpha\beta} &= \textstyle\sum_{\kappa=1}^{m} T_{\alpha\kappa}E_{\kappa\beta} \text{ if } \alpha \neq \beta, \\ E_{\alpha\alpha} &= 1 + \textstyle\sum_{\kappa=1}^{m} T_{\alpha\kappa}E_{\kappa\alpha} \text{ if } \alpha = \beta.\end{aligned}$$

This implies that $\boldsymbol{E} = \boldsymbol{I} + \boldsymbol{T}\boldsymbol{E}$, i.e:

$$(\boldsymbol{I} - \boldsymbol{T})\boldsymbol{E} = \boldsymbol{I}.$$

Hence the matrix $(\boldsymbol{I}-\boldsymbol{T})$ is non singular.

Assume now that $(\boldsymbol{I}-\boldsymbol{T})$ is non singular. It is well known (see for example Householder (1965), page 54) that for any matrix $\boldsymbol{M}$ such that $(\boldsymbol{I}-\boldsymbol{M})$ is non singular, the series $\sum_{k\geq 0}\boldsymbol{M}^k$ converges if and only if the spectral radius $\rho(\boldsymbol{M})$ of $\boldsymbol{M}$ is strictly less than 1 and that in this case,

$$\sum_{k\geq 0}\boldsymbol{M}^k=(\boldsymbol{I}-\boldsymbol{M})^{-1}. \tag{1.6}$$

Moreover, for any strict submatrix $\boldsymbol{N}$ of a nondegenerate matrix $\boldsymbol{M}$, we have $\rho(\boldsymbol{N})<\rho(\boldsymbol{M})$. Since all stochastic matrices are non degenerate (i.e. have no line of zeros) and have spectral radius 1, this implies that $\rho(\boldsymbol{T})<\rho(\boldsymbol{P})=1$ and hence (1.6) holds for $\boldsymbol{T}$. Thus

$$\sum_{k\geq 0}\boldsymbol{T}^k=(\boldsymbol{I}-\boldsymbol{T})^{-1}. \tag{1.7}$$

Since $\sum_{k\geq 0}\left(\boldsymbol{T}^k\right)_{\alpha\beta}$ is the expected number of steps in phase β, given that the initial phase is α, we get the result.

■

Furthermore, from (1.4), one sees that the expected value of X is given by

$$\mathrm{E}[X]=\boldsymbol{\tau}\left(\sum_{k=0}^{\infty}\boldsymbol{T}^k\right)\mathbf{1}.$$

and hence if absorption from any state occurs almost surely, we see that the expected number of steps before absorption is given by

$$\mathrm{E}[X]=\boldsymbol{\tau}(\boldsymbol{I}-\boldsymbol{T})^{-1}\mathbf{1}. \tag{1.8}$$

1.3 Markov interpretation of the N-Player ruin problem

We consider the general N-player ruin problem defined in Section 1.1. Let us suppose that the players are numbered from 1 to N, with $x_1,\ldots,x_N$ being their initial fortunes, and define Σ as the total capital, i.e. $\Sigma=x_1+\ldots+x_N$. Also let X_i^t be the wealth of player i after t games. Since the total wealth remains constant throughout the sequence of games, we see that for each $t\geq 1$ one has

$$X_1^t+\ldots+X_N^t=\Sigma.$$

Hence the flow of capitals can be modeled as a N-dimensional Markov chain $\{J_t=(X_1^t,\ldots,X_N^t)\}$ on the lattice

$$\Delta=\{(X_1,\ldots,X_N)\in\mathbb{Z}^N \text{ such that } X_1+\ldots+X_N=\Sigma\}, \tag{1.9}$$

starting at $J_0 = (x_1, \dots, x_N)$. Clearly every state belonging to the subset of (1.9) given by

$$\mathrm{int}(\Delta) = \{\mathrm{X_1} + \dots + \mathrm{X_N} = \Sigma \text{ such that } \mathrm{X_i} > 0 \text{ for every } \mathrm{i} = 1 \dots \mathrm{N}\} \tag{1.10}$$

will be transient for J_t. We will refer to $\mathrm{int}(\Delta)$ as the *interior points* of Δ.

Remark 3 *There are as many points in* $\mathrm{int}(\Delta)$ *as there are vectors of* $\mathbb{R}^N$ *whose components are strictly positive integers summing to* Σ*. This is the same as the number of ways of distributing* Σ *indistinguishable balls between* N *urns in such a way that each urn contains at least one ball; i.e. there are*

$$p = \begin{pmatrix} \Sigma + N - 1 - N \\ \Sigma - N \end{pmatrix} = \begin{pmatrix} \Sigma - 1 \\ N - 1 \end{pmatrix}$$

transient states for J_t*.*

The ruin of a player corresponds to the event that the process J_t passes from an interior point to a point which has (at least) one of its coordinates negative or zero. Because we supposed that the sequence of games stops as soon as at least one player is ruined, we see that all these points must be absorbing for J_t. Hence every state in $\delta(\Delta) = \Delta \setminus \mathrm{int}(\Delta)$ is absorbing for J_t. However it is clear that all the points of Δ at which the same combination of coordinates is non positive define the same ruin event for the N-player problem (see Figure 1 for an illustration of this when $N = 3$ and $\Sigma = 7$). There are therefore as many non-equivalent absorbing states as there are ways of choosing subgroups of sizes 1 to $N-1$ among the N initial players. Hence there are only $2^N - 2$ non-equivalent absorbing states. Each of these corresponds to a collection of indices $J_\zeta = \{i_1, \dots, i_k\} \subset \{1, \dots, N\}$. For each collection J_ζ let us define the absorbing state

$$O_\zeta = \{X_1 + \dots + X_N = \Sigma \text{ such that } X_j \leq 0 \;\forall\, j \in J_\zeta \text{ and } X_l > 0 \;\forall\, l \notin J_\zeta\}.$$

Together the states O_ζ, $\zeta = 1, \dots, 2^N - 2$ form a partition of $\delta(\Delta)$. Because the number of absorbing states will often be used later on, it will be convenient (although maybe a bit confusing) to use the notation

$$n = 2^N - 2.$$

Example 5 *Let us consider the case* $N = 3$, $\Sigma = 7$*. The lattice defined by equation (1.9) can be laid out as a triangle in the plane, with 36 points, 15 of which are* interior points*. We partition the edge points into 6 absorbing states labeled* O_1 *to* O_6 *as is illustrated in Figure 3. Each of these states defines a different ruin event (the points* O_2, O_4 *and* O_6 *correspond to the ruin of a single player and* O_1, O_3 *and* O_5 *correspond to the 3 possible combinations of two players being ruined at the same time).*

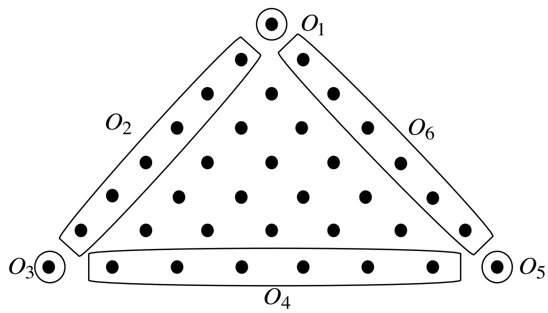

Figure 3: *Illustration of the lattice Δ, defined by equation (1.9), for $N = 3$ and $\Sigma = 7$.*

Note that all 3-Player ruin problems can be modeled as a random walk on a lattice of this form.

The phase-type distribution

Suppose that the $p = \begin{pmatrix} \Sigma - 1 \\ N - 1 \end{pmatrix}$ transient states as well as the $n = 2^N - 2$ absorbing states are ordered in a non-ambiguous way, and let $J_0 = \alpha \in \{1, \ldots, p\}$ be the initial point of J_t. Then with these notations one can write out the transition matrix of J_t.

For this we first denote by $\boldsymbol{r}^1, \ldots, \boldsymbol{r}^n$ the (column) vectors of one-step probabilities of absorption in $O_1, \ldots, O_n$ respectively, i.e.

$$r^{\zeta}_{\beta} = \mathrm{P}[J_t \in O_\zeta \mid J_{t-1} = \beta], \;\; \zeta = 1, \ldots, n, \beta = 1, \ldots, p.$$

Secondly we denote by $\boldsymbol{T}$ the $(p \times p)$ matrix corresponding to the transient states, i.e.

$$\boldsymbol{T}_{\alpha\beta} = \mathrm{P}[J_t = \beta \mid J_{t-1} = \alpha], \text{ for every } \alpha, \beta \in \{1, \ldots, p\}.$$

Then the transition matrix of the random walk corresponding to the ruin problem can be written as

$$\boldsymbol{P} = \begin{pmatrix} \boldsymbol{T} & \boldsymbol{r}^1 & \ldots & \boldsymbol{r}^n \\ \boldsymbol{0} & & \boldsymbol{I} & \end{pmatrix}, \tag{1.11}$$

where $\boldsymbol{I}$ is the $n \times n$ identity matrix. These arguments yield the following proposition.

Proposition 1.4 *Let X be the time till absorption of J_t into one of the absorbing states $O_1, \ldots, O_n$. Let $\boldsymbol{\tau}$ be the initial probability vector of J_t. Then*

$$X \sim PH^p(\boldsymbol{\tau}, \boldsymbol{r}^1, \ldots, \boldsymbol{r}^n, \boldsymbol{T}).$$

Expected duration of the game until ruin

From Proposition 1.3 we know that if the games are well defined, then starting with any initial distribution of fortunes, absorption into one of the states O_ζ will occur eventually and hence the matrix $(\boldsymbol{I}-\boldsymbol{T})$ is non singular. The expected duration of the N-player game is given by (1.8), i.e.

$$\mathrm{E}[X]=\boldsymbol{\tau}(\boldsymbol{I}-\boldsymbol{T})^{-1}\mathbf{1}, \tag{1.12}$$

and we have therefore answered the second question asked by the N-player ruin problem.

Ruin probabilities

The probabilities of ruin in any absorbing state O_ζ starting from any point α in the interior of Δ are given by the following proposition.

Proposition 1.5 *Starting from any point α interior to the grid, the probability of absorption of J_t into O_ζ $(\zeta=1,\ldots,n)$, is given by*

$$\mathrm{P}_\alpha[\text{ruin in }\zeta]=\boldsymbol{\tau}(\boldsymbol{I}-\boldsymbol{T})^{-1}\boldsymbol{r}^{\boldsymbol{\zeta}} \tag{1.13}$$

where $\zeta=1,\ldots,N$.

Proof: We use the same notations as in the proof of Proposition 1.2.

$$\begin{aligned}\mathrm{P}_\alpha[\text{ruin in }\zeta] &= \textstyle\sum_{k=1}^{\infty}\mathrm{P}_\alpha[J_k\in O_\zeta]\\ &= \textstyle\sum_{k=1}^{\infty}\boldsymbol{\tau}\boldsymbol{T}^{k-1}\boldsymbol{r}^{\zeta}\\ &= \boldsymbol{\tau}\left(\textstyle\sum_{k=1}^{\infty}\boldsymbol{T}^{k-1}\right)\boldsymbol{r}^{\zeta}\end{aligned}$$

Using the identity $\sum_{k=1}^{\infty}\boldsymbol{T}^{k-1}=(\boldsymbol{I}-\boldsymbol{T})^{-1}$, we obtain (1.13). ∎

Example 6 *Let us consider the N-player ruin problem for $N=3$ and $\Sigma=7$ (as in Example 5) when the games are defined by the symmetric ruin problem we introduced in Example 1 of the Introduction. The first thing we notice is that, because of the specific definition of the transitions between states, the interior points of the grid Δ can be partitioned into three disjoint subsets of states which we denote by C_1,C_2 and C_3 (see Figure 4). Each of these subsets C_j, $j=1,2,3$, has the property that, starting from a state belonging to C_j, the process J_t will stay in C_j until absorption. The transition matrix of J_t on the set of interior points will therefore be of the form*

$$\boldsymbol{T}=\begin{pmatrix}\boldsymbol{B}_1 & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{B}_2 & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \boldsymbol{B}_3\end{pmatrix},$$

where the entries in the submatrices B_i are given by the transition probabilities between phases belonging to the class C_i. From Proposition 1.4 we know that $(\boldsymbol{I}-\boldsymbol{T})^{-1}_{\alpha\beta}$ gives the expected number of steps in phase β before absorption, given that the initial phase is α.

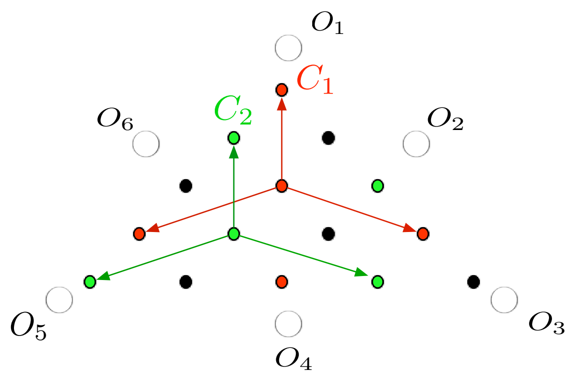

Figure 4: *The set of interior points is partitioned into three disjoint classes, C_1 (in red), C_2 (in green) and C_3 (in black). Once the process starts in one of these classes, it will stay there until absorption.*

Ordering the phases belonging to C_1 in clockwise direction starting from the highest point (see Figure 4), we see that

$$\boldsymbol{B_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix},$$

and that

$$(\boldsymbol{I}-\boldsymbol{B}_1)^{-1}\mathbf{1} = \left(1, \frac{8}{5}, \frac{9}{5}, \frac{8}{5}, \frac{12}{5}\right)'. \tag{1.14}$$

The same computation can be performed for $\boldsymbol{B}_2$ and $\boldsymbol{B}_3$.

Stirzaker (1994) showed that when the players start with initial distribution (a, b, c), then the expected number of games before the ruin of a player is given by

$$E[X] = \frac{abc}{\Sigma - 2}. \tag{1.15}$$

One can easily check numerically that this result concurs with ours although we do not see any probabilistic argument which would yield (1.15) through equation (1.12).

Applications to modified ruin problems

Any probability distribution with finite support on $\mathbb{N}$ is a discrete PH distribution (see Theorem 2.6.5 of Latouche and Ramaswami (1999).). Hence it is not a surprise that, translating the N-player problem into a random walk on a finite state space, the absorption probabilities are given by the distribution of PH random variables. In particular this also implies that any modified ruin problem which allows for the same kind of interpretation as above will also be solved by use of the same methods. We consider two such examples.

The N-player problem with capital constraints

An interesting modification of the N-player problem is that of computing the probability of, say, player 1 being ruined first while the other players still have certain defined assets. This problem is clearly equivalent to that of computing the probability that J_t exits the border through a specific collection of edge points, say $M = \{p_1, p_2, \ldots, p_K\} \subset \delta(\Delta)$. Hence, rewriting the ruin vector $\boldsymbol{r}_1$ as $(\boldsymbol{r}_1^M, \boldsymbol{r}_1^{M^c})$ with $(\boldsymbol{r}_1^M)_\alpha = \mathrm{P}[J_t \in M \mid J_{t-1} = \alpha]$ for $\alpha \in \{1, \ldots, p\}$, we see that it suffices to replace $\boldsymbol{r}_1$ by $\boldsymbol{r}_1^M$ in (1.13) to obtain the result.

The expected time until one player has everything

Consider a sequence of games which will only lead to ruin without debt (i.e. each player loses at most one unit per game). Now imagine that instead of stopping the sequence of games as soon as a ruin event occurs, we allow the players who have not been ruined to continue playing, and this until a single player is left (with all the money). The question we ask now is: what is the expected number of games that will be played until a single player is left? In the notations of the previous sections, we see that this is equivalent to looking for the expected number of steps until the random walk $J_k \in \Delta$ reaches one of the vertices $(\Sigma, 0, \ldots, 0)$, $\ldots$, $(0, \ldots, 0, \Sigma)$. The major difference between this situation and that which we described above is that now the only absorbing states for the random walk J_k are the vertices of Δ; the other edge points are partitioned into absorbing classes inside each of which the random walk will continue to take place.

Example 7 *If $N = 4$, then Δ is a tetrahedron. Starting from any point in the interior of Δ, we see that the ruin of a single player corresponds to J_k reaching the border at a point of the form $\boldsymbol{y} = (y_1, \ldots, y_4)$ such that $\sum_i y_i = \Sigma$ and only one y_i is set to zero. After such a ruin event, we see that the problem reduces to that of a random walk on a triangle, and hence the 4-player ruin problem reduces to a 3-player ruin problem. Likewise, once two players are ruined, the problem reduces to a 2-player ruin problem.*

Although our methods are à priori always applicable to this new situation, it is much harder to obtain elegant solutions and therefore we do not include these results.

1.4 Ruin by folding

In order to obtain explicit results for any given N-player ruin problem, one needs to write out the transition matrix $\boldsymbol{P}$ of the process J_t, and then apply equations (1.12) and (1.13). The drawback of this procedure is that, even for relatively small values of Σ and N, the matrices involved in these operations are very large and therefore the number of computations needed to obtain explicit results becomes rapidly intractable.

Hence we need to reduce the sizes of the matrices, and for this we will use their intrinsic properties. Indeed, one notices that for each choice of a game, only certain transitions will be possible (see e.g. Example 6 in which the set of transient states was partitioned into three subsets) and hence the matrix $\boldsymbol{T}$ will be sparse. This in turn implies that we do not need to work with the full matrix $\boldsymbol{T}$ but only with certain non identically nil submatrices. We will use this property in order to obtain an efficient algorithm for computing the key measures involved in the N-player problem. This will be done by use of what is known as a *folding* of the grid Δ (see Ye and Li (1994)).

1.4.1 Folding the grid

Definition 1.6 *We partition $\Delta = \{X_1 + \dots X_N = \Sigma\}$ into levels L_j, $j \geq 1$, each level being defined as the set of points such that at least one of their coordinates is equal to j and all their other coordinates are greater than j, i.e.*

$$L_j = \{X_1 + \ldots + X_N = \Sigma,\ X_i \geq j\ \forall i \text{ and } X_{j_0} = j \text{ for some } j_0\}$$

Example 8 *From Example 5 we know that when $N = 3$ and $\Sigma = 7$, we have a random walk on a set with 15 transient interior points, and 6 absorbing edge point. We can rearrange these points into two levels (see Figure 5), which are represented by concentric equilateral triangles. The exterior level (which is made up of the $n = 2^N - 2$ absorbing points) is not counted as a true level.*

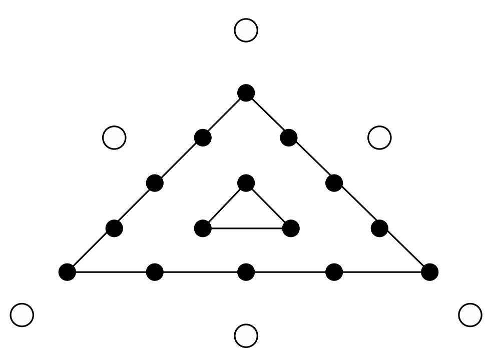

Figure 5: *When $N = 3$ and $\Sigma = 7$, the lattice defined by (1.9) is partitioned into two interior levels and a ruin level.*

The next proposition serves to count the number of levels, and the number of states on each level. For this, let us use the well known Euclidean algorithm (see e.g. Courant and Robbins (1978)) to divide Σ by N and obtain a unique decomposition of Σ into

$$\Sigma = Nk + l \text{ with } 0 \leq l \leq N - 1.$$

Proposition 1.7

1. *The grid is divided into k levels (without counting the ruin level).*
2. *Each level L_j with $j < k$ has*

$$b_j = \binom{\Sigma - Nj + N - 1}{\Sigma - Nj} - \binom{\Sigma - Nj - 1}{\Sigma - N(j+1)} \tag{1.16}$$

points. The number of points on level L_k is given by

$$b_k = \binom{l + N - 1}{l}.$$

Proof: We first prove statement 1. To do so, let us fix some integer j and consider level L_j. We want to find a necessary and sufficient condition for the level to be non-empty. Since $X_i \geq j \ \forall i$ and at least one player has capital $X = j$, we see that the condition $\{X_1 + \ldots + X_N = \Sigma\}$ will be satisfied if and only if there exist $N-1$ non negative integers $\tilde{x}_i, \ i = 1 \ldots N-1$ such that $j + (j + \tilde{x}_1) + \ldots + (j + \tilde{x}_N) = \Sigma$, i.e.

$$\tilde{x}_1 + \ldots + \tilde{x}_{N-1} = \Sigma - Nj$$

These $\tilde{x}_i$ will exist if and only if $\Sigma - Nj \geq 0$. By use of the unique decomposition of Σ into $Nk + l$, we see from the previous inequality that

$$N(j - k) \leq l$$

is a necessary and sufficient condition for the non emptiness of level L_j. Since $0 \leq l \leq N - 1$, this is satisfied if and only if $0 \leq j \leq k$. When j is equal to zero, we are at level 0, therefore there are only k different non-empty non-ruin levels.

For the second part of the proposition, let us take $1 \leq j \leq k - 1$. Clearly the number of points on level L_j will be given by

$$\#(L_j) = \#\{x_i \geq j, \ \forall i\} - \#\{x_i \geq j + 1, \ \forall i\}.$$

Equation (1.16) is then obtained by using the same urn model arguments as those used in the classification of states in Section 1.

This argument does not apply when we are looking for the number of points on the last level (L_k). However, subtracting from the total number of points the sum of the number of points on levels of lower order yields the result. ■

1.4.2 An imbedded Quasi Birth and Death Process

For a fixed ruin problem it is clear that level transitions of the random walk will only be possible from a level L_j to certain L_{j+k}'s independently of j. We are going to exploit this property. However, to make matters and notations less cumbersome, we will concentrate our attention on ruin games for which the random walk does not have transitions of more than one level at a time. Under this hypothesis, we therefore restrict our choice or ruin games to those studied for instance by Stirzaker (1994) and Bruss et al. (2003). The arguments we give will however be in principle adaptable to the different situations which arise in practice.

Rewriting the transition matrix

Every point α in the grid is defined by two coordinates $\alpha = (j, l)$ where $1 \leq j \leq k$ represents the level on which the point is and $1 \leq l \leq b_j$ the position of the point on this level. With the restriction we just imposed, it is clear that, starting from L_j, the random walk can only either stay on L_j or go up one level or go down one level. Therefore we define for each level Lj, $j \geq 1$ the submatrices $\boldsymbol{A}_0^{(j)}$, $\boldsymbol{A}_1^{(j)}$ and $\boldsymbol{A}_2^{(j)}$ in which the entries are given by

$$\begin{cases} (A_0^{(j)})_{\alpha\beta} = \mathrm{P}[J_{t+1} = \beta \in L_{j+1} \mid J_t = \alpha \in L_j] & \text{i.e. } L_j \to L_{j+1} \\ (A_1^{(j)})_{\alpha\beta} = \mathrm{P}[J_{t+1} = \beta \in L_j \mid J_t = \alpha \in L_j] & \text{i.e. } L_j \to L_j \\ (A_2^{(j)})_{\alpha\beta} = \mathrm{P}[J_{t+1} = \beta \in L_{j-1} \mid J_t = \alpha \in L_j] & \text{i.e. } L_j \to L_{j-1} \end{cases}$$

Each of these matrices represents the admissible transitions between levels. Since transitions to the absorbing states are only possible from the exterior level L_1, if we rearrange the transition matrix so as to group all points which are on the same level, $\boldsymbol{P}$ becomes

$$\boldsymbol{P} = \left[\begin{array}{cccccc|c} \boldsymbol{A}_1^{(k)} & \boldsymbol{A}_2^{(k)} & \boldsymbol{0} & \dots & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{A}_0^{(k-1)} & \boldsymbol{A}_1^{(k-1)} & \boldsymbol{A}_2^{(k-1)} & \dots & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_0^{(k-2)} & \boldsymbol{A}_1^{(k-2)} & \dots & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{A}_1^{(2)} & \boldsymbol{A}_2^{(2)} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{A}_0^{(1)} & \boldsymbol{A}_1^{(1)} & \boldsymbol{R} \\ \hline \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} \end{array}\right] \tag{1.17}$$

Remark 4 *The notations $\boldsymbol{A}_0, \boldsymbol{A}_1, \boldsymbol{A}_2$ have been chosen in order to stay consistent with the notations used in the standard text books on this subject (see e.g. Latouche and Ramaswami (1999)). However, our convention of enumerating the levels in decreasing order generates some minor differences between the form of the transient submatrix of (1.17) and that of a nonhomogeneous QBD.*

Example 9 *Let us consider the symmetric 3-player ruin problem we already studied in Example 6, this time taking $\Sigma = 10$. The process J_t then runs on a set with three levels of, respectively, 3, 12 and 21 points and a ruin level of 6 points. We fix the counting of the points on each level as starting from the highest point of the equilateral triangle and proceeding clockwise from there on. As an illustration we write the level-transition matrices starting from L_0 and L_1.*

$$\boldsymbol{A}_1^3 = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}, \qquad \boldsymbol{A}_2^3 = \begin{bmatrix} 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 \end{bmatrix}$$

$$\boldsymbol{A}_0^2 = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{A}_1^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 \end{bmatrix}$$

$$\boldsymbol{A}_2^2 = \begin{bmatrix} 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 \end{bmatrix}$$

For obvious space constraints, we do not write down the last three submatrices (i.e. $\boldsymbol{A}_i^{(1)}$, $i = 0, 1, 2$). They follow the same structure.

For $j = 1, \ldots, k$ and $l = 1, \ldots, n$ we define the *ruin vectors* $\boldsymbol{r}_{\boldsymbol{l}}^{(j)}$ in which the entries are given by

$$(r_l^{(j)})_i = \mathrm{P}[\text{ final ruin in } O_l \text{ starting from } (j, i)\ ], \text{ for } i = 1, \ldots, b_j.$$

Solving the ruin problem means finding the values of the ruin vectors.

Proposition 1.8 *For $l = 1, \ldots, n$,*

$$\begin{cases} \boldsymbol{r}_{\boldsymbol{l}}^{(1)} = (\boldsymbol{I} - \boldsymbol{A}_1^{(1)})^{-1} \left(R_{\bullet l} + \boldsymbol{A}_0^{(1)} \boldsymbol{r}_{\boldsymbol{l}}^{(2)} \right) \\ \boldsymbol{r}_{\boldsymbol{l}}^{(j)} = (\boldsymbol{I} - \boldsymbol{A}_1^{(j)})^{-1} \left(\boldsymbol{A}_2^{(j)} \boldsymbol{r}_{\boldsymbol{l}}^{(j-1)} + \boldsymbol{A}_0^{(j)} \boldsymbol{r}_{\boldsymbol{l}}^{(j+1)} \right), \ j = 1, \ldots k-1 \\ \boldsymbol{r}_{\boldsymbol{l}}^{(k)} = (\boldsymbol{I} - \boldsymbol{A}_1^{(k)})^{-1} \left(\boldsymbol{A}_2^{(k)} \boldsymbol{r}_{\boldsymbol{l}}^{(k-1)} \right) \end{cases} \tag{1.18}$$

where $R_{\bullet l} = (R_{1l}, \ldots, R_{b_1 l})'$.

Proof:

$$\begin{aligned}(r_l^{(j)})_i &= \mathrm{P}[\text{final ruin in } O_l \text{ starting from } (j,i)] \\ &= \sum_{t=1}^{b_j} (\boldsymbol{A}_1^{(j)})_{it}(r_l^{(j)})_t + \sum_{t=1}^{b_{j+1}} (\boldsymbol{A}_0^{(j)})_{it}(r_l^{(j+1)})_t + \sum_{t=1}^{b_{j-1}} (\boldsymbol{A}_2^{(j)})_{it}(r_l^{(j-1)})_t,\end{aligned}$$

i.e.

$$\begin{aligned}\boldsymbol{r}_{\boldsymbol{l}}^{(j)} &= \boldsymbol{A}_1^{(j)}\boldsymbol{r}_{\boldsymbol{l}}^{(j)} + \boldsymbol{A}_0^{(j)}\boldsymbol{r}_{\boldsymbol{l}}^{(j+1)} + \boldsymbol{A}_2^{(j)}\boldsymbol{r}_{\boldsymbol{l}}^{(j-1)} \\ &= \left(\boldsymbol{A}_1^{(j)}\right)^2 \boldsymbol{r}_{\boldsymbol{l}}^{(j)} + \left(\boldsymbol{I} + \boldsymbol{A}_1^{(j)}\right)\left(\boldsymbol{A}_0^{(j)}\boldsymbol{r}_{\boldsymbol{l}}^{(j+1)} + \boldsymbol{A}_2^{(j)}\boldsymbol{r}_{\boldsymbol{l}}^{(j-1)}\right) \\ &= \lim_{K\to\infty}\left[(\boldsymbol{A}_1^{(j)})^K \boldsymbol{r}_{\boldsymbol{l}}^{(j)} + \left(\sum_{t=0}^{K}(\boldsymbol{A}_1^{(j)})^t\right)\left(\boldsymbol{A}_0^{(j)}\boldsymbol{r}_{\boldsymbol{l}}^{(j+1)} + \boldsymbol{A}_2^{(j)}\boldsymbol{r}_{\boldsymbol{l}}^{(j-1)}\right)\right]\end{aligned}$$

From arguments given previously, we know that $\lim_{K\to\infty}(\boldsymbol{A}_1^{(j)})^K = 0$ and that

$$\sum_{t=0}^{\infty}(\boldsymbol{A}_1^{(j)})^t = (\boldsymbol{I} - \boldsymbol{A}_1^{(j)})^{-1}.$$

This yields the first equation. The two other equations are proved with similar arguments. ■

Equations like (1.18) and those appearing later on in this chapter allow for an explicit interpretation. To facilitate the understanding of this interpretation we give a detailed explanation of the second equation in (1.18) for some fixed intermediate $j \in \{2, \dots, k-1\}$, i.e.

$$\boldsymbol{r}_{\boldsymbol{l}}^{(j)} = (\boldsymbol{I} - \boldsymbol{A}_1^{(j)})^{-1}\left(\boldsymbol{A}_2^{(j)}\boldsymbol{r}_{\boldsymbol{l}}^{(j-1)} + \boldsymbol{A}_0^{(j)}\boldsymbol{r}_{\boldsymbol{l}}^{(j+1)}\right).$$

We condition on the first level visited after leaving L_j. Once the random walk leaves L_j it has two options. Either it goes up to L_{j+1} (which happens with probability $\boldsymbol{A}_0^{(j)}$) and is absorbed in O_l from there (which happens with probability $\boldsymbol{r}_{\boldsymbol{l}}^{(j+1)}$). This explains the term $\boldsymbol{A}_0^{(j)}\boldsymbol{r}_{\boldsymbol{l}}^{(j+1)}$. Or it goes down to L_{j-1} (which happens with probability $\boldsymbol{A}_2^{(j)}$) and is absorbed in O_l from there (which happens with probability $\boldsymbol{r}_{\boldsymbol{l}}^{(j-1)}$). This explains the term $\boldsymbol{A}_2^{(j)}\boldsymbol{r}_{\boldsymbol{l}}^{(j-1)}$. Before leaving L_j and being ruined, there is a probability that it first returns a number of times to L_j. This happens with probability $\sum_{i\geq 1}(\boldsymbol{A}_1^{(j)})^i$. This explains the presence of the term $(\boldsymbol{I} - \boldsymbol{A}_1^{(j)})^{-1}$.

Hence, we may summarize the meaning of these equations by saying that absorption probabilities are mutually expressed in terms of 'neighboring' absorption probabilities.

1.5 The Folding Algorithm

Suppose that we are solving the N-player ruin problem on a system with k levels. Let us consider the restriction of J_t to the set of even numbered levels $L_2, L_4, \dots, L_{2\lfloor k/2\rfloor}$. This yields a new random walk, for which transitions from level L_{2i} to level L_{2j} are given by the first hitting time of the initial random walk J_t from L_{2i} to L_{2j}. From the view point of J_t, we consider the transitions $P_{\alpha\beta}$ if and only if α and β are two states on the same level or if β is the first state on an even-numbered level that the random walk visits after leaving α. This new random walk runs on a grid with $\lfloor k/2 \rfloor$ levels.

Now, solving the ruin equations (1.13) for this smaller system, i.e. with smaller matrices, will yield the ruin vectors for even levels. And from Proposition 1.8 we see that we can apply equation (1.18) to determine the ruin vectors for the whole system.

This restriction of the initial random walk to a new set with half the number of levels is what we call the *folding* of the process. The new random walk is the *folded* random walk.

The transition matrix of the folded random walk is given by the following proposition (which is proved by using arguments similar to those used in Proposition 1.8).

Proposition 1.9 *Take $s = \lfloor k/2 \rfloor$. If $s \geq 3$, then define*

$$\left\{ \begin{array}{rcl} \boldsymbol{A}_0^{\star(j)} & = & \boldsymbol{A}_0^{(2j)}(\boldsymbol{I} - \boldsymbol{A}_1^{(2j+1)})^{-1}\boldsymbol{A}_0^{(2j+1)}, \\ \boldsymbol{A}_1^{\star(j)} & = & \boldsymbol{A}_1^{(2j)} + \boldsymbol{A}_2^{(2j)}(\boldsymbol{I} - \boldsymbol{A}_1^{(2j-1)})^{-1}\boldsymbol{A}_0^{(2j-1)} \\ & & +\, \boldsymbol{A}_0^{(2j)}(\boldsymbol{I} - \boldsymbol{A}_1^{(2j+1)})^{-1}\boldsymbol{A}_2^{(2j+1)}, \\ \boldsymbol{A}_2^{\star(j)} & = & \boldsymbol{A}_2^{(2j)}(\boldsymbol{I} - \boldsymbol{A}_1^{(2j-1)})^{-1}\boldsymbol{A}_2^{(2j-1)}. \end{array} \right. \tag{1.19}$$

for $j = 2 \ldots (s-1)$ if k is even or for $j = 2 \ldots s$ if k is odd.
Also, if k is even, define

$$\left\{ \begin{array}{l} \boldsymbol{A}_2^{\star(s)} = \boldsymbol{A}_2^{(k)}(\boldsymbol{I} - \boldsymbol{A}_1^{(k-1)})^{-1}\boldsymbol{A}_2^{(k-1)}, \\ \boldsymbol{A}_1^{\star(s)} = \boldsymbol{A}_1^{(k)} + \boldsymbol{A}_2^{(k-1)}(\boldsymbol{I} - \boldsymbol{A}_1^{(k-1)})^{-1}\boldsymbol{A}_0^{(k-1)}. \end{array} \right. \tag{1.20}$$

Finally define,

$$\left\{ \begin{array}{l} \boldsymbol{A}_1^{\star(1)} = \boldsymbol{A}_1^{(2)} + \boldsymbol{A}_0^{(2)}(\boldsymbol{I} - \boldsymbol{A}_1^{(3)})^{-1}\boldsymbol{A}_2^{(3)} + \boldsymbol{A}_2^{(2)}(\boldsymbol{I} - \boldsymbol{A}_1^{(1)})^{-1}\boldsymbol{A}_0^{(1)}, \\ \boldsymbol{A}_0^{\star(1)} = \boldsymbol{A}_0^{(2)}(\boldsymbol{I} - \boldsymbol{A}_1^{(3)})^{-1}\boldsymbol{A}_2^{(3)}, \\ \boldsymbol{R}^{\star} = \boldsymbol{A}_0^{(2)}(\boldsymbol{I} - \boldsymbol{A}_1^{(1)})^{-1}\boldsymbol{R}. \end{array} \right. \tag{1.21}$$

With these notations, the transition matrix of the process restricted to even levels is given by

$$P = \left[\begin{array}{cccccc|c} \boldsymbol{A}_1^{\star(s)} & \boldsymbol{A}_0^{\star(s)} & \boldsymbol{0} & \ldots & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{A}_0^{\star(s-1)} & \boldsymbol{A}_1^{\star(s-1)} & \boldsymbol{A}_2^{\star(s-1)} & \ldots & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_0^{\star(s-2)} & \boldsymbol{A}_1^{\star(s-2)} & \ldots & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \ldots & \boldsymbol{A}_1^{\star(2)} & \boldsymbol{A}_2^{\star(2)} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \ldots & \boldsymbol{A}_0^{\star(2)} & \boldsymbol{A}_1^{\star(2)} & \boldsymbol{R}^{\star} \\ \hline \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \ldots & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} \end{array} \right]$$

■

Example 10 *If $N = 3$ and $\Sigma = 9$, then we have 3 levels. After folding, we are left with a system with one level (see Figure 6)).*

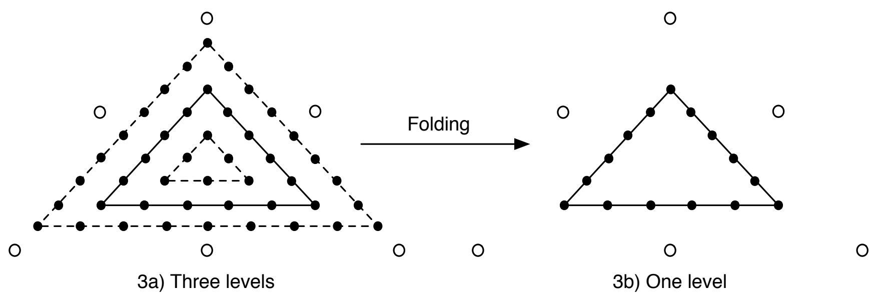

Figure 6: *Folding of the lattice defined by (1.9) when $N = 3$ and $\Sigma = 9$.*

If, after the initial folding, the ruin problem is considered on a set which still has more than two levels, then one can repeat the folding and obtain a random walk on a smaller set. The folding can be repeated over and over until the set on which the random walk is running has only one or two levels left. The ruin equations (1.13) can be solved on this set, and with these results we can apply recursively equations (1.18) in order to obtain the ruin vectors for the whole system.

It is now clear how one can devise a recursive algorithm which will compute the ruin probabilities. Let $\Sigma = Nk + l$ (Euclidean division of Σ by N) and $s = \lfloor k/2 \rfloor$.

1. Apply recursively Proposition 1.9 until $s \leq 2$.
2. Apply equation (1.13) to compute the ruin probabilities associated with the last system.
3. Use equation (1.8) recursively to compute the ruin vectors for each level of the game.

Now it is well known that the inversion of a $p \times p$ matrix takes in the order of p^3 operations and the multiplication of a $p \times q$ matrix with a $q \times r$ matrix takes in the order of pqr operations. Therefore, direct inversion of the ruin problem using equation (1.13) takes in the order of

$$p^3 = \begin{pmatrix} \Sigma - 1 \\ N - 1 \end{pmatrix}^3$$

operations. The next proposition gives a rough upper bound for the number of operations involved in the folding algorithm.

Proposition 1.10 *The number of operations involved in the folding algorithm is bounded above by*

$$\frac{\Sigma}{(\Sigma - 1)^3} N^5 p^3.$$

Remark 5 *For fixed N and $\Sigma > N$, this simple upper bound already shows considerable savings in the number of operations, of $O(\Sigma^2)$.*

Proof: We use the same notations as in Proposition 1.7. After each folding, the new random walk runs on a set with $\lfloor k/2 \rfloor$ levels. Therefore, after at most $\lfloor \log_2 k \rfloor$ foldings, there will be strictly less than 3 levels left. We denote by $N(l)$ be the number of operations involved in the lth folding. We will only take into account multiplications and inversions appearing in (1.19). For each $j \in \{1, \dots, k\}$ there are 12 such operations. Counting the number of operations in the same way as above and using the fact that the b_j's are decreasing in j, it is then straightforward to see that

$$N(j) \leq 12 \frac{k}{2^j} b_{2^{j-1}}^3.$$

This implies that the total number of operations involved in the folding of the process is in the order of

$$k \sum_{j=1}^{\lfloor \log_2 k \rfloor} \frac{1}{2^j} b_{2^{j-1}}^3 \leq k b_1^3 \sum_{j=1}^{\lfloor \log_2 k \rfloor} \frac{1}{2^j} \leq k b_1^3.$$

One can show that

$$b_1 \leq N \begin{pmatrix} \Sigma - 2 \\ \Sigma - N \end{pmatrix} = N \frac{N-1}{\Sigma - 1} p,$$

and therefore a rough upper bound for the number of operations involved in the folding of the process is given by

$$k \left(N \frac{N-1}{\Sigma - 1} p \right)^3.$$

Application of equation (1.13) to obtain the ruin vectors on the folded set will not change the order of the number of operations. Also, the recursive application of (1.18) to get the ruin vectors of the whole system will take the same number of operations as the folding, and therefore does not either change the order of the number of operations. Using $k \leq \frac{\Sigma}{N}$, we therefore see that the number of operations demanded by the folding algorithm allows the upper bound

$$\frac{\Sigma}{(\Sigma - 1)^3} N^5 p^3.$$

■

Chapter 2

Conformal Transformations, Brownian Motion and the 3-Player Ruin Problem

2.1 Brownian motion approximation of the symmetric 3-player problem

We return to the symmetric 3-player problem which was presented in Example 1 of the Introduction. In this problem, three players own initially a, b and c euros, respectively. They play a sequence of fair games during each of which one randomly selected player wins and receives one unit from the other two players. We are interested in the probability that a given player is ruined first.

Let again $\Sigma = a + b + c$. As in Chapter 1, this ruin problem defines a random walk on the triangular lattice

$$\Delta = \{(X, Y, Z) \in \mathbb{N}^n \mid X + Y + Z = \Sigma\},$$

with transitions of the form

$$(a, b, c) \to \begin{cases} (a+2, b-1, c-1) \\ (a-1, b+2, c-1) \\ (a-1, b-1, c+2) \end{cases} . \tag{2.1}$$

Since such transitions can only lead to ruin without debt, we see that the notion of ruin of a player corresponds to the random walk hitting the boundary of Δ.

This random walk can obviously be scaled down to a random walk on a triangle Δ' in the plane with vertices $(-\frac{1}{2}, 0), (\frac{1}{2}, 0)$ and $(0, \frac{\sqrt{3}}{2})$ (see Figure 1).

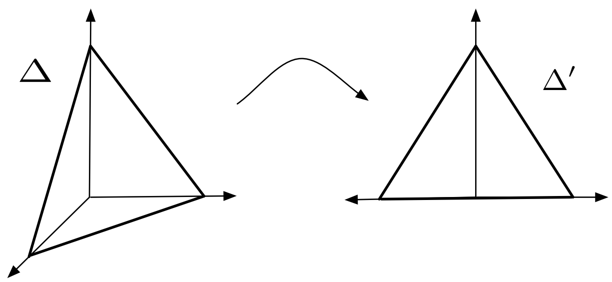

Figure 1: *Rescaling the grid* Δ.

Each vector (a, b, c) on Δ corresponds now to the vector $(\frac{a-b}{2\Sigma}, \frac{\sqrt{3}}{2\Sigma}c) \in \Delta'$, and, since every point (a, b, c) in Δ satisfies $a + b + c = \Sigma$, this transformation is bijective. It is now straightforward to apply this transformation to (2.1) and obtain that the transitions of the scaled random walk define transitions in Δ' of the form

$$(x, y) \to (x, y) + \frac{\sqrt{3}}{\Sigma}\Big(\cos((2k+1)\frac{\pi}{6}), -\sin((2k+1)\frac{\pi}{6})\Big) \tag{2.2}$$

for $k = 0, 1, 2$.

Now let $\{\lambda_i\}_{i\in\mathbb{N}}$ be a sequence of independent, identically distributed random vectors taking values $\big(\cos((2k+1)\pi/6), -\sin((2k+1)\pi/6)\big)$ for $k = 0, 1, 2$ with equal probability, and for each $(x, y) \in \mathbb{R}^2$ let $\{X_t^n,\ t \geq 0\}_{n\in\mathbb{N}}$ be the process defined by

$$X_t^n = (x, y) + \sqrt{2/n}\left(\sum_{i=1}^{\lfloor nt\rfloor} \lambda_i + (nt - \lfloor nt\rfloor)\lambda_{\lfloor nt\rfloor+1}\right).$$

For $n = \Sigma$ sufficiently large, the process X_t^Σ is a good approximation of the random walk on the grid Δ' defined by (2.2). Moreover it is well known (see for example Alabert et al. (2003) who perform the same kind of scaling in the setting of the three tower problem of Example 2) that the sequence of processes $\{X_t^n,\ t \geq 0\}_{n\in\mathbb{N}}$ converges in law to a standard Brownian motion without drift starting at (x, y). Hence, for large values of Σ, we see that the probability of the scaled random walk (starting at $(x, y) \in \Delta'$) hitting a given edge first is approximated by that of a Brownian motion (starting at (x, y)) hitting the same edge.

Therefore the hitting time problem for Brownian motions in an equilateral triangle can be seen as an asymptotic version of the symmetric three player problem.

2.2 Exit probabilities and conformal transformations

A region $U \subset \mathbb{C}$ is called *simply connected* if it is path-connected and every path between two points can be continuously transformed into every other. Informally, this means that U consists of one piece and doesn't have any "holes" that pass all the way through it. Also, recall that a mapping $f : U \to \mathbb{C}$ is *conformal* if it preserves the angle between two intersecting differentiable arcs. This is equivalent to f being holomorphic and having non-zero derivative everywhere on U. One of the fundamental building blocks of the theory of conformal transformations is the following theorem (see for example Rudin(1966) p. 273, or Bieberbach(1953) p. 128).

Theorem 2.1 *(Riemann's Theorem.)*
If U is a simply connected open region in the plane (other than the plane itself) then there exists a conformal transformation f of U onto D, where

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

It is uniquely determined up to the choice of three points in Γ and their images.

Since the inverse of a one-to-one conformal transformation is another conformal transformation, Theorem 2.1 also implies that any two simply connected regions of the plane are conformally equivalent, i.e. can be mapped onto one another by a one-to-one conformal transformation.

Let us consider the extended complex plane $\mathbb{C} \cup \infty$, i.e. the complex plane augmented by the point at infinity. An important family of conformal transformations is the *Möbius transformation.* These are mappings of the form

$$f(z) = \frac{az + b}{cz + d}$$

where a, b, c, d are complex numbers satisfying $ad - bc \neq 0$. Any such transformation is one to one from the extended plane to itself, and uniquely determined up to the choice of three distinct points and their three distinct images (see Bieberbach(1953) pages 24, 25, 26). In particular, for any $z_0 \in U$ such that $\mathrm{Im}(z_0) > 0$, the transformation

$$f_{z_0}(z) = e^{i\alpha} \frac{z - z_0}{z - \bar{z}_0}, \tag{2.3}$$

is a conformal one-to-one transformation of the upper half plane $\mathrm{Im}(z) \geq 0$ onto the unit disk which sends z_0 into the center of the disk.

The link between conformal transformations and the exit probabilities of a Brownian motion is given by the following theorem (see e.g. Lévy(1965), p. 254, Durrett (1984) or Bass (1995)).

Theorem 2.2 *(Lévy's Theorem.)*
The intrinsic properties of a Brownian motion remain invariant under conformal transformations.

Consequently, if a Brownian motion initially occupies a position z_0 interior to a contour Γ, the probability that it reaches Γ for the first time through an arc Γ' is proportional to the harmonic measure of Γ' seen from z_0. Hence, if U is the region bounded by Γ, and f_{z_0} is a conformal one-to-one transformation of U into the unit disk D which sends z_0 onto the centre of D, we see from Theorem 2.2 that the exit probability of the Brownian motion through Γ' will be given, by symmetry, by the length of $f_{z_0}(\Gamma')$ divided by 2π. This general principle is illustrated in Figure 2.

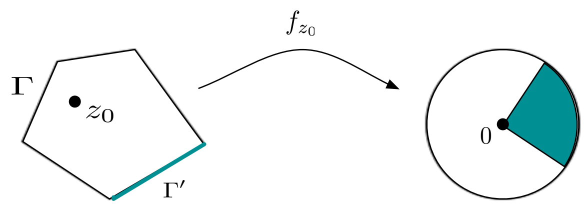

Figure 2: *The probability that a Brownian motion starting at z_0 exits Γ first through Γ' is given by the harmonic measure of Γ' seen from z_0.*

From Theorem 2.1, we know that there exists such a conformal transformation f_{z_0} in so far as the region bounded by Γ is an open simply connected region other than the plane itself.

2.3 The Schwarz-Christoffel transformation

This is a family of conformal transformations which map a canonical domain (unit disk, upper half plane...) conformally into the interior of a polygon. They were named after Hermann A. Schwarz and Elwin B. Christoffel, who discovered them independently. We shall now describe the construction of these transformations. These results are well known (see e.g. Nehari (1952) or Churchill et al. (1974)) and are stated without proofs.

Construction

Let P be a closed convex polygon in the complex plane, and denote its vertices (counterclockwise) by $w_1, w_2, \ldots, w_n$. Our aim is to construct a conformal mapping of the upper half plane onto P which sends the real axis $\text{Im}(z) = 0$ onto the border of P. For this let us choose $n-1$ real points $-\infty < x_1 < x_2 < \ldots < x_{n-1} < \infty$. The x_i's are called the *prevertices* of the polygon, and are, for the moment, supposed to be arbitrary.

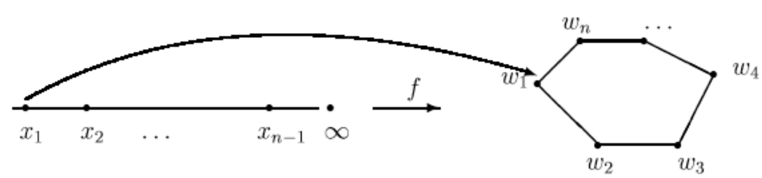

Figure 3: *Transformation of the upper half plane into a closed convex polygon P which sends the prevertices $x_1, \dots, x_{n-1}$ in that order onto the vertices of P, $w_1, \dots, w_{n-1}$.*

We will now obtain conditions which must be satisfied by any transformation of the upper half plane onto P which, for all $i = 1, \dots, n-1$, sends x_i onto w_i. With these conditions, we will derive a general formula for all such transformations; this will define the family of *Schwarz-Christoffel treansformations*.

Consider a smooth directed arc $C = z(t)$ in the complex z-plane and let v denote the unit vector tangent to C at a point $z_0 := z(t_0)$. Let τ denote the unit vector tangent to the image Γ of C in the complex w-plane under a transformation $w = f(z)$ at the corresponding point $w_0 := f(z_0)$. Suppose that the transformation f is analytic at z_0 and that $f'(z_o) \neq 0$ (in particular this implies that f is conformal).

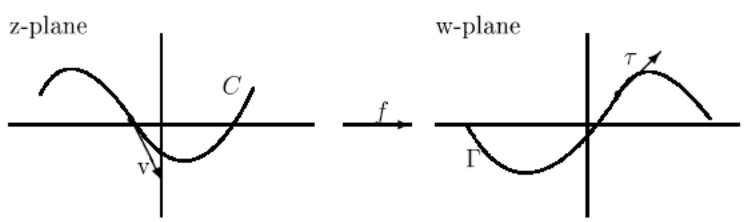

Figure 4: *The curve C in the complex z-plane is transformed into the curve Γ in the complex w-plane by a conformal mapping f. The vector τ is the unit vector tangent to Γ at the point $w_0 = f(z_0)$.*

The curve Γ is defined by $w(t) = f(z(t))$ at all points t. Hence we see that

$$\log(w'(t)) = \log(f'(z(t))) + \log(z'(t)),$$

and thus

$$\arg[\tau] = \arg[f'(z_0)] + \arg[v].$$

In particular, if C is a positively directed segment of the x-axis, we see that $v = 1$ and that $\arg[v] = 0$ at every $z_0 = x$ on C, i.e.

$$\arg[\tau] = \arg[f'(z_0)]. \tag{2.4}$$

If $f(z)$ has constant argument along that segment, it therefore follows that $\arg[\tau]$ is constant and that the image Γ of C is also a segment of a straight line.

Thus, in order to map the upper half plane onto P, we see that it suffices to obtain a transformation $w = f(z)$ whose derivative has constant argument along each of the segments $]x_k, x_{k+1}[$, and such that $\arg[f'(z)]$ changes value abruptly at each point $z = x_k$. Let us consider the properties of functions whose derivatives are given by

$$f'(z) = A(z - x_1)^{-k_1}(z - x_2)^{-k_2} \dots (z - x_{n-1})^{-k_{n-1}} \tag{2.5}$$

with $A \in \mathbb{C}$ and the k_i's $\in \mathbb{R}$. Such a function obviously satisfies:

$$\arg[f'(z)] = \arg[A] - k_1 \arg[z - x_1] - \dots - k_{n-1} \arg[z - x_{n-1}] \tag{2.6}$$

Now let us consider a point moving along the real axis $z = x \in \mathbb{R}$ in a positive direction, starting at $-\infty$ up to x_1. Since $z < x_i$ for all $i = 1 \dots n-1$, all the summands in equation (2.6) (except $\arg[A]$) are constant and equal to π. Hence on $[-\infty, x_1]$ we have

$$\arg[f'(z)] = \arg[A] - (k_1 + k_2 + k_3 + \dots + k_{n-1})\pi.$$

But as the point passes x_1, the term $\arg[z - x_1]$ reduces to zero, since $z - x_1 > 0 \in \mathbb{R}$. The other terms remain constant and equal to π. From equation (2.6) this implies that $\arg[f'(z)]$ increases abruptly at the prevertex x_1 by the angle $k_1\pi$ but remains constant from there up to x_2, and hence, on $[x_1, x_2]$ we have

$$\arg[f'(z)] = \arg[A] - (0 + k_2 + k_3 + \dots + k_{n-1})\pi.$$

As the moving point passes x_2 the argument of the derivative of our function increases again by the angle $k_2\pi$ and remains constant thereafter until x_3, i.e. for all $z \in [x_2, x_3]$,

$$\arg[f'(z)] = \arg[A] - (0 + 0 + k_3 + \dots + k_{n-1})\pi.$$

We can repeat the same argument at each prevertex x_i. From equation (2.4) it follows that the image of each segment $]x_i, x_{i+1}[$ is a line segment $]w_i, w_{i+1}[$ in the w-plane, and that the exterior angle at each change of direction z_i is given by $k_i\pi$. We note that from x_{n-1} to x_n all but one the summands in equation (2.6) vanish, which implies that $\arg[f'(z)] = \arg[A]$ along that segment. Setting $k_n\pi = 2\pi - (k_1 + \dots + k_{n-1})\pi$ we see that from $-\infty$ to x_1, $\arg[f'(z)] = \arg[A] + k_n\pi$. Therefore it suffices to set $k_i > 0 \ \forall i = \dots n$ to ensure that the image, through a function f whose derivative is given by equation (2.5), of a point moving in positive direction along the x-axis in the w-plane is a positively oriented closed convex polygon with vertices $w_1, \dots, w_n$ and exterior angles $k_1\pi, \dots, k_n\pi$.

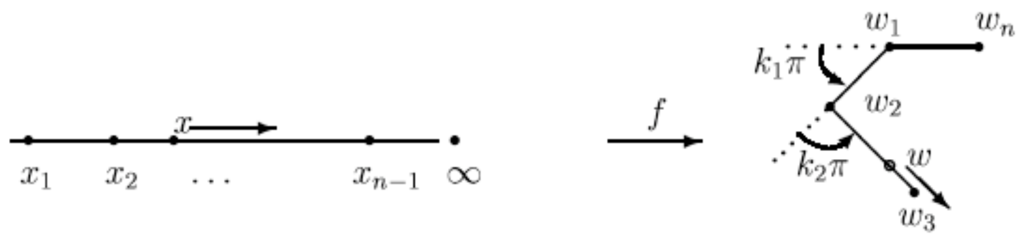

Figure 5: *The exterior angles of P at each vertex w_i are given by $k_i\pi$.*

It should be noted that if $k_1 + \ldots + k_{n-1} = 2$, then there is no vertex w_n since there is no change of direction at that point, and the image polygon has therefore $n-1$ sides. This implies that we can lift the restriction that one of the prevertices is the point at infinity, if we wish to do so. However, for practical reasons, this is rarely done.

These arguments explain the following theorem (of which a proof is given for example in Nehari (1952) and Churchill et al. (1974)).

Theorem 2.3 *(The Schwarz-Christoffel Formula.) Given a closed convex polygon P with vertices $w_1, \ldots, w_n$ and exterior angles $k_1\pi, \ldots, k_n\pi$ (taken in counterclockwise order), there exist $n-1$ real constants $x_1 < \ldots < x_{n-1}$ and two complex constants A, B such that the mapping*

$$f(z) = A\int_{z_0}^{z} (s-x_1)^{-k_1} \ldots (s-x_{n-1})^{-k_{n-1}} ds + B \tag{2.7}$$

is a conformal transformation of the upper half plane into the interior of P, which maps the real axis onto P, each x_i to the corresponding w_i and the point at infinity to w_n. This mapping is continuous throughout the upper half plane $y \geq 0$ and conformal except at the prevertices.

Mappings defined by (2.7) are known as Schwarz-Christoffel transformations.

Note that there is a one-to-one correspondence between points on the x-axis and points on P. Also, if z is some point interior to the upper half plane, and x_0 some point on the real axis (different from the x_i's), then, since f is conformal throughout the upper half plane, the angle formed by the vector joining x_0 and z must be preserved. Thus, the image of interior points of the upper half plane lies to the left of the polygon taken counterclockwise (see Figure 6).

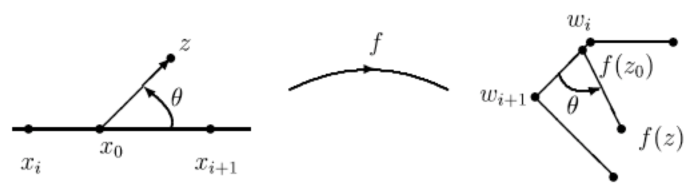

Figure 6: *The interior of the image polygon lies to the left of the border P.*

Lifting the restriction $x_n = \infty$ clearly adds one term to the integrand of (2.7), of the form $(s - x_n)^{-k_n}$.

Choice of constants

Let P be a polygon with vertices $w_1, \ldots, w_n$ and exterior angles $k_1, \ldots, k_n$. In order for f to map the entire x-axis onto P, we must have the following n equalities

$$f(x_1) = w_1, \ldots, f(x_n) = w_n.$$

Writing $f = AF + B$, one sees that the complex constant $A = |A|e^{i\arg[A]}$ comprises an arbitrary magnification factor $|A|$ and a rotation by the angle $\arg[A]$. In the same way, the constant $B = b_0 + ib_1$ represents an arbitrary translation without distortion through the vector $b_0 + ib_1$. Therefore, using

$$F(z) = \int_{z_0}^{z} (s - x_1)^{-k_1} \ldots (s - x_{n-1})^{-k_{n-1}} ds, \tag{2.8}$$

if we determine the x_i's in such a way that F maps the x-axis onto a polygon P' similar to P, we can then choose a magnification, rotation and translation through A and B to map P' onto P. Thus these two constants are also predetermined by the position and orientation of the polygon P, and we are left to choose the x_i's to ensure that the image through F of the x-axis is a polygon similar to P.

Knowing that there are an infinite number of ways to map the upper half plane onto itself, we anticipate some freedom in the choice of the x_i's. This is indeed the case. The image polygon P' through equation (2.8) has the same exterior angles as P. Therefore, it suffices to make sure that the $n-2$ connected sides of P' have a common ratio to the corresponding sides of P. This yields $n-3$ equations in the $n-1$ unknowns x_i. Therefore two of these, or two relations between them, can be chosen arbitrarily (provided of course that the corresponding system has $n-3$ real solutions). This means we have two degrees of freedom. When the condition $x_n = \infty$ is removed, we have in fact three degrees of freedom, as predicted by Riemann's theorem.

The remaining $n-3$ prevertices are then uniquely determined and can be obtained by solving a system of nonlinear equations. This is non-trivial and is known as the *Schwarz-Christoffel parameter problem*. See for example Howell (1990) for an overview of this problem and its inherent difficulty.

2.4 The Schwarz-Christoffel transformation as a tool for computing exit probabilities

2.4.1 Brownian motion in an infinite strip and a solution of the 2-player ruin problem

Let us take a planar Brownian motion starting at some point within an infinite strip which we suppose (without loss of generality) to be of width π. We are looking for an explicit link between the distance from the initial point $p_0 = x_0 + iy_0$ to the borders and the exit probability of the Brownian motion through one of the borders.

By viewing the strip as the limiting form of a rhombus with vertices $w_1 = i\pi, w_2 = -\infty, w_3 = 0, w_4 = \infty$ and corresponding exterior angles $k_1\pi = 0 = k_3\pi, k_2\pi = \pi = k_4\pi$, we can use the Schwarz-Christoffel transformation to determine a conformal transformation from the upper half plane into the infinite strip. Choose (arbitrarily) the prevertices $x_2 = 0, x_3 = 1, x_4 = \infty$. Since we have only three degrees of freedom, x_1 is left to be determined. We need a transformation f such that $f(x_1) = i\pi$, $f(x_2) = w_2$, $f(x_3) = 0$, and $f(x_4) = w_4$. This mapping has the derivative

$$\frac{df}{dz} = A(z-x_1)^0 z^{-1}(z-1)^0 = \frac{A}{z}$$

so that $f(z) = A\log(z) + B$.

In order to determine A, B and x_1, we must use the conditions on the prevertices, which yield $A = 1$, $B = 0$ and $x_1 = -1$, and the mapping we are looking for is given by

$$w = f(z) = \log(z).$$

The inverse of this transformation is therefore a conformal one to one transformation of the infinite strip into the upper half plane. Let us combine it with a Möbius transformation (see equation (2.3)) which maps $z_0 = f^{-1}(p_0) = e^{p_0}$ into the center of the unit disk. We have thus determined a one to one conformal transformation of the infinite strip into the unit disk which maps p_0 into its center; this transformation is given by

$$F(w) = i\left(\frac{e^w - e^{p_0}}{e^w - \overline{e^{p_0}}}\right). \tag{2.9}$$

Since the prevertices of our polygon are $x_1 = -1 = F(w_1), x_2 = 0 = F(w_2), x_3 = 1 = F(w_3)$ and $x_4 = \infty = F(w_4)$, from (2.9), we see that

$$\begin{aligned} F(w_2) &= \cos(\tfrac{\pi}{2} + 2y_0) + i\sin(\tfrac{\pi}{2} + 2y_0) \\ F(w_4) &= i \end{aligned}$$

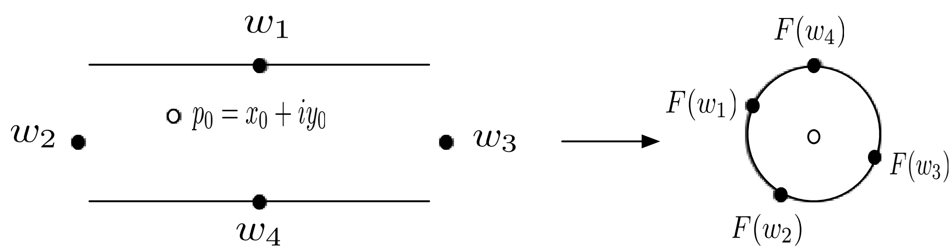

Figure 7: *Conformal transformation of the infinite strip into the unit disk, sending z_0 to the center of the disk.*

Therefore the probability that the Brownian motion will exit the strip through the upper edge is given by

$$\frac{\arg(F(w_2)) - \arg(F(W_4))}{2\pi} = \frac{y_0}{\pi}.$$

This probability is independent of x_0, as expected.

Now let a, b be two positive real constants and consider a Brownian motion starting at $p_0 = ib$ inside an infinite strip of width $a + b$. An immediate generalization of the previous arguments yields $\frac{b}{a+b}$ as the probability that the Brownian motion exits the strip through the upper edge first. This is, as expected, the same result as for the Brownian approximation of the 2-player ruin problem.

2.4.2 Brownian motion in a triangle and a solution of the symmetric 3-player ruin problem

Recall the Brownian motion approximation of the 3-player ruin problem which we described in Section 2.1. This is an exit problem for a Brownian motion starting at some point p_0 inside the equilateral triangle with vertices $[-\frac{1}{2}, \frac{1}{2}, i\frac{\sqrt{3}}{2}]$. After a change in coordinates, this problem can clearly be transformed into the same problem on the equilateral triangle with vertices $\Delta := [-1,\ 1,\ i\sqrt{3}]$ in the upper half complex plane (H^+). From Section 2.2 we know that if we construct a conformal transformation F_{p_0} from the triangle to the unit disk which maps the starting point p_0 into the center of the disk, then by computing the images of each of the three summits of the triangle through this mapping we will obtain the desired probabilities. (For example, the probability that the third player is ruined first is given by the length of the arc joining $F_{p_0}(-1)$ to $F_{p_0}(1)$, divided by 2π).

In order to construct an explicit conformal transformation of Δ into the unit disk, we will first construct a Schwarz-Christoffel transformation from the upper half plane H^+ into Δ, which we will denote by F. This transformation being one-to-one, we can take its inverse to get a conformal mapping from Δ into H^+. This inverse function maps the starting point p_0 into some point in H^+, say $z_0 = F^{-1}(p_0)$. We will then use a Möbius transformation , say M_{z_0}, to map H^+ into the unit disk, with z_0 being sent into the center of this disk (see Figure 8).

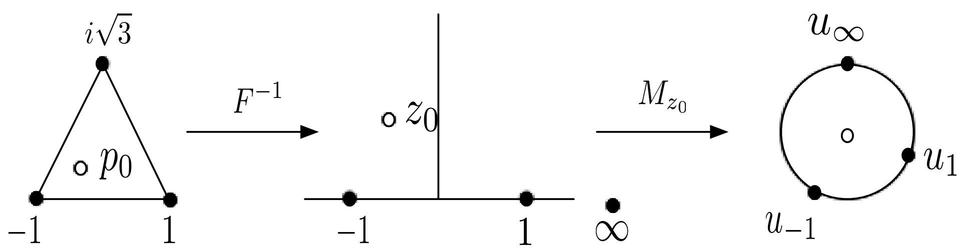

Figure 8: *The images of the vertices of Δ are on the unit circle, and the image of the starting point p_0 is at the center of the unit disk.*

The conformal transformation of the triangle into the unit disk which sends p_0 onto the its center (F_{p_0}) will be given by

$$F_{p_0} = M_{F^{-1}(p_0)} \circ F^{-1}, \tag{2.10}$$

i.e.

$$F_{p_0}(w) = i\left(\frac{F^{-1}(w) - z_0}{F^{-1}(w) - \bar{z}_0}\right).$$

Schwarz-Christoffel transformation of the upper half plane into Δ

Let us denote the exterior angles of Δ by $k_1\pi, k_2\pi$ and $k_3\pi$ respectively. Let $x_1, x_2 \in \mathbb{R}$ be the (arbitrary) prevertices of Δ, which are to be sent through F onto the vertices of Δ. From (2.7) we know that F will be of the form

$$w = F(z) = A\int_{z_1}^{z} (s - x_1)^{-k_1}(s - x_2)^{-k_2}ds + B, \tag{2.11}$$

with A, B some complex constants, and $z_1 \in H^+$. Since Δ was chosen to be equilateral, we have $k_1 = k_2 = \frac{2}{3}$. Hence choosing $x_1 = -x_2 = 1$ and $z_1 = 1$, equation (2.11) yields

$$w = F(z) = A\int_{1}^{z} (s - 1)^{-\frac{2}{3}}(s + 1)^{-\frac{2}{3}}ds + B. \tag{2.12}$$

Depending on the values of A and B, the transformation (2.12) maps the upper half plane onto the interior of any equilateral triangle in the complex plane. The choice of the vertices of Δ will determine the values of A and B.

From $F(1) = 1$ we immediately obtain that $B = 1$.

In order to determine the value of A, we must compute

$$F(-1) = -\int_{-1}^{1} (s^2 - 1)^{-\frac{2}{3}} ds + 1, \tag{2.13}$$

where the integration is performed on a path in the complex plane going to -1 to $+1$. First we notice that the function $f(s) = (s^2 - 1)^{-2/3}$ is holomorphic. Indeed, writing f in polar coordinates as $f(r, \theta) = (r^2 e^{2i\theta} - 1)^{-2/3}$, a direct computation shows that f satisfies the *Cauchy-Riemann equation* in polar coordinates, i.e. that

$$\frac{\partial f}{\partial r} = \frac{1}{ir}\frac{\partial f}{\partial \theta}.$$

Hence we know that the *Cauchy integral theorem* applies, i.e. the integral in (2.13) is the same along any path joining -1 to 1.

We choose a path of integration of the form $z = t$ along the real axis in the positive sense. Writing $s - 1$ as $|s-1|e^{i\phi}$ and $s + 1$ as $|s+1|e^{i\psi}$, we see that $\phi + \psi$ (the argument of $s^2 - 1$) remains constant throughout integration from -1 to 1 since $s + 1$ stays positive with zero argument, and $s - 1$ has constant argument π. Hence equation (2.13) becomes

$$-A\int_{-1}^{1} |s+1|^{-2/3}|s-1|^{-2/3} ds + 1 = -1,$$

or, equivalently,

$$A\int_{-1}^{1} (t^2 - 1)^{-2/3} dt = 2. \tag{2.14}$$

Direct integration yields

$$\int_{-1}^{1} (t^2 - 1)^{-2/3} dt = \frac{\sqrt{\pi}\Gamma(\frac{1}{3})}{\Gamma(\frac{5}{6})} = \frac{\Gamma(\frac{1}{3})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{3} + \frac{1}{2})} = B(1/2, 1/3),$$

where $B(\alpha, \beta)$ is the *beta function* given by

$$B(\alpha, \beta) := \int_0^1 t^{\alpha - 1}(1 - t)^{\beta - 1} dt.$$

This implies that

$$A = \frac{2}{B(\frac{1}{2}, \frac{1}{3})}.$$

Finally let us rewrite the constant $B = 1$ as

$$1 = \frac{B(\frac{1}{2},\frac{1}{3})}{B(\frac{1}{2},\frac{1}{3})} = \frac{2}{B(\frac{1}{2},\frac{1}{3})}\int_0^1 (s^2-1)^{-2/3}ds. \tag{2.15}$$

Combining the values of A and B given in equations (2.14) and (2.15) with the Schwarz-Christoffel transformation given in equation (2.12), we obtain

Proposition 2.4 *The transformation defined by*

$$w = F(z) = \frac{2}{B(\frac{1}{2},\frac{1}{3})}\int_0^z (s^2-1)^{-2/3}ds \tag{2.16}$$

is a conformal transformation of the upper half plane into an equilateral triangle with vertices ± 1 *and* $i\sqrt{3}$.

It should be noted that the inversion of the function given in equation (2.16) can, in general, not be expressed in terms of elementary functions, except for some specific choices of z.

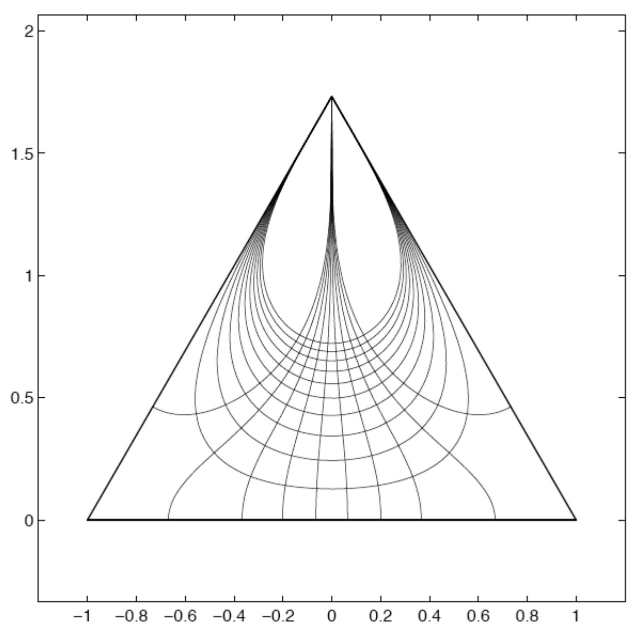

Figure 9: *The image of ten evenly spaced horizontal lines and ten evenly spaced vertical lines in* H^+ *by the mapping* F *defined in (2.16)*

Möbius transformation of the upper half plane into the unit disk and exit probabilities

Let $z_0 = F^{-1}(p_0) = x_0 + iy_0 \in H^+$ be the image of the starting point of the Brownian motion. Then we know that the exit probabilities of the Brownian motion through the edges of Δ will be proportional to the arc lengths of the images $M_{z_0}(F^{-1}(-1))$, $M_{z_0}(F^{-1}(1))$ and $M_{z_0}(F^{-1}(i\sqrt{3}))$. Now from the choice of the prevertices of the Schwarz-Christoffel transformation we know that $F^{-1}(1) = 1$, $F^{-1}(-1) = -1$ and $F^{-1}(i\sqrt{3}) = \infty$. Since the Möbius transformation of H^+ into the unit disk which sends z_0 into the center of this disk is given by

$$M_{z_0}(z) = i\left(\frac{z - z_0}{z - \overline{z_0}}\right),$$

we can compute $u_{-1} = M_{z_0}(-1)$, $u_1 = M_{z_0}(1)$ and $u_\infty = M_{z_0}(\infty)$ in terms of the value of x_0 and y_0. After straightforward computations this yields

$$\begin{aligned}
\theta_{-1} &= \arctan \frac{(1-x_0)^2 - y_0^2}{2y_0(1-x_0)} \\
\theta_1 &= \arctan -\frac{(1+x_0)^2 - y_0^2}{2y_0(1+x_0)} \\
\theta_\infty &= \tfrac{\pi}{2}.
\end{aligned} \tag{2.17}$$

Let p_1, p_2 and p_3 be the probabilities that the brownian motion exits the triangle through the edge whose vertices are, respectively, $[-1,1]$, $[1, i\sqrt{3}]$ and $[i\sqrt{3}, -1]$. Then from (2.17) we can conclude.

Proposition 2.5 *Let p_0 be the initial point of a Brownian motion inside an equilateral triangle with vertices $[-1, 1, i\sqrt{3}]$, and let $z_0 = F^{-1}(p_0)$. Also let $u_{-1} = M_{z_0}(-1)$ and $u_1 = M_{z_0}(1)$. Then the following equalities hold:*

$$\begin{aligned}
p_1 &= \frac{\theta_1 - \theta_{-1}}{2\pi} \\
p_2 &= \frac{1}{4} - \frac{\theta_1}{2\pi} \\
p_3 &= \frac{\theta_{-1}}{2\pi} - \frac{1}{4},
\end{aligned} \tag{2.18}$$

where θ_{-1} is the argument of u_{-1} and θ_1 is that of u_1.

The main difficulty lies in the computation of $z_0 = F^{-1}(p_0)$, because, as we have already mentioned, F can typically not be expressed in terms of elementary functions.

A special case: z_0 is purely imaginary

Equation (2.16) is tractable when z_0 is purely imaginary, i.e. when $z_0 = iy$ for some $y \in \mathbb{R}$. Indeed, in this case the transformation (2.16) becomes

$$\begin{aligned} w &= \frac{2}{B(1/2,1/3)} \int_0^{iy} (1-t^2)^{-2/3} dt \\ &= i\frac{2}{B(1/2,1/3)} \int_0^{y} (1+x^2)^{-2/3} dx \\ &= i\frac{1}{B(1/2,1/3)} \int_0^{y^2/(1+y^2)} u^{-1/2}(1-u)^{-5/6} du. \\ &= iB(1/2,1/6,\tfrac{y^2}{1+y^2})/B(1/2,1/3), \end{aligned}$$

where

$$B(\alpha,\beta,x) := \int_0^x t^{\alpha-1}(1-t)^{\beta-1} dt$$

is the incomplete Beta function.

Since $B(1/2,1/6) = \sqrt{3}B(1/2,1/3)$, we see that w converges to the top point $i\sqrt{3}$ of Δ, as y goes to infinity. Hence when $z_0 = iy$ is purely imaginary, we obtain

$$F(z_0) = i\sqrt{3}B(1/2,1/6,\frac{y^2}{1+y^2})/B(1/2,1/6). \tag{2.19}$$

Hence, if the Brownian motion starts at a point of the form $p_0 = it_0$ for $t_0 \in (0, i\sqrt{3})$, then $z_0 = F^{-1}(p_0)$ is also purely imaginary. Taking $y_0 = \text{Im } z_0$ we know from (2.19) that

$$\frac{t_0}{\sqrt{3}} = B(1/2,1/6,\frac{y_0^2}{1+y_0^2})/B(1/2,1/6).$$

Hence $\dfrac{y_0^2}{1+y_0^2}$ is the $\dfrac{t_0}{\sqrt{3}}$-quantile of the Beta distribution with parameters 1/2 and 1/6. For any value of t_0 we can therefore compute the corresponding y_0 and Proposition 2.5 yields the exit probabilities.

Numerical results for the 3-player ruin problem

Although we are not able to give a formal inversion of the function defined by 2.16, numerical integration will yield tight approximations. Moreover, Schwarz-Christoffel transformations have been the subject of intensive recent research and software developments.

All the numerical results that we need can be readily obtained by use of a Matlab toolbox which is freely available on the internet[1].

As an illustration, let us first consider the symmetric 3-player problem when all three players have the same initial distribution of capital. In our gambling model, this corresponds to starting the Brownian motion at the point $i\sqrt{3}/3$ inside the triangle with vertices $[-1, 1, i\sqrt{3}]$.

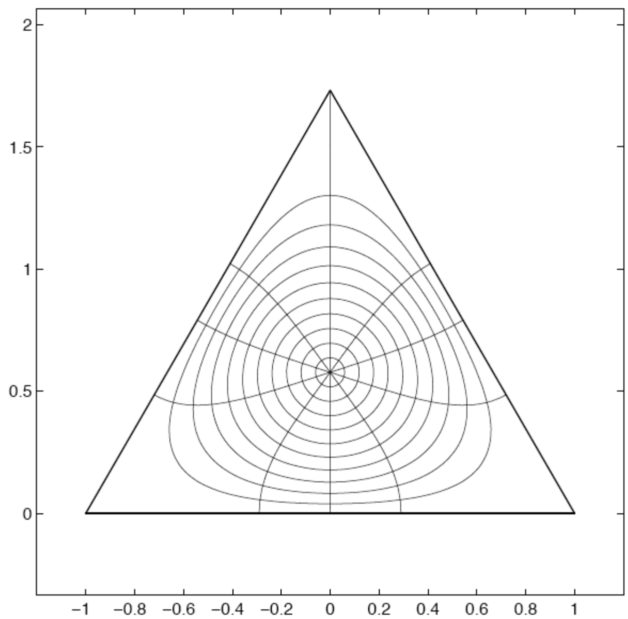

Figure 10: *The transformation of the triangle into the unit disk which sends $i\sqrt{3}/3$ into the center of the disk, along with the images of ten evenly spaced circles centered at the origin, and ten evenly spaced radii. All the intersections are orthogonal.*

Since the games are fair, we therefore expect the ruin probabilities to be equal. A direct computation of the images of the three vertices of Δ yields evenly spaced points on the unit disk as one would infer from Figure 10. Hence we indeed obtain equal ruin probabilities for each player.

The following table gives the output of our computations for specific choices of initial distributions, where a, b, c represent the assets of each player, Δ the point in the complex

[1]This Toolbox was designed for Matlab by Driscoll (see Driscoll(1996)) as an extension of a FORTRAN package developped by Trefethen (see Howell and Trefethen(1990)) in the early 1980's.

plane associated with this triplet, and p the probability that player 3 is ruined first.

Δ	a, b, c	p
$i\sqrt{3}/3$	$a = b = c$	0.33333
$i\sqrt{3}/5$	$a = b = 2c$	0.5617
$-1/8 + i5\sqrt{3}/8$	$a = \Sigma/8, b = \Sigma/4$	0.0534
$1/8 + 5\sqrt{3}/8$	$a = \Sigma/4, b = \Sigma/8$	0.0534

The three player ruin problem with capital constraints

Finally, we look at another version of the 3-player ruin problem. Let us recall the modification of the N-player ruin problem which was solved for the discrete case on page 19 . In this problem we aim to compute the probability of, say, player 3 being ruined, while players 1 and 2 still have certain defined assets. The solution to this problem is now straightforward since it suffices to view the triangle as a pentagon with four of its vertices aligned, and hence an appropriate choice of Schwarz-Christoffel transformation will yield the desired probabilites.

The following table gives the output of some computations for this modified problem. For each of these computations, we chose equal initial capital for all players, which means that the Brownian motion starts at the center.

Δ	constraints	p
$i\sqrt{(3)}/3$	$(-0.5, 0.5)$	0.2589
$i\sqrt{(3)}/3$	$(-1, -0.9)$	$1.8999.10^{-4}$
$i\sqrt{(3)}/3$	$(-0.9, -0.1)$	0.1284
$i\sqrt{(3)}/3$	$(-0.1, 1)$	0.2048
$i\sqrt{(3)}/3$	$(-0.6, 0.8)$	0.3197

It is clear that if one modifies the problem slightly so as to look for the probability of exit through a union of disjoint intervals, the probability is the sum of the probabilities of exit through each of the intervals and can be computed as before.

Final comments and Conclusion

The N-player ruin problem is a specific kind of exit problem for a stochastic process from a bounded domain. As we have outlined in the Introduction, this problem has been studied under different assumptions by a number of specialists, and several special cases are solved in the literature. We study this problem both in discrete and in continuous time.

In Chapter 1 we tackle the discrete problem. We obtain the key measures associated to it by means of a Markov chain interpretation. This yields the ruin probabilities and the expected number of games until a ruin occurs in terms of the transition matrix associated to the problem. This solution is computationally heavy. We then present an algorithm which yields, for smaller N, considerable savings in the number of operations required to compute the solution.

In Chapter 2 we study the continuous problem. Specifically, we set up the 3-player ruin problem as an exit problem for a Brownian motion inside a triangle. We describe the construction of a family of conformal transformations known as the Schwarz-Christoffel transformations. These transformations serve to compute exit probabilities of a Brownian motion from specific bounded domains and we show in detail how this can be done. We then solve a continuous version of the 2 and 3 player ruin problem by use of these transformations.

A natural question is whether this method still holds when there are more than three players. It turns out that the case $N > 3$ is substantially different from the case $N \leq 3$.

For the discrete problem, one sees that the elegant solutions to the symmetric 3-player problems that have been provided by use of Martingale arguments can seemingly not be generalized to the non-symmetric case or to higher dimensions (see Bruss et al. (2003)).

A similar breakdown occurs for the continuous problem. Indeed, we know that Riemann's theorem on conformal equivalence is only valid in the complex plane. Therefore, when the number of players is greater than 3, the conformal equivalence which is at the core of our results is no longer applicable. Hence a general solution of the asymptotic N-player ruin problem by use of conformal transformations cannot be hoped for when $N > 3$.

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Part II

On Robbins' Problem

Introduction

In the second part of this thesis, we study a problem of optimal stopping which was presented by Professor Herbert Robbins at the International Conference on Search and Selection in Real Time (Amherst, June 1990). In order to exemplify the significance of this problem, it is necessary that we first explain the context surrounding it, and, for this reason, we begin this work with a brief description of a class of sequential selection problems that are known as 'secretary problems'.

1 Historical background and related problems

Behind the denomination 'secretary problem' lies a simple real-life problem that made its way around the mathematical community, each new author bringing a different light on the implications and ramifications which lie behind this seemingly anecdotic mathematical game. The exact origin of this problem is obscure, and it seems that it had been known by many mathematicians before it appeared for the first time on print as a recreational problem in Martin Gardner's February 1960 column of the *Scientific American* (see Gardner (1960)). From that time on "it has been taken up and developed by a number of eminent probabilists and statisticians [...]" and it has spawned a whole class of problems which now "[...] constitute a 'field' of study within mathematics-probability-optimization[1]."

The oldest available scientific literature that addresses secretary problems explicitly dates back to 1960. Since then, many authors have approached such problems from different perspectives, and one can see from the survey paper by Freeman (1983), Petruccelli (1988) or Samuels (1991) how extensive this field has become.

A typical secretary problem starts with the following introduction. A decision maker sequentially observes realizations of random variables $X_1, X_2, \dots X_n$, where n is fixed. At each time $i \geq 1$ he must decide whether or not to reject the current observation X_i and examine the next observation, or to accept X_i and therefore reject all subsequent observations. The objective of the decision maker is to maximize a specified payoff function under certain hypothesis on the distribution of the X_i. What separates secretary problems from other sequential search and selection problems is that "the payoff [or cost] depends on the

[1]Ferguson (1984).

observations only through their relative ranks and not otherwise on their actual values[2]".

A rapid categorization of these problems can be given for example in terms of the knowledge that the observer is allowed to have on the distribution of the arrivals. When the distribution and the values of the observations are known to the observer, then he has 'full-information' in the sense that at each stage he knows as much as he can possibly know about the next observation. The extreme opposite of this are the 'no-information' problems, in which the observer is presented with values from a distribution of which all he knows is that all $n!$ rank orders are equally likely, and he is allowed to see only the relative ranks of the observations. The intermediate case, in which the observations are drawn from a distribution that is specified but contains a number of unknown parameters, is called the 'partial information' problem.

Since the decision maker must select an observation in order to maximize a predefined reward function that depends on the ranks of the observations, he must, at all times, balance the danger of stopping early and possibly missing better observations still to come, against that of going on too long and finding out that he has missed the better observations. We will see that, even when very little information is available to the decision maker, there exist strategies which yield surprisingly good performances.

Among these problems, there are four that stand out in a natural way. We may call these the secretary problems. The first three were solved successively by Lindley (1961), Chow, Moriguti, Robbins and Samuels (1964), and Gilbert and Mosteller (1966). The problem posed by Robbins in 1990 is the fourth problem of this kind. More than forty years after the resolution of the first three, several important questions concerning Robbins' problem are still open.

We now describe these four problems in more detail.

The no-information best-choice problem

Consider a situation where an employer has advertised an opening for a secretary. There are a known number, n, of applicants, and the employer interviews them one at a time. He is very specific about the qualities that are needed for the job so that, after each interview, he can rank the present applicant with respect to all previous applicants with no ties. The applicant must be told immediately after each interview whether or not he has been hired and there cannot be any regrets later on. Moreover if the first $n-1$ applicants have been rejected, then the employer is forced to hire the last one.

What selection strategy will maximize the probability of the employer selecting the best candidate? What is the maximal probability of choosing the best candidate? In particular, what is the limiting value of this probability when the number of applicants becomes infinite?

[2] Ferguson (1984)

These questions are the essence of the problem that has come to be known as the *classical no-information secretary problem* (abbreviated CSP), where the terminology "no-information" refers to the fact that the decisions of the employer must be based solely on the relative ranks of the different observations and not on their specific values.

The first solution of the CSP to be published in a scientific journal is due to Lindley (1961). It is obtained by simple backward recursion and states that if, for $r = 1, 2, \ldots, n-1$, we define

$$a_r = \frac{1}{r} + \frac{1}{r+1} + \ldots + \frac{1}{n-1},$$

then the optimal action is to ignore any candidate who is not the best so far, and, if the rth candidate is the relative best at the time at which it is observed, then he should be chosen if $a_r < 1$ and rejected if $a_r > 1$ (see also e.g. Gilbert and Mosteller (1966) or the survey papers Freeman (1983), Petruccelli (1988), or Samuels (1991)). Thus, if $r^\star$ is the first integer for which $a_{r-1} \geq 1 > a_r$, the optimal strategy is to reject the first $r^\star - 1$ applicants and then to accept the first applicant thereafter that is better than all previous applicants.

The probability of the employer selecting the best candidate with this policy is given by $(r^\star - 1)a_{r^\star-1}/n$, and integral approximation yields that for n large, $r^\star/n \approx e^{-1}$. With this result, one shows that the asymptotic optimal win probability is given by $1/e = 0.368...$. We see that with very little information, the employer still has a surprisingly high probability of obtaining the overall best applicant.

Remark 6 *This result is perhaps more striking when one notices that when there are, for example, 100 candidates, then it is (approximately) optimal to reject the first* 36 *applicants and to hire the first candidate thereafter who is relative best. The probability of selecting a record candidate is even much higher than* $1/e$*, typically over* 60%.

Remark 7 *An alternative solution is due to Dynkin (1963), who considers this problem as an application of the theory of Markov stopping times. In this setting, the optimal strategy is given by the one-stage look-ahead rule. There was a third different solution obtained by Rasche (1975). This solution is a corollary of the more general Odds-Theorem of optimal stopping, see Bruss (2000).*

The full-information best-choice problem

Consider the following situation. "An urn contain n tags, identical except that each has a different number printed on it. The tags are to be successively drawn at random without replacement (the $n!$ permutations are equally likely). Knowing the number of tags, a player must choose just one of the tags, his object being to choose the one with the largest number. The player's behavior is restricted because after each tag is drawn he must either choose it, thus ending the game, or permanently reject it. The problem is to find the strategy that maximizes the probability of obtaining the largest tag and to evaluate that probability[3]."

Let us suppose that the numbers on each tag (say $X_1, \ldots, X_n$) are a random sample from some specified continuous distribution (which we can take to be the uniform continuous distribution on $[0, 1]$ since we are only interested in the comparative quality of each tag). Clearly, this problem is equivalent to the CSP if the player only considers the relative ranks of the numbers on the tags, since the hypothesis of continuity of the distribution guarantees the absence of ties. But suppose now that at each stage i, the player is allowed to know the values $X_1, \ldots, X_i$. Then his decisions are to be based on a more informative data set, and thus the optimal win probability should be better than in the classical no-information problem.

As an illustration of this fact, let us consider the case $n = 2$ and let X_1, X_2 be the first and second numbers examined, respectively. In the no-information problem, the player does not have much of a choice, since the first arrival will always be of relative rank one (it is obviously the best so far) and thus the player will simply win with one chance out of two. Suppose now that the player is allowed full information on the problem and let us choose any number x between 0 and 1. We define the rule

$$\tau_x = \begin{cases} 1 \text{ if } X_1 > x \\ 2 \text{ otherwise.} \end{cases}$$

If both X_1 and X_2 are greater or smaller than x, this rule selects the larger of the two with probability 1/2. If not, then this rule necessarily selects the maximum of the two. Hence the win probability with the rule τ_x is given by

$$\mathrm{P}[X_{\tau_x} = \max(X_1, X_2)] = 1/2 + x - x^2,$$

which is always greater than 1/2 and equal to 3/4 for $x = 0.5$.

The optimal strategy for all n (say $\tau_n^\star$) was obtained by Gilbert and Mosteller (1966). These authors show that it is defined through a sequence of thresholds, which they call *decision numbers*, $b_0 = 0, b_1, b_2, \ldots$, not depending on n such that

$$\tau_n^\star = \min_{1 \leq i \leq n} \left\{ i : \; X_i = \max_{j \leq i} X_j \;\text{ and }\; X_i \geq b_{n-i} \right\}.$$

Each decision number b_m, $m = 1, 2, \ldots$ is solution to

$$\sum_{j=1}^{m} j^{-1} b_m^{-j} = 1 + \sum_{j=1}^{m} j^{-1},$$

where, as one would expect from the example we gave above, $b_1 = 1/2$. These numbers form an increasing sequence which goes to one as the number of draws becomes large.

[3]This presentation of the problem is the same as that in Gilbert and Mosteller (1966). It is equivalent to the presentation made by Gardner, see Gardner (1960), under the name 'game of Googol'.

Now let $w_n = \mathrm{P}[X_{\tau_n^\star} = \max\{X_1, \ldots, X_n\}]$ denote the win probability under the optimal strategy. Samuels (1982) showed that w_n is strictly decreasing in n and that

$$\lim_{n\to\infty} w_n \approx 0.580164...$$

Hence we see that there is an improvement of roughly 58% from the no-information to the full-information problem.

The no-information expected rank problem

We consider the same situation as in the classical secretary problem, in which an employer interviews n candidates for a job under the restriction that, at each interview, the only information he can work on is the relative ranks of the preceding applicants. Now suppose that instead of maximizing the probability of selecting the best, we consider the objective of minimizing the total expected rank of the selected candidate, where the overall best candidate is given rank one, the second best two, etc., and the worst rank n.

This objective is arguably more realistic than that of the CSP. Indeed, maximizing the probability of accepting only the best candidate implies a utility function that takes the value 1 if the best is accepted and 0 otherwise. Such 'nothing-but-the-best'[4] objectives are therefore very restrictive in comparison to real-life problems in which one could imagine that an employer would also be satisfied with a less perfect candidate. In this respect, a more appropriate utility function would be that which takes the value $n-i$ if the ith best candidate is accepted; maximizing the expected value of this utility function corresponds to minimizing the expected rank of the selected observation.

With this in mind, it is now easy to surmise that the optimal strategy for the best-choice problem is no longer optimal with respect to this new objective. Indeed, although this policy selects an arrival which has absolute rank 1 with a high probability, it also has a major drawback: it suffices for the overall best candidate to appear among the first $r^\star - 1$ applicants to ensure that this strategy never stops and thus selects the last candidate. Since the last candidate has expected rank $\frac{n+1}{2}$, we infer that this policy must be suboptimal in this setting (see Chapter 4 for more details on these arguments).

The optimal strategy for this problem can for example be obtained by the method of *backward induction* (see e.g. Chow et al. (1971)). Labeling the relative ranks of each applicant by $r_1, r_2, \ldots r_n$ respectively, a direct application of this method shows (see e.g. Lindley (1961) or Chow et al. (1966)) that the optimal strategy is given by a sequence of thresholds $s_1 \leq s_2 \leq \ldots \leq s_n = n$ such that it is optimal to stop on the first applicant whose relative rank satisfies $r_i \leq s_i$. However, it turns out that the recurrence equations which define the s_i's are extremely complicated and thus this result lends little insight into the asymptotic value of the optimal expected rank.

[4]This appellation is due to Lindley (1961).

A heuristic argument given in Lindley (1961) indicated that, by approximating these recurrence relations by a differential equation, the optimal expected rank should approach a finite limit as n goes to infinity. Chow et al. (1964) were able to make this rigorous and obtained then the limiting form of the expected rank under the optimal policy. For this they showed that the minimum expected rank for the n arrival problem is a strictly increasing function of n, and that it converges to

$$\prod_{j=1}^{\infty}\left(\frac{j+2}{j}\right)^{1/(j+1)} = 3.8695....$$

It is here of interest to point out that H. Robbins was co-author of this paper.

The full-information expected rank problem

This problem has the same formulation as the full-information best choice problem but instead of maximizing the probability of obtaining the best X_i, the objective is now to minimize the expected rank of the selected observation. One sees that this problem fits perfectly in the two-by-two pattern of the classical secretary problems. Surprisingly, although the three previous problems had been solved by the mid 60's, it was not until Professor Robbins' kindled the interest of the mathematical community (at the International Conference on Search and Selection in Real Time in 1990) that results were published on this problem. For this reason among others, it has been named in his honor (see Bruss and Ferguson (1993) and Assaf and Samuel-Cahn (1996); for a review see Bruss (2005)).

We defer a precise definition of this problem to Chapter 1. However, we can already deduce from the previous sections an upper bound on the optimal expected rank. Indeed, a player with full-information can only do better than a player with no information, since he can always choose to use a strategy which only considers the relative ranks of the arrivals. Therefore, letting $v(n)$ be the value of the optimal expected rank for the n-arrival full-information expected rank problem, we know that

$$\lim_{n\to\infty} v(n) \leq 3.8695...$$

Now, in light of the fact that the passage from no-information to full-information in the best choice problems yielded a 58% increase in the asymptotic win probability, it seems reasonable to hope that the improvement for the rank problem should be of the same order, i.e. that $\lim w(n) \approx 2.44$. We will see that the improvement is, in fact, better.

Before moving on, we need to relate Robbins' problem to one final example of selection problem which we will refer to as *Moser's problem*, in honor of Professor Leo Moser, who was the first to obtain its solution (see Moser (1956)). This problem is an extension of a problem posed by Cayley in 1875 (see Cayley (1875)). Although it does not satisfy our definition of a 'secretary problem', we will see that it yields some necessary intuitions on Robbins' problem.

Moser's problem

A player observes sequentially n random variables $X_1, \ldots, X_n$ which are known to be independent, identically and uniformly distributed on $[0, 1]$. His cost for stopping at time j is equal to the value of the observation, and no recall of preceding observations is permitted. His objective it to use an adapted stopping rule τ which minimizes $\mathrm{E}[X_\tau]$.

Moser (1956) shows that if we define recursively the sequence $(a_k)_{k\geq 0}$ by $a_0 = 1$ and

$$a_{j+1} = a_j - \frac{1}{2}a_j^2, \ j \geq 1,$$

then it is optimal to stop on X_j if $X_j \leq a_{n-j}$, i.e. the optimal strategy $\hat{\tau}_n$ is

$$\hat{\tau}_n = \min\{k \geq 1 \mid X_k \leq a_k\}.$$

The optimal return with this strategy is given by $\mathrm{E}[X_{\hat{\tau}_n}] = a_n$. These thresholds are asymptotically equal to

$$a_k \approx \frac{2}{n-k+1} \wedge 1,$$

and for large n,

$$a_n \approx \frac{2}{n + \log n + O(1)},$$

so that

$$\lim_{n\to\infty} \mathrm{E}[nX_{\hat{\tau}_n}] = 2.$$

Since the correlation between the values X_k and the ranks R_k tends to 1 as n goes to infinity (see Bruss and Ferguson (1993)), it would seem that the problem of minimizing the expected rank and that of maximizing the expected value should be largely equivalent. This is not true, as we will see in Chapter 1.

2 Poisson embedding of stopping problems

Let us now suppose that the arrivals occur in accordance to a stochastic counting process in continuous time on a fixed horizon (fixed time interval). The questions asked in the preceding examples clearly retain their interest in this more general setting, with the added difficulty that if the arrivals occur at random time then the number of observations must also be random, and thus there must be a learning process going on as the arrivals are observed. A special case is the best choice problem with a Poisson arrival process, in which i.i.d. random variables $X_1, X_2, \ldots$ are presented at the time points of a Poisson process of rate λ and the observer must make his decision before some fixed time T. If the observer selects the overall best observation, his reward is 1, and otherwise (even if no arrival is selected before T) it is set to zero. His objective is to maximize the expected reward (i.e. the probability of selecting the overall best).

The extension of the no-information best choice problem to this setting was first studied by Cowan and Zabcyk (1978) under the hypothesis that λ is known. They use the same approach as Dynkin (see Remark 7) and imbed a discrete Markov process in the continuous time problem. They then show that the optimal strategy is to accept the first arrival which is of relative rank one and whose arrival time t and arrival number m satisfy $\lambda t \leq x_m$, where x_m is the unique solution of the equation

$$\sum_{n=0}^{\infty} \frac{x^n}{n!(m+n)} = \sum_{n=1}^{\infty} \frac{x^n}{n!(m+n)} \sum_{k=1}^{n} \frac{1}{k+m-1}.$$

This is, as in the CSP, a one-stage look-ahead strategy. Bruss (1987) obtains a result that is somewhat more striking: if the intensity of the process is unknown, then it is optimal to accept (if possible) the first arrival of relative rank one which occurs after time $s^\star = T/e$. This strategy coincides with the asymptotic optimal strategy for the CSP.

Bruss (2000) proves a more general statement, via the Odds-theorem. This theorem states the following. If the optimal stopping problem can be expressed as that of stopping a sequence of independent indicators $I_1, I_2, \ldots, I_n$, then, letting $p_j = \mathrm{E}[I_j]$, $q_j = 1 - p_j$ and $r_j = p_j/q_j$, an optimal rule for stopping on the last success is to stop on the first index k with $I_k = 1$ and $k \geq s$ (if any). The integer s is defined by

$$s = \sup\{1, \sup\{1 \leq k \leq n : \sum_{j=k}^{n} r_j \geq 1\}\},$$

with $\sup\{\emptyset\} := -\infty$. This theorem provides a unifying framework for optimal stopping problems, even when the number of observations is random (see also Bruss and Paindaveine (2000) for a partial generalization). For example, if the arrival process follows an inhomogeneous Poisson process with intensity rate $\lambda(t)$ and if an observation has to be selected before time T, then it is optimal to stop on the first success (if any) which occurs after time

$$s = \sup\{0, \sup\{0 \leq t \leq T : \int_t^T \lambda(t)h(t)du \geq 1\}\},$$

where $h(t)$ is the success parameter function for an experiment occurring at time t. Here the product $\lambda(t)h(t)$ is supposed to have at most finitely many discontinuities on $[0, T]$. If we define the indicator function I_k to be equal to one if the kth observation is of relative rank one, then this theorem gives the optimal strategy for the no-information best choice problem.

Bruss and Rogers (1991) describe a general approach for the embedding of optimal selection problems in a Poisson process. This embedding of best choice problems has been studied for example in Gnedin (2002) or Gnedin (2006).

We have not found any reference in the literature to a Poisson embedding of the expected rank problem.

3 Preview of Part Two

In the second part of this thesis, we study a Poisson embedding of Robbins' problem. As we have outlined above, this is an optimal stopping problem in which the utility function is given by the expected rank of the selected problem, and because of the specific form of this objective, the results obtained on the Poisson best-choice problems are for the most part not applicable to our situation. We set out on this journey with the objective of contributing to the solution of Robbins' problem. We have been only partially successful in this respect. As we will show, the pathologies of the discrete problem remain predominant in the continuous setting, and although we are able to obtain a number of formal results, these are not sufficient to gain insight on the behavior of the optimal strategy for Robbins' problem.

Chapter 1.

We recall fundamental results on Robbins' problem. These results are due to Bruss and Ferguson (1993, 1996) and Assaf and Samuel-Cahn (1996). We start by giving a precise statement of the problem, and define the value function of the discrete problem $v(n)$ as the optimal expected rank obtainable for n arrivals. This function is increasing and bounded, which shows that the asymptotic problem makes sense, and we define $v = \lim_{n\to\infty} v(n)$. We then define the class of 'memoryless threshold rules', and cite a number of results which will be useful to us in subsequent chapters. These results are stated without proof. We also recall a number of interesting properties of the optimal strategy which serve to illustrate the inherent complexity of Robbins' problem.

Chapter 2.

We consider a continuous version of Robbins' Problem in which the observations follow a Poisson arrival process of homogeneous rate 1 on $[0, t] \times [0, 1]$. This problem will be called the Poisson embedded Robbins' problem. Translating the previous optimal selection problem in this setting we define the value function $w(t)$ as the optimal expected rank obtainable on the fixed horizon $[0, t]$. We prove some important results, including continuity and boundedness of the value function. We extend the definition of memoryless threshold rules to this setting, and obtain a number of results for these specific rules, including an integral expression for the asymptotic value. We derive the existence of optimal strategies for the Poisson problem, and obtain an integro-differential equation on $w(t)$, which we use to obtain estimates on the limiting value.

Chapter 3.

We compare the asymptotic value of Robbins' problem v with that of the Poisson embedded Robbins' problem. We show that the Poisson embedded problem yields an upper bound on the discrete problem. We also obtain an inequality in the other direction, which depends on the asymptotic behavior of the discrete optimal strategy. We explain why this inequality yields interesting conclusions, although we are unable to obtain sufficiently precise arguments to justify our intuition rigorously.

Chapter 4.

In our effort to contribute to the solution of Robbins' problem, we have studied the distribution of a number of specific strategies and obtained estimates on the limiting value v. This chapter is a review of the methods we used. Our aim was to construct a non-memoryless threshold strategy which would yield a strict improvement on the value obtained through the optimal memoryless threshold strategy. However, as we will see, we have gone into some depth on this subject and the improvements we obtain are non-conclusive. This is why we only give a brief overview of these results, and most of our computations are not included.

Chapter 1

The classical Robbins' Problem

There are so far only four papers specifically on Robbins' problem. These are Bruss and Ferguson (1993), Assaf and Samuel-Cahn (1996), Bruss and Ferguson (1996) and Bruss (2005). We now summarize the content of these works in order to outline some distinguishing features of the problem. These results will also serve as a foundation on which we will build the subsequent chapters of this thesis.

1.1 Robbins' Problem

A player observes sequentially n i.i.d random variables $X_1, \ldots, X_n$ distributed uniformly on $[0,1]$ and has to chose exactly one of them. The objective of the player is to minimize the expected rank of the chosen observation, where the best observation is given rank one, the second best rank two, etc., and the worst rank n. However, once a value is rejected, it cannot be recalled afterwards, so that at time k, only X_k can be selected, and the data on which the decision is made are the values of the arrivals up to time k. Let $\mathcal{F}_k = \sigma(X_1, \ldots, X_k)$ be the σ-algebra generated by $X_1, \ldots, X_k$. The *relative rank* of an arrival X_k is defined by

$$r_k = \sum_{j=1}^{k} \mathbf{1}_{\{X_j \le X_k\}},$$

and the *(absolute) rank* of the kth observation is defined by

$$R_k^\star = \sum_{j=1}^{n} \mathbf{1}_{\{X_j \le X_k\}}.$$

Since $R_k^\star$ is not $\mathcal{F}_k$-measurable, we replace it by

$$R_k = \mathrm{E}[R_k^\star | \mathcal{F}_k] = r_k + (n-k)X_k.$$

The objective of the player is to use a non anticipating strategy τ which minimizes $\mathrm{E}[R_\tau]$ (where this problem is clearly equivalent to that of minimizing $\mathrm{E}[R_\tau^\star]$ since the corresponding expressions are equal for all stopping rules τ). Now let $T_n = \{\tau : \{\tau = k\} \in \mathcal{F}_k,\ \forall k = 1, 2, \dots, n\}$ be the set of stopping rules adapted to $X_1, \dots, X_n$, and define the *value function* for n arrivals by

$$v(n) = \inf_{\tau \in T_n} E[R_\tau]. \tag{1.1}$$

Robbin's problem consists in studying the value function $v(n)$ defined by equation (1.1), the stopping rule $\tau^\star = \tau_n^\star$ which achieves $v(n)$ and the asymptotic value

$$v = \lim_{n \to \infty} v(n). \tag{1.2}$$

1.2 The optimal rule

Backward induction (see Chow. et al (1971)) guarantees the existence of an optimal strategy $\tau_n^\star \in T_n$ for all n, and provides, in principle, a way to compute it. However, even for small values of $n \geq 3$, computing the optimal strategy through backward induction is a formidable task and does not seem to give any intuition on the asymptotic value (see Assaf and Samuel-Cahn (1996) for the case $n = 3$). Moreover, Bruss and Ferguson (1996) prove that the optimal rule is a threshold rule of the form

$$\tau_n^\star = \inf\{1 \leq k \leq n : X_k \leq p_k^{(n)}(X_1, X_2, \dots, X_k)\},$$

where the functions $p_k^{(n)}(.)$ are *fully history dependent* in the sense that for each k, the value of the corresponding threshold depends on every arrival $X_1, \dots, X_k$. They also show that no nontrivial statistic of $X_1, \dots, X_k$ is sufficient to achieve the optimal value $v(n)$, and hence the optimal thresholds have an unbounded number of arguments and "figure in the list of most undesirable mathematical objects[1]".

Now although this property of the optimal rule seems to demonstrate that the problem is not tractable, Bruss and Ferguson (1993) also proved that this problem possesses some monotonic features. These authors prove that the optimal thresholds are stepwise-monotone-increasing in the sense that for each n and for all $k = 1, \dots, n-2$,

$$0 \leq p_k^{(n)}(X_1, X_2, \dots, X_k) \leq p_{k+1}^{(n)}(X_1, X_2, \dots, X_k, X_{k+1}) < p_n^{(n)} = 1$$

almost surely. They also show that the value function $v(n)$ is increasing in n. In particular this proves that the limiting value $v = \lim_{n\to\infty} v(n)$ exists, since, as we have already seen, the value function is bounded above for all n by the value function in the corresponding no-information problem (see Chow et al. (1964)). To show the monotonicity of $v(n)$,

[1]Bruss (2005)

they imagined a prophet whose only ability is that, at time 0, he can foresee the worst observation, that is, the value of the largest order statistic $X_{(n)}$. Let $v_p(n)$ denote the minimal expected rank that the prophet can obtain through the use of adapted strategies. Then clearly he can do better than an observer with no knowledge of the future of the process, so that $v_p(n) \leq v(n)$. Optimal behavior forces the prophet to reject $X_{(n)}$ and to solve Robbins' problem for $n-1$ i.i.d. observations uniformly distributed on $[0, X_{(n)}]$. This means that his value is equal to that of an observer with no prophetic abilities working with $n-1$ observations, i.e. it is equal to $v(n-1)$. Hence $v(n-1) \leq v(n)$.

1.3 Memoryless Threshold Strategies

Because of the complexity of $v(n)$ and the corresponding optimal threshold rule, the authors who have studied Robbins' Problem have introduced a class of considerably simpler strategies known as *memoryless threshold rules* (or strategies). These are rules which are defined for each n through a sequence of constants (called *threshold constants*) $0 \leq a_{n,1} \leq a_{n,2} \leq \ldots \leq a_{n,n} = 1$ by

$$\tau_n = \min\{k : X_k \leq a_{n,k}\}. \tag{1.3}$$

Remark 8 *The restriction $a_{n,n} = 1$ is necessary in order to ensure that rules defined by (1.3) stop for at least one of the arrivals X_k, and hence for exactly one. Also, Assaf and Samuel-Cahn (1996) prove that for any discrete memoryless threshold rule defined by a sequence which is not monotone increasing, there exists a rule determined by a monotone increasing sequence which yields a better value. Thus only monotone increasing sequences need to be considered.*

Now let M_n be the set of all such rules, and for all $\tau_n \in M_n$, let

$$V(\tau_n) = \mathrm{E}[R_{\tau_n}]. \tag{1.4}$$

$V(\tau_n)$ will often be called the value of Robbins' problem under the strategy τ_n. Straightforward computations yield the following lemma (see Bruss and Ferguson (1993) and Assaf and Samuel-Cahn (1996)).

Lemma 1.1 *Consider the threshold sequence $0 < a_1 \leq a_2 \leq \ldots \leq a_n = 1$ and let τ_n be the corresponding strategy. Then if $a_{k-1} < 1$,*

$$V(\tau_n) = 1 + \frac{1}{2}\sum_{k=1}^{n-1}(n-k)a_k^2\prod_{j=1}^{k-1}(1-a_j) + \frac{1}{2}\sum_{k=1}^{n}\prod_{j=1}^{k-1}(1-a_j)\sum_{j=1}^{k-1}\frac{(a_k-a_j)^2}{1-a_j} \tag{1.5}$$

where $0/0$ should be interpreted as 0 in the last sum.

We define the restricted value function $V(n)$ (with a capital 'V') as the optimal value of $V(\tau_n)$ among all $\tau_n \in M_n$ (i.e. it is the minimal expected rank attainable through a memoryless threshold rule) and we define the restricted asymptotic value

$$V = \lim_{n\to\infty} V(n). \tag{1.6}$$

Note that $V(n)$ gives an upper bound on $v(n)$ for all n, and hence $v \le V$.

We now recall a number of results which will be used without reference later on in the text. These results are given in Bruss and Ferguson (1993, 1996) and Assaf and Samuel-Cahn (1996) and hence are stated without proofs.

Theorem 1.2 *There exists an optimal rule $\tau_n^\star \in M_n$, i.e. there exists a memoryless threshold rule $\tau_n^\star$ for which $V(\tau_n^\star) = V(n)$. Moreover this rule is uniquely defined.*

Proof: See Bruss and Ferguson (1996), Lemma 2.1.

Theorem 1.3 *$V(n)$ is an increasing and bounded function of n and hence the limit $V = \lim_{n\to\infty} V(n)$ exists and is finite.*

Proof: See Assaf and Samuel-Cahn (1996), Theorem 2.4 or Bruss and Ferguson (1996) Theorem 2.

Theorem 1.4 *For any stopping rule τ_n let*

$$U_n(\tau_n) = \left(2\mathrm{E}[nX_{\tau_n}](1 + \mathrm{E}[\tau_n/n])\right)^{\frac{1}{2}}. \tag{1.7}$$

Also let $U^\star = \liminf U_n(\tau_n^\star)$ where $\tau_n^\star$ is the optimal memoryless threshold rule. Then

$$V = U^\star, \tag{1.8}$$

and

$$V \ge \liminf_{n\to\infty} \inf_{\tau_n \in M_n} U_n(\tau_n). \tag{1.9}$$

Proof: Assaf and Samuel Cahn (1996), Theorem 2.5.

We can immediately conclude from (1.9) that $V \ge 2$. This follows from Moser's problem, since the optimal strategy $\hat{\tau}_n$ for this problem satisfies our definition of a memoryless threshold rule and hence this is the rule which minimizes $\mathrm{E}[X_\tau]$ among all adapted strategies, i.e. $\mathrm{E}[X_{\hat{\tau}_n}] = \inf_{\tau\in M_n} \mathrm{E}[X_\tau]$. Therefore, we see that

$$\lim_{n\to\infty} \inf_{\tau_n \in M_n} U_n(\tau_n) \ge \lim_{n\to\infty} \left(\mathrm{E}[nX_{\hat{\tau}_n}]\right) = 2.$$

Moreover, since $\tau_n^\star$ the optimal strategy for Robbins' problem is not necessarily optimal with respect to Moser's problem we see that, for each n, $\mathrm{E}[X_{\tau_n^\star}] \ge \mathrm{E}[X_{\hat{\tau}_n}]$, and thus

$$\lim \mathrm{E}[nX_{\tau_n^\star}] \ge 2.$$

This property of the optimal strategy will be discussed in more detail in Chapter 3.

1.4 Upper bounds on the value function

We know that $M_n \subset T_n$ and thus the asymptotic value $\lim_{n\to\infty} V(\tau_n)$ for any sequence of memoryless threshold rules $(\tau_n)_{n\geq 1}$ gives an upper bound on v. For example, by considering a version of Moser's rule of the form

$$\tau_n = \min\left\{k \geq 1 : X_k \leq \frac{2}{n-k+2}\right\},$$

Bruss and Ferguson (1993) show by use of integral approximation of (1.5) that

$$v \leq \lim_{n\to\infty} V(\tau_n) = \frac{7}{3}.$$

Assaf and Samuel-Cahn (1996) obtain a limiting form of (1.5) for certain thresholds. For this they consider functions $r(.)$ with support $[0,1]$ which satisfy

$$\int_0^1 r(u)du = \infty, \tag{1.10}$$

and

$$\liminf_{z\to 1}(1-z)r(z) > 1. \tag{1.11}$$

To each such function $r(.)$, they associate a value $W_n(r)$ which is the expected rank obtained by using a threshold rule defined by a threshold sequence $a_{n,k}(r)$ defined by $a_{n,n} = 1$ and

$$a_{n,k}(r) = \min\left\{\frac{1}{n+1} r\left(\frac{k}{n+1}\right), 1\right\}, \quad k = 1, \ldots, n-1.$$

If $r(.)$ is increasing on $[0,1]$, then for each n the sequence $a_{n,k}(r)$ satisfies the conditions of Lemma 1.1 and direct integral approximation of (1.5) yields

$$\begin{aligned}\lim_{n\to\infty} W_n(r) &= 1 + \tfrac{1}{2}\int_0^1 (1-u)r^2(u)\bar{F}(u)du \\ &\quad + \tfrac{1}{2}\int_0^1 \int_0^u \bar{F}(u)(r(u)-r(z))^2 dz du,\end{aligned}$$

where $\bar{F}(z) = \exp\left\{-\int_0^z \mathrm{r(u)du}\right\}$.

As an illustration, they then first consider functions of the form $r(z) = c/(1-z)$ where $c > 1$ and show that

$$\lim_{n\to\infty} W_n(r) = 1 + \frac{1}{2}c + \frac{1}{c^2-1}.$$

This expression is minimal for $c = 1.9489...$ and yields the upper bound $v \leq 2.3318.....$ They also study generalizations of the previous functions, namely $r(z) = g(z)/(1-z)$ with $g(.)$ being some slowly varying function at $z = 1$. Using $g(z) = \sum_{j=0}^{m} c_j z^j$ they obtain

$$\lim_{n\to\infty} W_n(r) = 2.3267,$$

for $m = 2$ and $c_0 = 1.77, c_1 = 0.54$ and $c_2 = -0.27$. Such small improvements are however not unexpected. Indeed, Assaf and Samuel-Cahn obtain the following result:

$$\inf_{\tau_n \in M_n} U_n(\tau_n) = 2.29558...$$

(see Theorem 5.1 of Assaf and Samuel-Cahn (1996)). From (1.9) this implies that

$$2.29558... \leq V \leq 2.3267....$$

1.5 Lower bounds on the value function

The best lower bounds on V are due to Bruss and Ferguson (1993), who consider a sequence of truncated games which yield systematically lower payoff than the original problem. Their idea is as follows (see Section 4 of Bruss and Ferguson (1993)). They first consider a modification of Robbins' Problem in which the payoff for stopping at k is either 1 (if $R_k = 1$) or 2 (if $R_k > 1$). The problem then becomes one of finding a stopping rule which minimizes $\mathrm{P}[R_\tau = 1] + 2\mathrm{P}[R_\tau > 1] = 2 - \mathrm{P}[R_\tau = 1]$, i.e. we are now looking for a stopping rule τ which maximizes $\mathrm{P}[R_\tau = 1]$. This is just the full-information best-choice problem and has been solved by Gilbert and Mosteller (1966). The asymptotic value for this problem is 0.580164... and hence applying the optimal rule to $2 - \mathrm{P}[R_\tau = 1]$ one obtains an asymptotic value of 1.419386.... Since this modification of Robbins' Problem is clearly in favor of the decision-maker, his asymptotic value must be better than that of a decision-maker in the original problem and thus

$$v \geq 1.419836... .$$

Now suppose that instead of truncating at 2, one counts ranks 1 to m as their value and any higher rank as $m+1$ for $m = 1, \ldots, n-1$. This yields an increasing sequence of modified payoffs indexed by m, (say $R_k(n,m)$) such that for $m = n-1$ the payoff corresponds to the actual rank $R_k(n)$. Denoting for each m the value of the corresponding problem by $v_n^{(m)} = \inf_{\tau_n \in T_n} \mathrm{E}[R_\tau(n,m)]$, Bruss and Ferguson then show that the sequences $v_n^{(m)}$ are non decreasing and bounded in n and that

$$v^{(m)} \to v \text{ as } m \to \infty,$$

where $v^{(m)} = \lim_{n\to\infty} v_n^{(m)}$. Although these truncated problems represent considerable simplifications over the original problem, Bruss and Ferguson report that the computational aspects involved with this approach are still severe. Some computations were carried out for $m = 1$ to $m = 5$, and these pushed the lower bound up from 1.462 when $m = 1$ to 1.908 when $m = 5$.

1.6 Final comments

As a summary of the preceding results, we will say the following. First, although the optimal strategy for Robbins' problem exists, it is defined by thresholds which have an unbounded number of arguments and therefore it seems unlikely that it will ever be expressible in a closed tractable form. Therefore, the essence of Robbins' problem lies in determining the asymptotic value v. Secondly, although the optimal rules are of increasing complexity, the value function increases monotonically as the number of observations grows. Moreover its limit for n going to infinity exists and is finite, and satisfies

$$1.908... \leq v \leq 2.3267...$$

Now, as reported in Bruss (2005), Professor Robbins conjectured in 1990 that the asymptotic optimal value lies at about $v \approx 2$ but did not say anything about how close this estimate was (even when specifically asked later on). This conjecture is further supported by two facts. First, Bruss and Ferguson (1996) proved that for any finite n, there always exist strategies which yield strict improvement on the optimal memoryless value, so that we know that $v(n) < V(n)$ for all n. However, to our knowledge, nobody has so far been able to show that these improvements remain strictly positive in the limit as n tends to infinity. Secondly, the truncation method of Bruss and Ferguson (1993) described before yields a pattern of increasing lower bounds the extrapolation of which rather hints to a value around 1.97 than around 2.32. Therefore an interesting question directly related to Robbins problem' is to determine whether or not v is equal to V.

The work we have done in Chapters 2, 3 and 4 shows our efforts on these questions.

Chapter 2

The Poisson Embedded Robbins' Problem

2.1 Definition of the problem

We study a version of Robbins' problem for a random number of arrivals. The problem is as follows. A decision maker observes opportunities occurring according to a planar Poisson process of homogeneous rate 1. He inspects each option when the opportunity arises and has to chose exactly one *before a given time* t. Decisions are to be made immediately after each arrival, and no recall of preceding observations is permitted. The loss incurred by selecting an arrival is defined at time t as the total number of observations in $[0, t]$ which are smaller than the selected observation. If no decision has been reached before the given time t, then his loss is equal to some function of t, say $\Pi(t)$. At all times the decision maker has the knowledge of the full history of the process, and his objective is to use a non anticipating strategy which will minimize the expected loss.

Formal definition and notations

We denote the arrival process by $(T_1, X_1), (T_2, X_2), \ldots$, where the random variables $T_1 < T_2 < \ldots$ are interpreted as the arrival times of a homogeneous Poisson arrival-counting process $(N(s))_{s\geq 0}$ of rate 1, with associated i.i.d. random values $(X_k)_{k=1,2,\ldots}$. We suppose that the X_k's are independent of the T_k's and, without loss of generality, we assume their common distribution to be the uniform distribution on $[0, 1]$. Hence the two-dimensional process $(T_1, X_1), (T_2, X_2), \ldots$ is a planar Poisson process on the strip $[0, \infty) \times [0, 1]$ which we usually confine to $[0, t]$. We define the relative rank of an observation (T_k, X_k) by

$$r_k = \sum_{j=1}^{k} \mathbf{1}_{\{X_j \leq X_k\}}. \tag{2.1}$$

We now define a finite horizon $t > 0$. The absolute rank of the kth arrival is defined (with respect to the time horizon t) by

$$R_k^{(t)} = \sum_{j=1}^{N(t)} \mathbf{1}_{\{X_j \le X_k\}}, \tag{2.2}$$

where the sum is set to 0 if $N(t) = 0$. At each time t, $R_k^{(t)}$ is the absolute rank of the kth arrival among all arrivals which have occurred before the horizon t. The loss incurred by selecting the kth arrival is given by

$$R_k^{(t)} \mathbf{1}_{\{T_k \le t\}} + \Pi(t)\mathbf{1}_{\{T_k > t\}},$$

and the objective of the decision maker is to use a non-anticipating stopping rule adapted to the arrival process which minimizes the expected loss.

Now let $\mathcal{T}$ be the set of all $\mathbb{N}$-valued random variables (stopping rules) such that $\{\tau = k\}$ is measurable with respect to the σ-algebra $\mathcal{F}_k$ generated by $T_1 \ldots, T_k$ and $X_1, \ldots, X_k$. Note that for all $\tau \in \mathcal{T}$, the expected rank of an arrival selected before time t through τ satisfies $\mathrm{E}[R_\tau^{(t)}] = \mathrm{E}[\mathrm{E}[R_\tau^{(t)} \mid \mathcal{F}_\tau]] = \mathrm{E}[r_\tau + (t-\tau)X_\tau]$ so that, although the absolute ranks $R_k^{(t)}$ are not measurable with respect to $\mathcal{F}_k$, the problem of minimizing the loss among all adapted stopping rules is well defined via the problem of minimizing $\mathrm{E}[r_\tau + (t-\tau)X_\tau]$.

Finally let

$$\tilde{R}_\tau^{(t)} := R_\tau^{(t)} \mathbf{1}_{\{T_\tau \le t\}} + \Pi(t)\mathbf{1}_{\{T_\tau > t\}}. \tag{2.3}$$

The Poisson embedded Robbins' problem is to study the value function $w(t)$ defined by

$$w(t) = \inf_{\tau \in \mathcal{T}} \mathrm{E}\left[\tilde{R}_\tau^{(t)}\right] = \inf_{\tau \in \mathcal{T}} \mathrm{E}\left[R_\tau^{(t)} \mathbf{1}_{\{T_\tau \le t\}} + \Pi(t)\mathbf{1}_{\{T_\tau > t\}}\right], \tag{2.4}$$

including its asymptotic value

$$w = \lim_{t\to\infty} w(t), \tag{2.5}$$

if it exists, as well as the stopping rule $\tau^\star = \tau_t^\star \in \mathcal{T}$ which achieves this value.

Memoryless threshold rules

For each $t > 0$, we define the set of *threshold functions* on $[0, t]$ as the set of all functions $g_t : \mathbb{R} \to [0, 1] : s \mapsto g_t(s)$ such that $g_t(s) = 1\ \forall s \ge t$. To each such function we associate (uniquely) a *memoryless threshold rule* σ_t

$$\sigma_t = \inf\{i,\ i = 1, 2, \ldots \text{ such that } X_i \le g_t(T_i)\}, \tag{2.6}$$

and a value

$$W(\sigma_t) = \mathrm{E}[\tilde{R}_{\sigma_t}] = \mathrm{E}[R_{\sigma_t}\mathbf{1}_{\{T_{\sigma_t} \le t\}} + \Pi(t)\mathbf{1}_{\{T_{\sigma_t} > t\}}]. \tag{2.7}$$

Let $\mathcal{M}_t$ be the set of all such rules (note the indexing in t). We define the restricted value function $W(t)$ as the minimal value of $W(\sigma_t)$ obtainable on $\mathcal{M}_t$, i.e.

$$W(t) = \inf_{\sigma_t \in \mathcal{M}_t} W(\sigma_t). \tag{2.8}$$

We also denote the restricted asymptotic value (if it exists) by

$$W = \lim_{t \to \infty} W(t). \tag{2.9}$$

Clearly, for all t, we have $\mathcal{M}_t \subset \mathcal{T}$. Therefore, as in the discrete problem, the restricted optimal value $W(t)$ gives an upper bound on the optimal value $w(t)$ for all t, and thus the corresponding limits, if they exist, must satisfy $w \le W$.

Prerequisites

The following properties of the homogeneous Poisson process of rate 1 will be used without reference.

1. The number of observations in each bounded domain has Poisson distribution with mean equal to the area of the domain.

2. The random variables counting the number of observations in disjoint domains are independent.

See for example Snyder and Miller (1991).

The penalty function

The function $\Pi(t)$ reflects the loss incurred for selecting no observation before time t, and hence we will often call it the *penalty function*. It might seem more appropriate to have chosen this function to be dependent on the history of the process and, in this respect, a reasonable choice of penalty for selecting no observation before time t would have been the rank of the last observation before time t. This would have ensured more symmetry between the Poisson embedded problem and the discrete problem. However, choosing the loss function to be a random variable implies a great deal of complications and it is therefore convenient to choose $\Pi(t)$ to depend only on the horizon t. A first estimate on the behavior of this function is given by the expected rank of the last observation, which can be obtained by conditioning on the total number of arrivals in $[0, t]$. For this, let us temporarily denote the rank of the last observation by R. Then

$$\mathrm{E}[R] = \sum_{k=1}^{\infty} \mathrm{P}[N(t) = k]\mathrm{E}[R \mid N(t) = k].$$

Given $N(t) = k$, the expected rank of the last observation is equal to $\frac{k+1}{2}$ and a direct computation yields

$$\mathrm{E}[R] = \frac{1}{2}(t + 1 - e^{-t}).$$

This is a Lipschitz continuous increasing function on $[0, \infty)$, which is nil at $t = 0$. For this reason, although we choose to keep the penalty function $\Pi(.)$ unspecified throughout the text, we will suppose that it is a Lipschitz continuous increasing function which is nil at $t = 0$ and approximately linear in t (i.e. $\lim_{t\to\infty} \Pi(t)/t = \alpha,\ \alpha \in (0, \infty)$).

Now if there are no arrivals in $[0, t]$ then $\tilde{R}_\tau^{(t)} = \Pi(t)$. Also, the decision maker always has the choice of refusing every arrival before t, so that we obtain

$$w(t) \leq \Pi(t),$$

for all $t > 0$. This implies that $\lim_{t\to 0} w(t) \leq \Pi(0)$. Setting $\Pi(0) = 0$, we get the initial condition $w(0) = 0$.

2.2 Properties of the value functions

Our first result exploits the relative simplicity of the memoryless threshold strategies to obtain a rough upper bound on $W(t)$ and hence on $w(t)$.

Proposition 2.1 *The value functions are bounded on* $]0, \infty]$, *and satisfy*

$$0 \leq w(t) \leq W(t) \leq 2.33182$$

for all t.

Proof: The lower bound is clear. For the upper bound, let us consider the memoryless threshold strategies defined for all $c > 1$ by

$$\tau_c^{(t)} = \inf\left\{ i \geq 1 \text{ such that } X_i \leq \varphi_c^{(t)}(T_i) := \frac{c}{t - T_i + c} \wedge 1 \right\}. \tag{2.10}$$

Note that such strategies are similar to the asymptotic optimal strategies for Moser's problem. Clearly $\tau_c^{(t)} \in \mathcal{M}_t \subset \mathcal{T}$ for all t and $c > 0$, so that

$$w(t) \leq W(t) \leq W(\tau_c^{(t)}).$$

A direct computation of $W(\tau_c^{(t)})$ (which is performed in more generality in Section 2.3) shows that this function is bounded for all $c > 1$ by $1 + \frac{c}{2} + \frac{1}{c^2-1}$. Differentiating this last expression with respect to c, we obtain that it is minimal for $c = 1.9469\ldots$. This yields the upper bound which we stated above. ∎

We now prove that the value functions $w(.)$ and $W(.)$ are continuous on $\mathbb{R}^+$. This is not unexpected since, for small positive δ, the number of arrivals in $[t, t+\delta]$ is equal to 0 with a probability that is close to one and hence it seems intuitively clear that the difference between the value obtained by acting optimally on $[0, t+\delta]$ should be close to that obtained by acting optimally on $[0, t]$.

Proposition 2.2 *The value function $w(t)$ is uniformly continuous on $[0, \infty)$.*

Proof: Let $t > 0$ and fix some constant $\delta > 0$. We consider the Poisson embedded Robbins' problem with horizon $t + \delta$. By conditioning on the number of arrivals in $[0, \delta]$, say $N(0, \delta)$, we get

$$w(t+\delta) \geq e^{-\delta} \inf_{\tau \in \mathcal{T}} \mathrm{E}[\tilde{R}_\tau^{(t+\delta)} \mid N(0,\delta) = 0] + \delta e^{-\delta} \inf_{\tau \in \mathcal{T}} \mathrm{E}[\tilde{R}_\tau^{(t+\delta)} \mid N(0,\delta) = 1]. \tag{2.11}$$

We study the summands of (2.11) separately.

Assertion 1:

$$\inf_{\tau \in \mathcal{T}} \mathrm{E}[\tilde{R}_\tau^{(t+\delta)} \mid N(0,\delta) = 0] \geq w(t). \tag{2.12}$$

Proof: Suppose the contrary to (2.12), i.e. that there exists a strategy $\tau_0 \in \mathcal{T}$ for which

$$\mathrm{E}[\tilde{R}_{\tau_0}^{(t+\delta)} \mid N(0,\delta) = 0] < w(t).$$

Now let us define a strategy $\tilde{\tau}$ which acts on $[0, t]$ as τ_0 acts on $[0, t+\delta]$ conditionally to there being no arrivals in $[0, \delta]$. From the homogeneity of the arrival process we see that this strategy is well defined on $[0, t]$ and satisfies

$$\mathrm{E}[\tilde{R}_{\tilde{\tau}}^{(t)}] = \mathrm{E}[R_{\tau_0}^{(t+\delta)} \mathbf{1}_{\{T_{\tau_0} \leq t+\delta\}} + \Pi(t) \mathbf{1}_{\{T_{\tau_0} > t+\delta\}} \mid N(0,\delta) = 0].$$

Because the penalty function is supposed to be increasing, we have

$$\mathrm{E}[R_\tau^{(t+\delta)} \mathbf{1}_{\{T_\tau \leq t+\delta\}} + \Pi(t) \mathbf{1}_{\{T_\tau > t+\delta\}} \mid N(0,\delta) = 0] \leq \mathrm{E}[\tilde{R}_\tau^{(t+\delta)} \mid N(0,\delta) = 0]$$

for all $\tau \in \mathcal{T}$. Therefore we have obtained a strategy which yields an expected loss at time t which is strictly smaller than $w(t)$. This contradicts the definition of $w(t)$ as the smallest possible expected loss at time t among all adapted strategies, and hence (2.12) must hold.

$\square$

Assertion 2:

$$\inf_{\tau \in \mathcal{T}} \mathrm{E}[\tilde{R}_\tau^{(t+\delta)} \mid N(0,\delta) = 1] \geq \int_0^1 \min\{1 + xt, w(t)\}\, dx. \tag{2.13}$$

Proof: Conditioning on the value X of the first (and only) arrival in $[0,\delta]$, we see that

$$\inf_{\tau\in\mathcal{T}} \mathrm{E}[\tilde{R}_\tau^{(t+\delta)} \mid N(0,\delta)=1] \geq \int_0^1 \inf_{\tau\in\mathcal{T}} \mathrm{E}[\tilde{R}_\tau^{(t+\delta)} \mid N(0,\delta)=1, X=x]dx. \tag{2.14}$$

The optimality principle (see e.g. Ferguson (2000)) tells us that an optimal action given x is to select this arrival if its expected rank is smaller than the optimal value obtainable by refusing it, and to refuse it otherwise. Selecting x yields an expected loss of $1+xt$, and refusing it yields an expected loss given by

$$E(x,\delta) = \inf_{\tau\in\mathcal{T}, T_\tau>\delta} \left\{ \mathrm{E}[\tilde{R}_\tau^{(t+\delta)} \mid N(0,\delta)=1, X=x] \right\},$$

where the infimum is taken over all adapted strategies for which $T_\tau > \delta$ almost surely. Therefore

$$\inf_{\tau\in\mathcal{T}} \mathrm{E}[\tilde{R}_\tau^{(t+\delta)} \mid N(0,\delta)=1, X=x] = \min\left\{1+xt, E(x,\delta)\right\}.$$

Let us rewrite $R_\tau^{(t+\delta)}$ as

$$R_\tau^{(t+\delta)} = \sum_{i=1}^{N(t+\delta)} \mathbf{1}_{\{X_i\leq X_\tau\}} = \mathbf{1}_{\{X_1\leq X_\tau\}} + R_\tau^{(\delta,t+\delta)}$$

where $R_\tau^{(\delta,t+\delta)}$ represents the rank of the selected observation among all arrivals in $[\delta, t+\delta]$. With this notation we see that the expected loss under a strategy which refuses the first arrival satisfies

$$\begin{aligned}\mathrm{E}[\tilde{R}_\tau^{(t+\delta)} \mid N(0,\delta)=1, X=x] &= \mathrm{E}[\tilde{R}_\tau^{(\delta,t+\delta)} \mid N(0,\delta)=1, X=x] \\ &\quad + \mathrm{P}[X_\tau > x, T_\tau < t \mid N(0,\delta)=1, X=x],\end{aligned}$$

with

$$\tilde{R}_\tau^{(\delta,t+\delta)} = R_\tau^{(\delta,t+\delta)} \mathbf{1}_{\{T_\tau<t+\delta\}} + \Pi(t+\delta)\mathbf{1}_{\{T_\tau\geq t+\delta\}}.$$

Since $\mathrm{P}[X_\tau > x, T_\tau < t \mid N(0,\delta)=1, X=x] \geq 0$ for all $x\in[0,1]$, this yields from (2.14)

$$\begin{aligned}&\inf\nolimits_{\tau\in\mathcal{T}} \mathrm{E}[\tilde{R}_\tau^{(t+\delta)} \mid N(0,\delta)=1] \\ &\quad \geq \int_0^1 \min\left\{1+xt, \inf_{\tau\in\mathcal{T}, T_\tau>\delta} \mathrm{E}[\tilde{R}_\tau^{(\delta,t+\delta)} \mid N(0,\delta)=1, X=x]\right\} dx.\end{aligned} \tag{2.15}$$

Applying the same arguments as in the proof of Assertion 1, we see that the infimum appearing in (2.15) must be greater than $w(t)$, and hence (2.13) holds.

□

Combining (2.12) and (2.13) with (2.11) we obtain

$$w(t+\delta) \geq e^{-\delta} w(t) + \delta e^{-\delta} \int_0^1 \min\{1+xt, w(t)\}\, dx. \tag{2.16}$$

Since $1+xt$ is increasing in x and $1+xt = w(t)$ for $x = \frac{w(t)-1}{t}$, we obtain after some rearrangement

$$\int_0^1 \min\{1+xt, w(t)\}\, dx = w(t) - \frac{1}{2}\frac{(w(t)-1)^2}{t}.$$

Substituting this back into (2.16) yields

$$w(t+\delta) \geq e^{-\delta}(1+\delta) w(t) - \frac{1}{2}\delta e^{-\delta}\frac{(w(t)-1)^2}{t}.$$

Since $w(t) \leq 3$, and $e^{-\delta} \geq 1-\delta$, this yields

$$\begin{aligned} w(t+\delta) &\geq w(t) - \delta^2 w(t) - \delta e^{-\delta}\tfrac{1}{2}\frac{(w(t)-1)^2}{t} \\ &\geq w(t) - 3\delta^2 - 2\delta\frac{e^{-\delta}}{t}, \end{aligned}$$

and thus

$$w(t+\delta) - w(t) \geq -\delta\left(3\delta + 2\frac{e^{-\delta}}{t}\right). \tag{2.17}$$

We now obtain an upper bound on the difference $w(t+\delta) - w(t)$. For this, let $\mathcal{K}_t$ be the subset of $\mathcal{T}$ consisting of all the strategies which disregard any event occurring in $(t, t+\delta)$. Clearly

$$w(t+\delta) = \inf_{\mathcal{T}} \mathrm{E}[\tilde{R}_\tau^{(t+\delta)}] \leq \inf_{\mathcal{K}_t} \mathrm{E}[\tilde{R}_\tau^{(t+\delta)}] \tag{2.18}$$

Now take $\tau \in \mathcal{K}_t$. Then $\mathbf{1}_{\{T_\tau \leq t+\delta\}} = \mathbf{1}_{\{T_\tau \leq t\}}$ almost surely. Since the rank $R_\tau^{(t+\delta)}$ of the selected arrival (evaluated with respect to the number of observations in $[0, t+\delta]$) cannot increase from t to $t+\delta$ by more than the number of arrivals in $(t, t+\delta)$, this yields

$$\begin{aligned} \mathrm{E}[R_\tau^{(t+\delta)}\mathbf{1}_{\{T_\tau \leq t+\delta\}}] &\leq \mathrm{E}[R_\tau^{(t)}\mathbf{1}_{\{T_\tau \leq t\}}] + \mathrm{E}[\text{number of arrivals in } (t, t+\delta)] \\ &\leq \mathrm{E}[R_\tau^{(t)}\mathbf{1}_{\{T_\tau \leq t\}}] + \delta. \end{aligned}$$

This inequality holds for all $\tau \in \mathcal{K}_t$. Therefore, it follows from (2.18) that

$$w(t+\delta) \leq \inf_{\mathcal{K}_t}\left\{\mathrm{E}[R_\tau^{(t)}\mathbf{1}_{\{T_\tau < t\}}] + \Pi(t+\delta)\mathrm{P}[T_\tau \geq t]\right\} + \delta. \tag{2.19}$$

In order to be able to compare the infimum appearing in (2.19) to $w(t)$, we need $\Pi(t+\delta)$ to be evaluated at time t instead of time $t+\delta$. Let us denote $\Pi(t+\delta)-\Pi(t)$ by $\Delta\Pi(\delta)$. Then

$$\begin{aligned}
&\inf_{\mathcal{K}_t}\left\{\mathrm{E}[R_\tau^{(t)}\mathbf{1}_{\{T_\tau\le t\}}]+\Pi(t+\delta)\mathrm{P}[T_\tau>t]\right\}\\
&\quad=\inf_{\mathcal{K}_t}\left\{\mathrm{E}[R_\tau^{(t)}\mathbf{1}_{\{T_\tau\le t\}}]+\Pi(t+\delta)\mathrm{P}[T_\tau>t]\right\}+\inf_{\mathcal{K}_t}\left\{-\Delta\Pi(\delta)\mathrm{P}[T_\tau>t]\right\}\\
&\qquad\qquad-\inf_{\mathcal{K}_t}\left\{-\Delta\Pi(\delta)\mathrm{P}[T_\tau>t]\right\}\\
&\quad\le\inf_{\mathcal{K}_t}\left\{\mathrm{E}[R_\tau^{(t)}\mathbf{1}_{\{T_\tau\le t\}}]+\Pi(t)\mathrm{P}[T_\tau>t]\right\}-\inf_{\mathcal{K}_t}\left\{-\Delta\Pi(\delta)\mathrm{P}[\tau\ge t]\right\}\\
&\quad\le\inf_{\mathcal{K}_t}\left\{\mathrm{E}[R_\tau^{(t)}\mathbf{1}_{\{T_\tau\le t\}}+\Pi(t)\mathbf{1}_{\{T_\tau>t\}}]\right\}+\Delta\Pi(\delta)\sup_{\mathcal{K}_t}\left\{\mathrm{P}[T_\tau\ge t]\right\}.
\end{aligned}\tag{2.20}$$

Now $\sup_{\mathcal{K}_t}\{\mathrm{P}[T_\tau\ge t]\}$ must be equal to 1 because it suffices to choose a strategy which ignores all the arrivals in $[0,t+\delta]$ to ensure that it stops almost surely after the horizon t. Hence, from (2.20), we obtain

$$\inf_{\mathcal{K}_t}\{\mathrm{E}[R_\tau^{(t)}\mathbf{1}_{\{T_\tau\le t\}}+\Pi(t+\delta)\mathbf{1}_{\{T_\tau>t\}}]\}\le\inf_{\mathcal{K}_t}\mathrm{E}[\tilde{R}_\tau^{(t)}]+\Delta\Pi(\delta)\tag{2.21}$$

Clearly from the definition of $\mathcal{K}_t$, we have

$$\inf_{\mathcal{K}_t}\mathrm{E}[\tilde{R}_\tau^{(t)}]=w(t).$$

Therefore, substituting (2.21) in the right-hand side of (2.19), we get $w(t+\delta)\le w(t)+\Delta\Pi(\delta)+\delta$, and, since $\Pi(t)$ is Lipschitz continuous, this implies that there must exist a constant $L>0$ such that

$$w(t+\delta)-w(t)\le(L+1)\delta.\tag{2.22}$$

Combining (2.17), and (2.22) we see that for all $t>0$,

$$-\delta\left(3\delta+2\frac{e^{-\delta}}{t}\right)\le w(t+\delta)-w(t)\le(L+1)\delta.$$

Taking the limit as δ goes to zero on both sides of this inequality yields the continuity of $w(t)$ on $(0,\infty)$. By definition of the initial value $w(0)=0$, we also know that this function is continuous at 0. Finally, if $t\ge1$, and δ is sufficiently small, then from (2.17) we obtain

$$w(t+\delta)-w(t)\ge-5\delta.\tag{2.23}$$

Taking $M=\max\{L+1,5\}$, this yields

$$|w(t+\delta)-w(t)|\le M\delta$$

for all $t\ge1$ and all δ sufficiently small. This last inequality proves that $w(t)$ is locally Lipschitz continuous on $[1,\infty)$. Since it is continuous on $[0,1]$, Proposition 2.2 is established.

■

If we restrict our attention to $\mathcal{M}_t$, i.e. the set of memoryless threshold rules on $[0, t]$, we see that the first part of the proof of Proposition 2.2 holds for $W(t)$ with only minor changes. The second part of this proof also holds. To see this, it suffices to define $\mathcal{K}_t$ as the set of memoryless threshold rules associated to threshold functions which are identically nil on $[t, t+\delta]$, and everything runs smoothly. This yields

Proposition 2.3 *The value function $W(t)$ restricted to the class of memoryless threshold rules is uniformly continuous on $[0, \infty)$.*

We now prove the existence of optimal strategies for the Poisson embedded Problem. This is intuitively clear since we have shown that the value functions $w(t)$ and $W(t)$ are well defined and bounded, so that we should be able to compare the expected rank of each arrival to the best obtainable value and thus decide at each arrival whether or not it is optimal to stop. The point is that this comparison is possible at any arrival time, and so leads to an almost surely unique optimal strategy.

Proposition 2.4 *For each t there exists a stopping rule $\tau_t^\star$ in $\mathcal{T}$ such that*

$$w(t) = \mathrm{E}\left[\tilde{R}^{(t)}_{\tau_t^\star}\right], \tag{2.24}$$

and a stopping rule $\sigma_t^\star \in \mathcal{M}_t$ such that

$$W(t) = \mathrm{E}\left[\tilde{R}^{(t)}_{\sigma_t^\star}\right]. \tag{2.25}$$

Proof: Fix $t > 0$, and suppose that there is an arrival of value X_i at time T_i, $0 < T_i < t$, $i \geq 1$. Let $E(i, t)$ be the expected loss incurred by refusing this arrival and continuing optimally thereafter, i.e. $E(i, t)$ is the minimal expected rank obtainable under the history $\mathcal{F}_i$ by using strategies which stop almost surely after the ith arrival. It is given by

$$E(i, t) = \inf_{\tau \in \mathcal{T}, \tau > i} \mathrm{E}[\tilde{R}^{(t)}_\tau \mid \mathcal{F}_i],$$

where the infimum is taken over the set of all stopping rules $\tau \in \mathcal{T}$ such that $\mathrm{P}[\tau > i] = 1$. For all $i \geq 0$ and every history $\mathcal{F}_i$, we see that $E(i, t)$ is well defined for all horizons $t > 0$. Using arguments similar to those appearing in the proof of Proposition 2.2 we see that it satisfies the upper bound

$$E(i, t) \leq r_i + w(t - T_i) + \Pi(t) - \Pi(T_i).$$

From the optimality principle, we know that it can only be optimal to stop on an arrival X_i if the expected loss incurred by selecting X_i is smaller than the expected loss incurred by refusing X_i. Hence, if we define the rule $\tau_t^\star$ by

$$\begin{cases} \tau_t^\star = i & \text{if} \quad \mathrm{E}[\tilde{R}^{(t)}_i \mid \mathcal{F}_i] \leq E(i, t) \\ \tau_t^\star > i & \text{if} \quad \mathrm{E}[\tilde{R}^{(t)}_i \mid \mathcal{F}_i] > E(i, t), \end{cases} \tag{2.26}$$

then $\tau_t^\star$ belongs to $\mathcal{T}$ and must be optimal for each time t.

Minor adaptations of these arguments show that the same result holds for the restricted problem. ■

Remark 9 *An alternative proof of Proposition 2.4 is provided by known general results on stopping problems (see e.g. Ferguson (2000), Chapter 3) which can be adapted to the Poisson embedded Robbins' problem (restricted or not) in order to show the existence of optimal rules in this problem.*

We have thus far shown that $w(t)$ and $W(t)$ are bounded continuous functions of t. This does not guarantee the existence of the limit $w = \lim_{t\to\infty} w(t)$, nor that of $W = \lim_{t\to\infty} W(t)$. However, it seems intuitively clear that both $w(t)$ and $W(t)$ are increasing functions of t. Indeed, as the time horizon t gets large, it seems clear that the task of the decision maker gets more difficult, so that the optimal rank he can obtain must be increasing with t. This conjecture is further strengthened by the fact that the value functions in the discrete case are increasing in n.

To prove this claim, we initially tried to adapt the prophet trick of Bruss and Ferguson (1993), in which a player (called a *half-prophet*) is told before the game which arrival will be the worst (see Chapter 1). However, this approach wasn't appropriate to our situation - partly because of problems with the penalty function. We have then tried many different approaches, and none of these yielded the desired inequalities. We have put a great deal of thought into proving this seemingly obvious claim and we weren't able to make these arguments precise. We will further address this aspect of Robbins' problem in Chapter 3.

2.3 Memoryless threshold rules

A memoryless threshold rule is a strategy for which the decision to stop on an arrival depends exclusively on whether or not the value of this arrival is smaller than a given threshold. The thresholds must be fixed in advance, that is they must be independent of the history of the process. As we have mentioned in Chapter 1, this class of strategies was introduced in the discrete setting by Bruss and Ferguson (1993) and Assaf and Samuel-Cahn (1996) and a number of results are known, including upper and lower bounds on the restricted value function. Now we have already proved that the restricted value function $W(t)$ is an increasing function of t. In this section we will show that other interesting properties of the discrete problem can be nicely transposed to the continuous problem. We will also show how, under certain conditions, the memoryless rules in the continuous and the discrete cases yield the same limiting values.

2.3.1 Threshold functions

Let $g_t(.)$ be a threshold function on $[0, t]$ and take $\sigma_t \in \mathcal{M}_t$ to be the corresponding memoryless threshold rule (see equation (2.6)). Now define the function $\mu_t(s)$ for $s \geq 0$ by

$$\mu_t(s) = \int_0^s g_t(u)du. \tag{2.27}$$

From the properties of homogeneous Poisson processes we see that for all $s \in [0, t]$,

$$\mathrm{P}[T_{\sigma_t} \geq s] = e^{-\mu_t(s)},$$

and hence the density of T_{σ_t} is given on $(0, t)$ by

$$f_{T_{\sigma_t}}(s) = g_t(s)e^{-\mu_t(s)}. \tag{2.28}$$

Our next result shows that, as in the discrete case, only increasing thresholds need to be considered. Intuitively, this simply translates the fact that if it is optimal to accept an arrival of value x at time s, then it should also be optimal to accept an arrival of smaller value at later times $s' \geq s$.

Proposition 2.5 *Let $\sigma_t^\star \in \mathcal{M}_t$ be the optimal memoryless threshold rule and let $g_t^\star(.)$ be the corresponding threshold function. Then for all $0 \leq s \leq s' \leq t$, $g_t^\star(s) \leq g_t^\star(s')$.*

Proof: Suppose that there is an arrival X_i at time T_i, and define $E(i, t)$ as the minimal expected rank obtainable with memoryless strategies which stop almost surely after the ith arrival, i.e.

$$E(i, t) = \inf_{\sigma_t \in \mathcal{M}_t, \sigma_t > i} \mathrm{E}[\tilde{R}_{\sigma_t}^{(t)} \mid \mathcal{F}_i],$$

where the infimum is taken over the set of all stopping rules $\sigma_t \in \mathcal{M}_t$ such that $\mathrm{P}[\sigma_t > i] = 1$. We know that it is optimal to stop on an arrival (T_i, X_i) if and only if

$$\mathrm{E}[\tilde{R}_i^{(t)} \mid \mathcal{F}_i] \leq E(i, t).$$

Since $\mathrm{E}[\tilde{R}_i^{(t)} \mid \mathcal{F}_i] = r_i + (t - T_i)X_i$, this implies that it is optimal to stop on (T_i, X_i) if and only if it satisfies

$$r_i + (t - T_i)X_i \leq E(i, t). \tag{2.29}$$

Now suppose that $\sigma_t^\star$ is optimal but that $g_t^\star(.)$ is not increasing on $[0, t]$ (as illustrated in Figure 1).

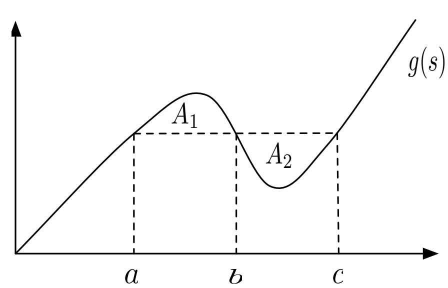

Figure 1: *The threshold function is not monotone increasing and hence we can define the areas A_1 and A_2.*

Since $g_t^\star(.)$ is not increasing, it must be possible to choose areas A_1, A_2 and a, b, c as illustrated in Figure 1. By definition of $\sigma_t^\star$, any arrival in A_1 will be accepted, and any arrival in A_2 will be rejected. Now suppose that the ith arrival (T_i, X_i) lies in A_1. Then this arrival is accepted and must be an optimal choice in the class of memoryless strategies, so that it satisfies equation (2.29), which yields

$$r_i + (t - T_i)X_i \leq E(i, t). \tag{2.30}$$

If the next arrival (T_{i+1}, X_{i+1}) lies in A_2, then, although $X_{i+1} \leq X_i$, it will not be selected by $\sigma_t^\star$. Hence it is not optimal to stop on this arrival, and

$$r_{i+1} + (t - T_{i+1})X_{i+1} > E(i + 1, t). \tag{2.31}$$

Since $X_{i+1} \leq X_i$, we must have $r_{i+1} \leq r_i$. Also, under fixed history up to time $i - 1$, we see that $E(i, t) \leq E(i + 1, t)$. Therefore, from (2.30) and (2.31) we obtain

$$E(i, t) \leq E(i + 1, t) < r_{i+1} + (t - T_{i+1})X_{i+1} \leq r_i + (t - T_i)X_i,$$

which in turn yields

$$r_i + (t - T_i)X_i > E(i, t). \tag{2.32}$$

Hence, if the optimal strategy $\sigma_t^\star$ is defined through a non monotone increasing threshold function, we see that there is a positive probability of there being a realization of the process for which equations (2.30) and (2.32) must hold at the same time. This yields a contradiction. ■

From now on we will only consider threshold functions $g_t(.)$ that are monotone increasing on $[0, t]$.

2.3.2 The value of memoryless threshold rules

Equation (2.28) gives the density of the arrival time of an observation selected through a memoryless threshold rule. Hence, conditioning on the arrival time, we see that for all $\sigma_t \in \mathcal{M}_t$,

$$\mathrm{E}[R_\tau^{(t)} \mathbf{1}_{\{T_{\sigma_t} \leq t\}}] = \int_0^t \mathrm{E}[R_\tau^{(t)} \mid T_{\sigma_t} = s] f_{T_{\sigma_t}}(s) ds. \tag{2.33}$$

We now use this equation to obtain an integral version of (1.5) for memoryless threshold rules.

Proposition 2.6 *Let $g_t(s)$ be a continuous increasing threshold function, and let $\sigma_t \in \mathcal{M}_t$ be the corresponding memoryless threshold rule. Let $\mu_t(s) = \int_0^s g_t(u)du$. Then*

$$\begin{aligned} W(\sigma_t) &= 1 + (\Pi(t) - 1)e^{-\mu_t(t)} + \tfrac{1}{2}\int_0^t g_t(s)^2(t-s)e^{-\mu_t(s)}ds \\ &\quad + \tfrac{1}{2}\int_0^t \int_0^s \frac{(g_t(s) - g_t(u))^2}{1 - g_t(u)} du\, e^{-\mu_t(s)} ds \end{aligned} \tag{2.34}$$

Proof: Recall that for a memoryless threshold rule σ_t defined by a function g_t, the density of T_{σ_t} is given on $(0, t)$ by

$$f_{T_{\sigma_t}}(s) = g_t(s)e^{-\mu_t(s)}. \tag{2.35}$$

Now choose $s \in (0, t)$ and suppose that $T_{\sigma_t} = s$. Then, conditionally to $X_{N(s)} = x \in [0, g_t(s)]$, the relative rank $r_{N(s)}$ is given by the number of arrivals in A_1 and A_2 (see Figure 1) , and

$$\mathrm{E}[R_{\sigma_t}^{(t)} \mid T_{\sigma_t} = s, X_{\sigma_t} = x] =$$

$$\begin{cases} 1 + x(t-s) & \text{if } 0 \le x \le g_t(0) \\ 1 + x(t-s) + \displaystyle\int_0^{g_t^{-1}(x)} \frac{x - g_t(u)}{1 - g_t(u)} du & \text{if } g_t(0) \le x \le g_t(s). \end{cases} \tag{2.36}$$

where the second part of (2.36) holds because we know that if we haven't stopped before s then there can have been no arrivals under the curve before s so that, conditionally to $T_{\sigma_t} = s$, the value of any arrival occurring at time $0 \le u \le g_t^{-1}(x)$ is uniformly distributed on $[g_t(u), 1]$.

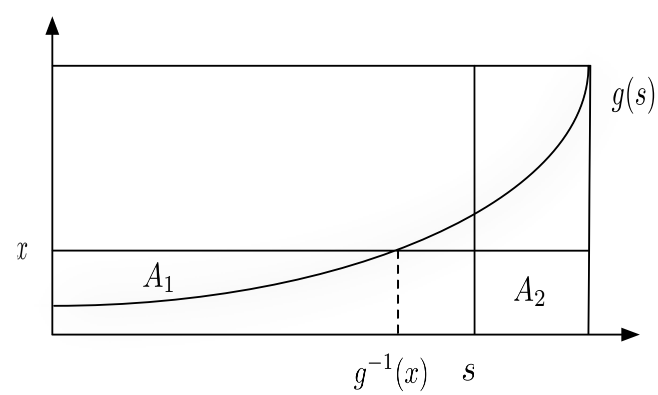

Figure 2: Smaller arrivals can only occur in A_1 and A_2.

Now, conditionally to $T_{\sigma_t} = s$, we know that the arrivals are distributed uniformly on $[0, g_t(s)]$. Therefore, integrating (2.36) yields

$$\begin{aligned} \mathrm{E}[R^{(t)}_{\sigma_t} \mid T_{\sigma_t} = s] &= 1 + \tfrac{1}{g_t(s)} \int_0^{g_t(s)} x(t-s)dx \\ &\quad + \tfrac{1}{g_t(s)} \int_{g_t(0)}^{g_t(s)} \int_0^{g_t^{-1}(x)} \frac{x - g_t(u)}{1 - g_t(u)} du dx. \end{aligned} \tag{2.37}$$

Using

$$W(\sigma_t) = \int_0^t \mathrm{E}[R^{(t)}_{\sigma_t} | T_{\sigma_t} = s] f_{T_{\sigma_t}}(s) ds + \Pi(t)\mathrm{P}[T_{\sigma_t} \geq t],$$

straightforward rearrangement and integration of (2.37) yields (2.34). ■

Equation (2.34) still holds if the threshold function g_t is not continuous on $[0, t]$. For example, take a threshold rule for the discrete n-arrival problem $\sigma_n \in M_n$ defined by a sequence $(a_i)_{1 \leq i \leq n}$ satisfying the conditions given in Lemma 1.1. Then $V(\sigma_n)$ is given by (1.5). Now define the threshold function $g_n(.)$ on $[0, n]$ by

$$g_n(s) = \sum_{i=0}^{n-1} a_{i+1} \mathbf{1}_{[i,i+1)}(s)$$

and let $\tilde{\sigma}_n \in \mathcal{M}_n$ be the corresponding threshold strategy. Then a direct computation of (2.34) yields

$$\begin{aligned} W(\tilde{\sigma}_n) = \;& 1 + \tfrac{1}{2} \sum_{k=1}^{n-1} a_k^2 (n-k) \prod_{i=1}^{k-1} e^{-a_i} \left(\frac{1 - e^{-a_k}}{a_k} \right) \\ & + \tfrac{1}{2} \sum_{k=1}^{n} \prod_{i=1}^{k-1} e^{-a_i} \left(\frac{1 - e^{-a_k}}{a_k} \right) \sum_{j=1}^{k-1} \frac{(a_k - a_j)^2}{1 - a_j} \\ & + \tfrac{1}{2} \sum_{k=1}^{n} \prod_{i=1}^{k-1} e^{-a_i} \left(a_k - 1 + e^{-a_k} \right) + (\Pi(n) - 1) \prod_{i=1}^{n} e^{-a_i} \end{aligned} \tag{2.38}$$

Equation (2.38) is very similar to equation (1.5). However, we have not been able to use this similarity to compare the discrete and the Poisson embedded problems, because no upper or lower bounds seem to appear clearly out of (2.38).

2.3.3 Asymptotic values

Take $g_n(.)$ and σ_n as above. If there have been no satisfactory arrivals for σ_n before time n, then the loss of the decision maker is given by $\tilde{R}_{\sigma_n} = \Pi(n)$ and thus

$$W(\sigma_n) \geq \Pi(n) \mathrm{e}^{-\mu_n(n)}, \tag{2.39}$$

where $\mu_n(s)$ is, as before, defined for all $0 \le s \le n$ by $\mu_n(s) = \int_0^s g_n(u)du$. Since we are interested in optimal values, and since the penalty function is chosen to be asymptotically linear in the horizon t, we see from (2.39) that we can restrict our attention without loss of generality to sequences of threshold functions which satisfy

$$\lim_{n\to\infty} n\mathrm{e}^{-\mu_n(n)} = 0. \tag{2.40}$$

Now let $(g_n(.))_{n\ge1}$ be a sequence of strict monotone increasing threshold functions on $[0,n]$ (i.e. for each n, the function $g_n(.)$ is a strictly increasing threshold function with horizon n) and let $\sigma_n \in \mathcal{M}_n$ be the corresponding sequence of memoryless threshold strategies. Clearly, for all n, we have

$$w(n) \le W(n) \le W(\sigma_n).$$

We shall show that, under general conditions on the threshold sequence, we can use equation (2.34) to obtain $\lim_{n\to\infty} W(\sigma_n)$.

We first define the functions

$$h_n(u) = ng_n(nu)$$

for $u \in [0,1]$. A change of variables in (2.34) yields

$$W(\sigma_n) = 1 + (\Pi(n)-1)e^{-\mu_t(n)} + \frac{1}{2}\int_0^1 d_n^1(s)ds + \frac{1}{2}\int_0^1\int_0^s d_n^2(s,u)duds \tag{2.41}$$

where

$$\begin{cases} d_n^1(s) = h_n(s)^2(1-s)e^{-\int_0^s h_n(v)dv} \\ \\ d_n^2(s,u) = \dfrac{(h_n(s)-h_n(u))^2}{1-h_n(u)/n}\, e^{-\int_0^s h_n(v)dv} \end{cases}$$

Now suppose that the sequence $g_n(.)$ satisfies (2.40) and that, for all $u \in (0,1)$, the sequence $h_n(u)$ converges. We can define the limit function

$$g(u) = \lim_{n\to\infty} h_n(u) = \lim_{n\to\infty} ng_n(nu).$$

Note that this function is unbounded in $u = 1$. In order to interchange the limit and the integration appearing in (2.41), we need some stronger assumptions on the sequence of thresholds $g_n(.)$. We will impose two conditions.

(C1) For every $s \in (0,1)$, $h_n(s)$ increases monotonically as it approaches $g(s)$.

(C2) The sequence of functions $h_n(s)$ is uniformly convergent on every interval $[0,a]$, for $a < 1$.

With these assumptions, a version of the dominated convergence theorem applies to $d_n^1(.)$ and $d_n^2(.,.)$ so that

$$\lim_{n\to\infty}\int_0^1 d_n^1(s)ds = \int_0^1 \lim_{n\to\infty} d_n^1(s)ds = \int_0^1 d_1(s)ds,$$

with

$$d_1(s) = g(s)^2(1-s)e^{-\int_0^s g(u)du},$$

and also

$$\lim_{n\to\infty}\int_0^1\int_0^s d_n^2(s,u)duds = \int_0^1\int_0^s d_2(s,u)duds,$$

with

$$d_2(s,u) = (g(s)-g(u))^2\, e^{-\int_0^s g(u)du}.$$

Hence, taking the limit for n going to infinity in (2.34), we get

$$\lim_{n\to\infty} W(\sigma_n) = 1 + \frac{1}{2}\int_0^1 d_1(s)ds + \frac{1}{2}\int_0^1\int_0^s d_2(s,u)duds =: L(g).$$

Likewise, if for each $n \geq 1$ we define the threshold sequence $a_i = g_n(i)$, $i = 1,\ldots,n$ and let $\tau_n \in M_n$ be the discrete stopping rule defined through these thresholds, then (1.5) applies and gives $V(\tau_n)$ as the Lebesgue integral of suitably chosen step functions. Under the same conditions on the sequence $g_n(.)$ as above, we see that we can take the limit for n going to infinity of $V(\tau_n)$ and this also yields $L(g)$.

This explains the following proposition.

Proposition 2.7 *Let $(g_n(.))_{n\geq 1}$ be a sequence of threshold functions satisfying (2.40), such that $\lim_{n\to\infty} ng_n(u)$ exists and is finite for all $u \in (0,1)$. Define the function*

$$g(u) = \lim_{n\to\infty} ng_n(u).$$

Let $\sigma_n \in \mathcal{M}_n$ be the sequence of memoryless threshold rules (for the Poisson embedded problem with horizon n) defined, for each n, by $g_n(.)$ and let $\tau_n \in M_n$ be the sequence of memoryless threshold rules (for the discrete n-arrival Robbins' problem) defined, for each n, by the threshold sequence $(g_n(i))_{i=1,\ldots,n}$. Then, under assumptions C1 and C2,

$$\lim_{n\to\infty} W(\sigma_n) = \lim_{n\to\infty} V(\tau_n) = L(g) \tag{2.42}$$

where

$$\begin{aligned} L(g) &= 1 + \tfrac{1}{2}\int_0^1 g(u)^2(1-u)e^{-\int_0^u g(x)dx}du \\ &\quad + \tfrac{1}{2}\int_0^1\int_0^u (g(u)-g(v))^2 dv\, e^{-\int_0^u g(x)dx}du. \end{aligned} \tag{2.43}$$

Remark 10 *This is the same integral expression as that obtained by Assaf and Samuel-Cahn (1996), see Section 1.4.*

Example 1 *Let* $g_n(s) = \dfrac{c}{n-s+c}$, *with* $c > 1$. *This sequence satisfies the conditions imposed above with* $g(u) = \dfrac{c}{1-u}$. *Applying (2.42) yields*

$$L(c) = 1 + \frac{c}{2} + \frac{1}{c^2-1}.$$

Therefore for all $c > 1$ *and all* $t \in [0,\infty)$ *we get* $w(t) \leq W(t) \leq L(c)$. *This expression is minimal for* $c = 1.94697$ *and yields the upper bound*

$$w(t) \leq 2.33183.$$

This upper bound has already been obtained in Bruss and Ferguson (1993) and Assaf and Samuel Cahn (1996) for the discrete n *arrival problem.*

2.4 A differential equation on the value function

Let $w(t|x)$ denote the optimal value conditioned on a first (artificial) arrival at time 0 with value x, which cannot be selected, i.e.

$$w(t|x) = \inf_{\tau\in\mathcal{T}} \left\{ \mathrm{E}[R_\tau^{(t)} \mathbf{1}_{\{T_\tau \leq t\}}] + \Pi(t)\mathrm{P}[T_\tau > t] + \mathrm{P}[X_\tau \geq x, T_\tau \leq t] \right\}. \tag{2.44}$$

For fixed t, this function is monotone decreasing in x on $[0,1]$, with

$$w(t) + 1 \geq w(t \mid 0) \geq w(t \mid x) \geq w(t \mid 1) = w(t),\ 0 \leq x \leq 1.$$

Using the same arguments as in Proposition 2.2, we can also show that for fixed $x \in [0,1]$, $w(t \mid x)$ is continuous in t and that (if the limit $w = \lim_{t\to\infty} w(t)$) exists, then

$$\lim_{t\to\infty} w(t \mid x) = w,$$

independently of x.

Now let $\tau_t^\star$ be the optimal strategy for the Poisson embedded problem with horizon t, and let us suppose that $\Pi(t)$ is increasing, Lipschitz and differentiable in t. A dynamic programming approach to the Poisson embedded Robbins' problem yields the following result.

Theorem 2.8 *Let* I *be an open bounded interval* $I \subset (0,\infty)$. *Then for almost all* $t \in I$ *(i.e. except on a set of Lebesgue measure 0), the value function* $w(t)$ *is differentiable and satisfies*

$$w'(t) + w(t) = \int_0^1 \min\{1 + xt, w(t \mid x)\}dx + \chi(t), \tag{2.45}$$

where $\chi(t)$ *satisfies* $0 \leq \chi(t) \leq 1$ *for all* $t \geq 0$, *and* $\lim_{t\to\infty} t\chi(t) = 0$.

Before proving this theorem we need two preparatory lemmas on the properties of the distribution of the optimal strategy.

Lemma 2.9 *For all* $t > 0$,

$$\lim_{\Delta t \to 0^+} \mathrm{E}[\tilde{R}^{(t)}_{\tau^\star_t} | N(\Delta t) = 1] = \int_0^1 \min\{1 + xt, w(t \mid x)\} dx.$$

Proof: To show this, let us fix $\Delta t > 0$ and suppose that there is a unique arrival of value X in $[0, \Delta t]$. Conditioning on X which is uniformly distributed on $[0,1]$, we see that

$$\mathrm{E}[\tilde{R}^{(t)}_{\tau^\star_t} \mid N(\Delta t) = 1] = \int_0^1 \mathrm{E}[\tilde{R}^{(t)}_{\tau^\star_t} \mid N(\Delta t) = 1, X = x] dx. \tag{2.46}$$

Since $\tau^\star_t$ is optimal for the horizon t, the optimality principle must apply and thus

$$\begin{aligned} &\mathrm{E}[\tilde{R}^{(t)}_{\tau^\star_t} \mid N(\Delta t) = 1, X = x] \\ &= \min\{1 + x(t - \Delta t), \inf_{\tau \in \mathcal{T}, T_\tau > \Delta t} \mathrm{E}[\tilde{R}^{(t)}_\tau \mid N(\Delta t) = 1, X = x]\} \end{aligned} \tag{2.47}$$

where the first argument of the minimum appearing in the rhs of (2.47) is given by expected loss incurred by selecting $X = x$. This is equal to the expected number of arrivals in $[\Delta t, t] \times [x, 1]$, i.e. $1 + x(t - \Delta t)$.

Assertion 1:

$$\inf_{\tau \in \mathcal{T}, T_\tau > \Delta t} \mathrm{E}[\tilde{R}^{(t)}_\tau \mid N(\Delta t) = 1, X = x] \geq w(t - \Delta t \mid x). \tag{2.48}$$

Proof: Since $\Pi(t)$ is increasing, we can write

$$\begin{aligned} &\inf_{\tau \in \mathcal{T}, T_\tau > \Delta t} \mathrm{E}[\tilde{R}^{(t)}_\tau \mid N(\Delta t) = 1, X = x] \\ &\quad = \inf_{\tau \in \mathcal{T}, T_\tau > \Delta t} \mathrm{E}[R^{(t)}_\tau \mathbf{1}_{\{T_\tau \leq t\}} + \Pi(t) \mathbf{1}_{\{T_\tau > t\}} \mid N(\Delta t) = 1, X = x] \\ &\quad \geq \inf_{\tau \in \mathcal{T}, T_\tau > \Delta t} \mathrm{E}[R^{(t)}_\tau \mathbf{1}_{\{T_\tau \leq t\}} + \Pi(t - \Delta t) \mathbf{1}_{\{T_\tau > t\}} \mid N(\Delta t) = 1, X = x]. \end{aligned}$$

Now recall that the rank of an observation selected before the time horizon t through any strategy τ is given (at time t) by

$$R^{(t)}_\tau = \mathbf{1}_{\{X_\tau > X_1\}} + \sum_{j=2}^{N(t)} \mathbf{1}_{\{X_\tau > X_j\}}.$$

Hence, since the arrival process is homogeneous (and therefore the distribution of the arrivals is the same on $[\Delta t, t]$ as on $[0, t-\Delta t]$) we get

$$\begin{aligned}
&\inf_{\tau\in\mathcal{T}, T_\tau>\Delta t} \mathrm{E}[R_\tau^{(t)}\mathbf{1}_{\{T_\tau\le t\}} + \Pi(t-\Delta t)\mathbf{1}_{\{T_\tau>t\}} \mid N(\Delta t)=1, X=x] \\
&= \inf_{\tau\in\mathcal{T}} \mathrm{E}[R_\tau^{(t-\Delta t)}\mathbf{1}_{\{T_\tau\le t-\Delta t\}} + \Pi(t-\Delta t)\mathbf{1}_{\{T_\tau>t-\Delta t\}} + \mathbf{1}_{\{X_\tau\ge x, T_\tau\le t-\Delta t\}}] \qquad (2.49)\\
&= w(t-\Delta t \mid x).
\end{aligned}$$

Therefore

$$\inf_{\tau\in\mathcal{T}, T_\tau>\Delta t} \mathrm{E}[\tilde{R}_\tau^{(t)} \mid N(\Delta t)=1, X=x] \ge w(t-\Delta t \mid x)$$

This proves Assertion 1.

□

Assertion 2: Let $\Delta\Pi(t,\Delta t) = \Pi(t) - \Pi(t-\Delta t)$. Then

$$\inf_{\tau\in\mathcal{T}, T_\tau>\Delta t} \mathrm{E}[\tilde{R}_\tau^{(t)} \mid N(\Delta t)=1, X=x] \le w(t-\Delta t \mid x) + \Delta\Pi(t,\Delta t). \qquad (2.50)$$

Proof: By use of the same inequalities as in equation (2.20) on page 74, we see that

$$\begin{aligned}
&\inf_{\tau\in\mathcal{T}, T_\tau>\Delta t} \mathrm{E}[R_\tau^{(t)}\mathbf{1}_{\{T_\tau\le t\}} + \Pi(t)\mathbf{1}_{T_\tau>t} \mid N(\Delta t)=1, X=x] \\
&\le \inf_{\tau\in\mathcal{T}, T_\tau>\Delta t} \mathrm{E}[R_\tau^{(t)}\mathbf{1}_{\{T_\tau\le t-\Delta t\}} + \Pi(t-\Delta t)\mathbf{1}_{\{T_\tau>t-\Delta t\}} \mid N(\Delta t)=1, X=x] \\
&\quad +\Delta\Pi(t,\Delta t)\sup_{\tau\in\mathcal{T}, T_\tau>\Delta t} \mathrm{P}[T_\tau > t].
\end{aligned}$$

The supremum appearing in the last equation is clearly equal to 1, and hence from (2.49) we get

$$\begin{aligned}
&\inf_{\tau\in\mathcal{T}, T_\tau>\Delta t} \mathrm{E}[R_\tau^{(t)}\mathbf{1}_{\{T_\tau\le t\}} + \Pi(t)\mathbf{1}_{T_\tau>t} \mid N(\Delta t)=1, X=x] \\
&\le w(t-\Delta t \mid x) + \Delta\Pi(t,\Delta t).
\end{aligned}$$

□

From Assertions 1 and 2 we know that for all $\Delta t > 0$ and for all horizons t, we have the bounds

$$\mathrm{E}[\tilde{R}_{\tau_t^\star}^{(t)} \mid N(\Delta t)=1] \le \int_0^1 \min\{1+x(t-\Delta t), w(t-\Delta t \mid x) + \Delta\Pi(t,\Delta t)\}dx.$$

and

$$\mathrm{E}[\tilde{R}_{\tau_t^\star}^{(t)} \mid N(\Delta t)=1] \ge \int_0^1 \min\{1+x(t-\Delta t), w(t-\Delta t \mid x)\}dx$$

We now take the limit of both inequalities for $\Delta t \to 0^+$. Since we can interchange limits and integrals in both expressions, we see from the continuity of $w(t \mid x)$ and that of $\Pi(t)$ that the upper bound converges to the same limit as the lower bound and thus the lemma holds. ■

Lemma 2.10 *For all $t > 0$ and all $\Delta t > 0$,*

$$\mathrm{E}[\tilde{R}^{(t)}_{\tau^\star_t} | N(\Delta t) = 0] = w(t - \Delta t) + \chi(t, \Delta t). \tag{2.51}$$

where $\chi(t, \Delta t)$ is a positive function which satisfies $\chi(t, 0) = 0$ and

$$\Pi'(t)\mathrm{P}[T_{\tau^\star_t} > t] \leq \lim_{\Delta t \to 0^+} \frac{\chi(t, \Delta t)}{\Delta t} \leq \Pi'(t) \lim_{\Delta t \to 0^+} \mathrm{P}[T_{\tau^\star_{t-\Delta t}} > t - \Delta t].$$

Proof: Let $\Delta\Pi(t, \Delta t)$ be as above. From the homogeneity of the arrival process we see that the optimal strategy $\tau^\star_t$ conditioned on there being no arrivals in $[0, \Delta t]$ can be applied as a (suboptimal) strategy on $[0, t - \Delta t]$, and thus

$$w(t - \Delta t) \leq \mathrm{E}[R^{(t)}_{\tau^\star_t} \mathbf{1}_{\{T_{\tau^\star_t} \leq t\}} \mid N(\Delta t) = 0] + \Pi(t - \Delta t)\mathrm{P}[T_{\tau^\star_t} \geq t \mid N(\Delta t) = 0].$$

Rearranging this last equation, we obtain

$$w(t - \Delta t) + \Delta\Pi(t, \Delta t)\mathrm{P}[T_{\tau^\star_t} > t \mid N(\Delta t) = 0] \leq \mathrm{E}[\tilde{R}^{(t)}_{\tau^\star_t} | N(\Delta t) = 0]. \tag{2.52}$$

Likewise, let us consider a strategy $\tilde{\tau}$ on $[0, t]$ which ignores every arrival in $[0, \Delta t]$ and applies $\tau^\star_{t-\Delta t}$ on $[\Delta t, t]$. Then from the homogeneity of the arrival process we see that

$$\begin{aligned}
&\mathrm{E}[\tilde{R}^{(t)}_{\tilde{\tau}} | N(\Delta t) = 0] \\
&\quad = \mathrm{E}[R^{(t-\Delta t)}_{\tau^\star_{t-\Delta t}} \mathbf{1}_{\{\tau^\star_{t-\Delta t} \leq t - \Delta t\}}] + \Pi(t)\mathrm{P}[T_{\tau^\star_{t-\Delta t}} > t - \Delta t] && (2.53) \\
&\quad = w(t - \Delta t) + (\Pi(t) - \Pi(t - \Delta t))\mathrm{P}[T_{\tau^\star_{t-\Delta t}} > t - \Delta t]
\end{aligned}$$

Since $\tau^\star_t$ is optimal, we know that

$$\mathrm{E}[\tilde{R}^{(t)}_{\tau^\star_t} | N(\Delta t) = 0] \leq \mathrm{E}[\tilde{R}^{(t)}_{\tilde{\tau}} | N(\Delta t) = 0],$$

so that from (2.53) we obtain

$$\mathrm{E}[\tilde{R}^{(t)}_{\tau^\star_t} | N(\Delta t) = 0] \leq w(t - \Delta t) + (\Pi(t) - \Pi(t - \Delta t))\mathrm{P}[T_{\tau^\star_{t-\Delta t}} > t - \Delta t]. \tag{2.54}$$

Now define

$$\chi(t, \Delta t) := \mathrm{E}[\tilde{R}^{(t)}_{\tau^\star_t} | N(\Delta t) = 0] - w(t - \Delta t).$$

Combining (2.52) and (2.54), we see that

$$\Delta\Pi(t, \Delta t)\mathrm{P}[T_{\tau^\star_t} > t \mid N(\Delta t) = 0] \leq \chi(t, \Delta t) \leq \Delta\Pi(t, \Delta t)\mathrm{P}[T_{\tau^\star_{t-\Delta t}} > t - \Delta t]. \tag{2.55}$$

and therefore from the continuity of the penalty function we get that $\chi(t, 0) = 0$ by taking the limit for $\Delta t \to 0^+$. This proves the first statement of Lemma 2.10.

Now divide (2.55) by Δt, and take the limit for $\Delta t \to 0^+$ on both sides of this inequality. Clearly $\mathrm{P}[T_{\tau^\star_t} > t \mid N(\Delta t) = 0] \geq \mathrm{P}[T_{\tau^\star_t} > t]$. From the differentiability of $\Pi(t)$, we therefore see that

$$\Pi'(t)\mathrm{P}[T_{\tau^\star_t} > t] \leq \lim_{\Delta t \to 0^+} \frac{\chi(t, \Delta t)}{\Delta t} \leq \Pi'(t) \lim_{\Delta t \to 0^+} \mathrm{P}[T_{\tau^\star_{t-\Delta t}} > t - \Delta t], \tag{2.56}$$

if this limit exists. ∎

Proof of Theorem 2.8

Fix $\Delta t > 0$. Conditioning on the number of arrivals in $[0, \Delta t]$ we get

$$\begin{aligned} w(t) = \mathrm{E}[\tilde{R}_{\tau_t^\star}] &= \mathrm{P}[N(\Delta t) = 0]\,\mathrm{E}[\tilde{R}^{(t)}_{\tau_t^\star} | N(\Delta t) = 0] \\ &\quad + \mathrm{P}[N(\Delta t) = 1]\,\mathrm{E}[\tilde{R}^{(t)}_{\tau_t^\star} | N(\Delta t) = 1] \\ &\quad + \mathrm{P}[N(\Delta t) \geq 2]\,\mathrm{E}[\tilde{R}^{(t)}_{\tau_t^\star} | N(\Delta t) \geq 2]. \end{aligned} \tag{2.57}$$

Since $w(t)$ is bounded, $\mathrm{E}[\tilde{R}^{(t)}_{\tau_t^\star} | N(\Delta t) \geq k]$ must also be bounded for all $k \geq 0$. Hence, for Δt sufficiently small, (2.57) becomes

$$w(t) = (1 - \Delta t)\,\mathrm{E}[\tilde{R}^{(t)}_{\tau_t^\star} | N(\Delta t) = 0] + \Delta t\,\mathrm{E}[\tilde{R}^{(t)}_{\tau_t^\star} | N(\Delta t) = 1] + o(\Delta t). \tag{2.58}$$

Using Lemma 2.10 we obtain, after straightforward manipulations of (2.58),

$$\frac{w(t) - w(t - \Delta t)}{\Delta t} - \frac{\chi(t, \Delta t)}{\Delta t} = -w(t - \Delta t) + \mathrm{E}[\tilde{R}^{(t)}_{\tau_t^\star} | N(\Delta t) = 1] + \frac{o(\Delta t)}{\Delta t}.$$

Let Δt go to zero on both sides of this equation. The continuity of $w(t)$ and Lemma 2.9 guarantee the existence of the limit of the rhs for $\Delta t \to 0^+$, and thus the limit of the lhs must also exist. Now recall the final arguments in the proof of Proposition 2.2. We saw that $w(t)$ is uniformly continuous on $\mathbb{R}^+$, and that it satisfied a local Lipschitz-condition on $(1, \infty)$. Therefore $w(t)$ must be Lipschitz on any compact subinterval $\bar{I} \subset (1, \infty)$. Hence Rademacher's Theorem (see e.g. Heinonen (2004)) applies, and $w(t)$ must be differentiable almost everywhere on I, i.e. there exists a subset $\Omega \subset I$ of Lebesgue measure 1 such that for all $t \in \Omega$, the limit $\lim_{\Delta t \to 0^+} (w(t) - w(t - \Delta t))/\Delta t$ exists and is finite. Consequently the limit $\lim_{\Delta t \to 0^+} \chi(t, \Delta t)/\Delta t$ must also exist for all $t \in \Omega$. Now recall that

$$w(t) \geq \Pi(t)\mathrm{P}[T_{\tau_t^\star} > t].$$

Since $w(t)$ is bounded and $\Pi(t)$ is increasing and unbounded, this implies that $\mathrm{P}[T_{\tau_t^\star} > t]$ must go to zero faster than t goes to infinity. Hence, from equation (2.56), the function $\chi(t) := \lim_{\Delta t \to 0^+} \chi(t, \Delta t)/\Delta t$ satisfies $\lim_{t \to \infty} t\chi(t) = 0$. ∎

Remark 11 *Note that if we were able to prove that*

$$\lim_{\Delta t \to 0} \mathrm{P}[T_{\tau^\star_{t-\Delta t}} > t - \Delta t] = \mathrm{P}[T_{\tau_t^\star} > t], \tag{2.59}$$

then we would not need the local Lipschitzness of $w(t)$ to obtain a differential equation. Indeed, we could then use this result to prove that $w(t)$ is a differentiable function, and that it satisfies (2.45) everywhere on $\mathbb{R}^+$. Unfortunately, we do not know the distribution of the optimal stopping time, and hence (2.59) remains an open question.

Properties of $w(t \mid x)$

If $w(t \mid x)$ were known, then equation (2.45) would be solvable, at least numerically, and this solution would yield the value function for the Poisson embedded problem. This defines a new secondary aim into Robbins' problem, namely to estimate the difference between $w(t \mid x)$ and $w(t)$ sufficiently precisely in order to be able to use (2.45) to obtain estimates on $w(t)$. This problem turns out to share the same difficulties as Robbins' problem itself. We are, however, able to give some rough estimates on $w(t \mid x)$.

Proposition 2.11 *For all* $x \in [0, 1]$,

$$w(t) \leq w(t|x) \leq w(t) + 1 - e^{-(1-x)t}$$

Proof: The first inequality is evident. To show the second one, we need to express $w(t \mid x)$ in terms of $w(t)$. This is done in the following way.

$$\begin{aligned} w(t|x) &= w(t|x) + \inf_{\tau \in \mathcal{T}_t} \{-\mathrm{P}[X_\tau \geq x, T_\tau < t]\} - \inf_{\tau \in \mathcal{T}_t} \{-\mathrm{P}[X_\tau \geq x, T_\tau < t]\} \\ &\leq w(t) + \sup_{\tau \in \mathcal{T}_t} \{\mathrm{P}[X_\tau \geq x, T_\tau < t]\}. \end{aligned}$$

Now let $\tau^\star = \inf_i \{X_i > x\}$. This is a stopping time which stops on the first arrival, if any, over x. It is clear that $\tau^\star$ yields the supremum appearing in the previous inequality, therefore

$$\begin{aligned} w(t|x) &\leq w(t) + \mathrm{P}[X_{\tau^\star} \geq x, T_{\tau^\star} < t] \\ &\leq w(t) + \mathrm{P}[T_{\tau^\star} < t] \\ &= w(t) + 1 - \mathrm{P}[\text{ there is no arrival in } [0,t] \times [x,1]] \\ &= w(t) + 1 - e^{-(1-x)t} \end{aligned}$$

■

We define the difference function

$$h(t, x) = w(t \mid x) - w(t).$$

For each $x \in [0, 1]$, $h(t, x)$ is the difference between two continuous functions and thus is continuous in t. Moreover, this function is decreasing in x and satisfies

$$0 \leq h(t, x) \leq 1 - e^{-(1-x)t}.$$

Since estimates on $h(t, x)$ yield estimates on $w(t \mid x)$, it is natural to consider $h(t, x)$ for specific strategies. In this spirit, for every strategy $\tau \in \mathcal{T}$, we define the function

$$h_\tau(t, x) = \mathrm{P}[X_\tau > x, T_\tau < t].$$

This function has some interesting properties.

Proposition 2.12 *Let $g_t(.)$ be a threshold function and let τ be the corresponding memoryless threshold rule. Let $h_\tau(t,x)$ be defined by $h_\tau(t,x) = \mathrm{P}[X_\tau > x, T_\tau < t]$. Then*

$$h_\tau(t,x) = \begin{cases} 1 - e^{-\mu_t(t)} - x\displaystyle\int_0^t e^{-\mu_t(s)}ds & 0 \le x \le g_t(0) \\ \\ 1 - e^{-\mu_t(t)} - x\displaystyle\int_{g_t^{-1}(0)}^t e^{-\mu_t(s)}ds & g_t(0) \le x \le 1 \end{cases}$$

Moreover this functions satisfies

$$\begin{array}{ll} h_\tau(t,x) = 1 - e^{-\mu(t)} - x\mathrm{E}[T_\tau] & \text{if } 0 \le x \le g_t(0) \\ h_\tau(t,x) > 1 - e^{-\mu(t)} - x\mathrm{E}[T_\tau] & \text{if } g_t(0) < x \le 1 \end{array}$$

Proof: From the definition of τ we know that, conditionally to $T_\tau = s \in [0,t)$, X_τ is distributed uniformly on $[0, g_t(s)]$. Hence, using the density of T_τ which is given by (2.35), we see that if $0 \le x \le g_t(0)$, then

$$\begin{aligned} \mathrm{P}[X_\tau > x, T_\tau < t] &= \int_0^t \frac{g_t(s) - x}{g_t(s)} f_{T_\tau}(s)ds \\ &= 1 - e^{-\mu(t)} - x\int_0^t e^{-\mu_t(s)}ds \\ &= 1 - e^{-\mu(t)} - x\mathrm{E}[T_\tau]. \end{aligned}$$

Likewise, if $g_t(0) \le x \le 1$, then we see from the definition of a threshold rule that $\mathrm{P}[X_\tau > x \mid T_\tau = s]$ will be identically nil for all $s \in [0, g_t^{-1}(x)]$. Therefore

$$\begin{aligned} \mathrm{P}[X_\tau > x, T_\tau < t] &= \int_0^{g_t^{-1}(x)} f_{T_\tau}(s)ds + \int_{g_t^{-1}(x)}^t \frac{g_t(s) - x}{g_t(s)} f_{T_\tau}(s)ds \\ &= 1 - e^{-\mu_t(t)} - x\int_{g_t^{-1}(0)}^t e^{-\mu_t(s)}ds \\ &\ge 1 - e^{-\mu_t(t)} - x\int_0^t e^{-\mu_t(s)}ds. \end{aligned}$$

This yields the result. ∎

Estimates on the limiting value

The function $\chi(t)$ is a nuisance parameter of equation (2.45). However, it is uniformly bounded by an $o(1/t)$, and will not play any role asymptotically.

Now choose some constant $c > 1$. Then for all $s > 0$ in which the differential equation is satisfied, we can write

$$\begin{aligned} w'(s) + w(s) &\leq \int_0^{\frac{c}{c+s}} (1 + xs)\mathrm{dx} + \int_{\frac{c}{c+s}}^1 (w(s) + h(s,x))\mathrm{dx} + \chi(s) \\ &\leq w(s) - \tfrac{c}{c+s} w(s) + H(s,c) \end{aligned}$$

where

$$H(s,c) = \int_0^{\frac{c}{c+s}} (1 + xs)\mathrm{dx} + \int_{\frac{c}{c+s}}^1 h(s,x)\mathrm{dx} + \chi(s). \tag{2.60}$$

Hence

$$w'(s) + \frac{c}{s+c} w(s) \leq H(s,c), \tag{2.61}$$

Multiplying both sides of (2.61) by $(c+s)^c$, we get

$$((c+s)^c w(s))' \leq (c+s)^c H(s,c),$$

which after integration yields

$$w(t) \leq (c+t)^{-c} \int_0^t (c+s)^c H(s,c)\mathrm{ds}. \tag{2.62}$$

Example 2
1. We saw that $h(t,x) \leq 1 - e^{-(1-x)t}$. Applying (2.62) to $1 - e^{-(1-x)t}$ and taking the limit of this expression for $t \to \infty$ yields the trivial upper bound $w \leq \infty$. In fact, one can show that any upper bound on $h(t,x)$ which is not asymptotically equivalent to zero will always yield from (2.62) a trivial upper bound on $w(t)$.
2. If τ is the memoryless threshold strategy defined by the threshold function $g_t(s) = \frac{c}{t-s+c}$ (see Example 1), then an explicit computation of $h_\tau(t,x) = \mathrm{P}[X_{N(\tau)} > x, T_\tau < t]$ yields

$$h_\tau(t,x) = \begin{cases} 1 + g_t(0)^c(\frac{c}{c+1}x - 1) - \frac{c}{c+1}\frac{x}{g_t(0)} & 0 \leq x \leq g_t(0) \\ \\ g_t(0)^c(\frac{c}{c+1}x - 1) + \frac{1}{c+1}\left(\frac{g_t(0)}{x}\right)^c & g_t(0) \leq x \leq 1 \end{cases} \tag{2.63}$$

Applying (2.62) to $h_\tau(t,x)$, and taking the limit for $t \to \infty$ we see that this yields, as before, the upper bound

$$w \leq 1 + \frac{c}{2} + \frac{1}{c^2 - 1}.$$

Acknowledgments

We are grateful to Freddy Delbaen for his help to shorten the proof of Lemma 2.4, and to Michael Drmota for equation 2.62.

Chapter 3

Comparison of the Classical and the Poisson Embedded Robbins' Problem

3.1 Introduction

Although the Poisson embedded problem is interesting in its own right, we introduced this problem in order to obtain information on the discrete problem. In this chapter we will show that the Poisson model yields upper bounds on the discrete problem. We will also see that we were only able obtain a one sided comparison of the two problems (namely $v \leq w$) and that our proof of the inequality in the other direction requires information on the optimal discrete rule for which we have found no justification.

3.2 Upper and lower bounds

Proposition 3.1 *For all $\epsilon > 0$ there exists $t^\star > 0$ such that for all $t \geq t^\star$,*

$$w(t) > v - \epsilon.$$

Proof: Fix $\epsilon > 0$. Let us consider the problem with horizon t, but assume that the optimal stopper (say Q) is told in advance the number of arrivals which will occur in $[0, t]$. Let $w_Q(t)$ be the expected optimal value for Q. Clearly he can only do better than a decision maker without information, and thus $w_Q(t) \leq w(t)$. To be precise, let $\tilde{\mathcal{F}}_s$ be the enlarged σ-algebra generated by $N(u)$, $X_{N(u)}, T_{N(u)}$, $0 \leq u \leq s$ and $N(t)$ together and let $\tilde{\mathcal{T}}$ be the set of stopping times adapted to $(\tilde{\mathcal{F}}_s)_{0 \leq s \leq t}$. Then, by construction, Q is taking the infimum over $\tilde{\mathcal{T}} \supset \mathcal{T}$ and thus

$$w(t) = \inf_{\tau \in \mathcal{T}} \mathrm{E}[\tilde{R}^{(t)}_\tau] \geq w_Q(t) = \inf_{\sigma \in \tilde{\mathcal{T}}_t} \mathrm{E}[\tilde{R}^{(t)}_\sigma]. \tag{3.1}$$

Conditioning on $N(t)$, we get

$$\inf_{\sigma\in\tilde{\mathcal{T}}} \mathrm{E}[\tilde{R}_\sigma^{(t)}] \geq \sum_{k=0}^{\infty} \mathrm{P}[N(t)=k] \inf_{\sigma\in\tilde{\mathcal{T}}} \mathrm{E}[\tilde{R}_\sigma^{(t)} \mid N(t)=k]. \tag{3.2}$$

Conditionally to $\{N(t)=k\}$, Q will be told that there are exactly k arrivals. Hence, if the penalty function is greater than the expected rank of the last arrival, then the best he can do is apply the discrete optimal strategy σ_k (i.e. the strategy which is optimal for exactly k arrivals), and the value he will obtain will be equal to the discrete optimal value for k arrivals, $v(k)$. Note that if $k \geq \Pi(t)$ this is no longer true. From (3.1) and (3.2) this yields

$$w(t) \geq \sum_{k=0}^{\lfloor \Pi(t) \rfloor} \mathrm{P}[N(t)=k] v(k), \tag{3.3}$$

and this inequality holds for all $t>0$.

Since $v(k)$ increases towards v, we know that there exists $m_0 \in \mathbb{N}$ such that $v(m) > v-\epsilon$ for all $m \geq m_0$. Let $t_0 = \inf\{t \in \mathbb{R}$ such that $\Pi(t) > m_0\}$ (since $\Pi(t)$ is strictly increasing, t_0 is uniquely defined). We have therefore shown that for all $t \geq t_0$,

$$w(t) \geq (v-\epsilon) \sum_{k=m_0}^{\lfloor \Pi(t) \rfloor} \mathrm{P}[N(t)=k],$$

or, equivalently

$$w(t) \geq (v-\epsilon)\left(\mathrm{P}[N(t)\geq m_0] - \mathrm{P}[N(t) > \lfloor \Pi(t) \rfloor]\right). \tag{3.4}$$

Now, since $(N(t))_{t\geq 0}$ is a Poisson process of constant positive rate, we know that if $\Pi(t) \sim t + t^\gamma$ for some $\gamma \in (0,1)$, then $\mathrm{P}[N(t) \geq \Pi(t)] \to 0$ as t goes to ∞. Hence there exists t_1 such that for all $t \geq t_1$, $\mathrm{P}[N(t)\geq \Pi(t)] < \epsilon$. From (3.4) this shows that for all $t \geq \max\{t_0, t_1\}$,

$$w(t) \geq (v-\epsilon)\left(\mathrm{P}[N(t)\geq m_0] - \epsilon\right).$$

Also, m_0 is fixed with respect to t (it only depends on ϵ). Therefore we know that there must exist t_2 such that for all $t \geq t_2$, $\mathrm{P}[N(t) \geq m_0] > 1-\epsilon$. Therefore, for all $t \geq \max\{t_0, t_1, t_2\}$ we know that

$$w(t) \geq (v-\epsilon)(1-2\epsilon) = v - 2v\epsilon - \epsilon + \epsilon^2.$$

Since $v \leq 3$, we have shown that for all $\epsilon > 0$ there exists $t^\star \in \mathbb{R}$ such that for all $t \geq t^\star$, $w(t) \geq v - 7\epsilon$. The choice of ϵ being arbitrary, this yields the result. ∎

The arguments in this proof also hold if we restrict our attention to the set of memoryless strategies $\mathcal{M}_t$. Hence Proposition 3.1 holds for the asymptotic memoryless values W and V, and we obtain the following

Proposition 3.2 *Let W be the minimal expected rank obtainable through memoryless thresholds in the Poisson embedded Robbins' Problem, and let V be its discrete counterpart. Then there exists $t^\star \in \mathbb{R}$ such that for all $t \geq t^\star$,*

$$W(t) > V - \epsilon.$$

These results do not prove that the limits w and W exist. They only show that, by choosing the penalty function to be sufficiently large in order to ensure that it is (nearly) never optimal to refuse all observations, the Poisson embedded problem yields upper bounds on v and V. What we would need to prove the existence of w and W is an inequality of the same form but in the other direction. As we have mentioned in the introduction to this chapter, we will only be able to obtain this up to a certain point.

We first need a technical lemma on the estimates of the tail probabilities for Poisson processes. We have not found this result in the literature, and hence we include its proof.

Lemma 3.3 *Let $N(n)$ be the number of arrivals of a Poisson process of rate 1 on $[0, n] \times [0, 1]$. Let α be some scalar such that $\frac{1}{2} < \alpha < \frac{2}{3}$. Then*

$$n\mathrm{P}[N(n) < n - n^\alpha] \to 0, \ \textit{as n goes to infinity.} \tag{3.5}$$

Proof: $N(n)$ is a Poisson random variable of mean and variance n so that, by the central limit theorem, $(N(n) - n)/\sqrt{n}$ converges in law to a standard normal distribution $\mathcal{N}(0, 1)$. By choosing α between $\frac{1}{2}$ and $\frac{2}{3}$ we ensure that $n^{\alpha - \frac{1}{2}}$ increases to ∞ as $n \to \infty$ and that $(n^{\alpha-\frac{1}{2}})^3/\sqrt{n} = n^{3\alpha - 2}$ decreases to 0 as $n \to \infty$. Therefore we can apply a theorem on normal approximation (see Feller (1968), p.193) to get

$$\mathrm{P}\left[\frac{N(n) - n}{\sqrt{n}} > n^{\alpha - \frac{1}{2}}\right] \sim \frac{1}{\sqrt{2\pi}} \frac{1}{n^{\alpha - \frac{1}{2}}} e^{-\frac{1}{2}(n^{\alpha - \frac{1}{2}})^2}. \tag{3.6}$$

For n sufficiently large we can use the approximate symmetry of the distribution of $(N(n) - n)/\sqrt{n}$ to obtain from (3.6)

$$\mathrm{P}\left[N(n) < n - n^\alpha\right] = \mathrm{P}\left[\frac{N(n) - n}{\sqrt{n}} < -n^{\alpha - \frac{1}{2}}\right] \sim \frac{1}{\sqrt{2\pi}} \frac{1}{n^{\alpha - \frac{1}{2}}} e^{-\frac{1}{2}(n^{\alpha - \frac{1}{2}})^2}.$$

Hence altogether

$$n\mathrm{P}\left[N(n) < n - n^\alpha\right] \sim \frac{1}{\sqrt{2\pi}} n^{\frac{3}{2} - \alpha} e^{-\frac{1}{2}(n^{\alpha - \frac{1}{2}})^2} < \frac{1}{\sqrt{2\pi}} \frac{n}{e^{\frac{1}{2}n^{2\alpha - 1}}}$$

which tends to 0 as n tends to infinity.

∎

Proposition 3.4 *Let $\beta_n = \lfloor n - n^\alpha \rfloor$ (where $\lfloor x \rfloor$ denotes the floor of x) for $\alpha \in (0,1)$, and let $\tau^\star_{\beta_n} \in T_{\beta_n}$ be the optimal strategy for the discrete problem with β_n arrivals. For all $\alpha \in (\frac{1}{2}, \frac{2}{3})$, and for all $\epsilon > 0$,*

$$w(n) \leq v(\beta_n) + \mathrm{E}[X_{\tau^\star_{\beta_n}}]\left(n^\alpha + 1 + \frac{\epsilon}{2}\right) \tag{3.7}$$

for all n sufficiently large.

Proof: A strategy for the discrete case defines a (suboptimal) strategy for the continuous case, so that we can consider $\tau^\star_{\beta_n} \in T_n$ as a strategy acting on $[0,n]$. Let $\sigma_{\beta_n} \in \mathcal{T}_n$ be this strategy. Now let $\tilde{w}(\beta_n)$ be the value obtained by applying the β_n-optimal strategy σ_{β_n} on the continuous time interval $[0,n]$, i.e. $\tilde{w}(\beta_n)$ is the value of a strategy which is optimal if and only if there are exactly β_n arrivals on $[0,n]$. Also let $\tilde{w}(\beta_n|E)$ denote the expected loss under a β_n-optimal strategy conditioned on the event E. Since σ_{β_n} is suboptimal, this yields

$$w(n) \leq \tilde{w}(\beta_n), \tag{3.8}$$

with

$$\tilde{w}(\beta_n) = \tilde{w}(\beta_n|N(n) < \beta_n)\mathrm{P}[N(n) < \beta_n] + \tilde{w}(\beta_n|N(n) \geq \beta_n)\mathrm{P}[N(n) \geq \beta_n]. \tag{3.9}$$

Now if $N(n) < \beta_n$, then there is a positive probability that the β_n-optimal strategy σ_{β_n} does not stop on any arrival within the given time, and that the player loses the penalty. Hence the only immediate upper bound we can obtain on $\tilde{w}(\beta_n|N(n) < \beta_n)$ is given by

$$\tilde{w}(\beta_n|N(n) < \beta_n) \leq \Pi(n).$$

From Lemma 3.3 we see that $\Pi(n)\mathrm{P}[N(n) < \beta_n] \to 0$ as n goes to infinity, (even when $\Pi(n) \sim n + n^\gamma$ with $\gamma \in [0,1]$) so that for n sufficiently large,

$$\tilde{w}(\beta_n|N(n) < \beta_n)\mathrm{P}[N(n) < \beta_n] < \frac{\epsilon}{2}. \tag{3.10}$$

On the other hand, suppose that $N(n) \geq \beta_n$. Then

$$\tilde{w}(\beta_n|N(n) \geq \beta_n) = v(\beta_n) + \mathrm{E}[X_{\sigma_{\beta_n}}(N(n) - \beta_n)|N(n) \geq \beta_n]. \tag{3.11}$$

Since the β_n-optimal strategy stops almost surely not later than the β_nth arrival, we see that, given $N(n) \geq \beta_n$, $X_{\sigma_{\beta_n}}$ (i.e. the expected value of an arrival selected by use of a strategy which is optimal in the discrete β_n arrival case) is independent of $X_{\beta_n+1}, X_{\beta_n+2}, \ldots$, so that

$$\mathrm{E}[X_{\sigma_{\beta_n}} \mid N(n) \geq \beta_n] = \mathrm{E}[X_{\sigma_{\beta_n}}],$$

and since σ_{β_n} (which acts in continuous time) is equivalent to the discrete β_n-optimal strategy $\tau^\star_{\beta_n}$, the right hand side of (3.9) becomes

$$v(\beta_n) + \mathrm{E}[X_{\tau^\star_{\beta_n}}](\mathrm{E}[N(n)|N(n) \geq \beta_n] - \beta_n). \tag{3.12}$$

Furthermore,

$$\mathrm{E}[N(n)|N(n) \geq \beta_n] = \sum_{k=\beta_n}^{\infty} k \frac{\mathrm{P}[N(n) = k]}{\mathrm{P}[N(n) \geq \beta_n]} \leq \frac{\mathrm{E}[N(n)]}{1 - \mathrm{P}[N(n) < \beta_n]}. \tag{3.13}$$

Now we know from Lemma 3.3 that $\mathrm{P}[N(n) < \beta_n]$ is small for large values of n. Hence, since $\frac{1}{1-\delta} < 1 + 2\delta$ for sufficiently small values of $\delta > 0$, (3.13) yields

$$\mathrm{E}[N(n)|N(n) \geq \beta_n] \leq \mathrm{E}[N(n)](1 + 2\mathrm{P}[N(n) < \beta_n]) = n + 2n\mathrm{P}[N(n) < \beta_n].$$

Applying Lemma 3.3 to this last equation, we see that for all n sufficiently large,

$$\mathrm{E}[N(n)|N(n) \geq \beta_n] < n + \frac{\epsilon}{2}, \tag{3.14}$$

and hence (3.11) and (3.12) yield

$$\tilde{w}(\beta_n|N(n) \geq \beta_n) \leq v(\beta_n) + \mathrm{E}[X_{\tau^\star_{\beta_n}}](n - \beta_n + \frac{\epsilon}{2}). \tag{3.15}$$

Combining (3.8), (3.10) and (3.15) we obtain

$$w(n) \leq \tilde{w}(\beta_n) \leq v(\beta_n) + \mathrm{E}[X_{\tau^\star_{\beta_n}}](n - \beta_n + \frac{\epsilon}{2}).$$

Since $\beta_n = \lfloor n - n^\alpha \rfloor$, we see that $n - \beta_n + \frac{\epsilon}{2} < n^\alpha + 1 + \frac{\epsilon}{2}$. These final remarks conclude the proof. ∎

The arguments in Lemma 3.4 also hold if we restrict our attention to the memoryless strategies in M_t and $\mathcal{M}_t$, Hence we obtain the following corollary.

Corollary 3.5 *Take β_n as above and let $\tau^\star_{\beta_n} \in M_{\beta_n}$ be the optimal memoryless threshold strategy for β_n arrivals. If $\alpha \in (\frac{1}{2}, \frac{2}{3})$, then, for all $\epsilon > 0$, there exists n sufficiently large to ensure that*

$$W(n) \leq V(\beta_n) + \mathrm{E}[X_{\tau^\star_{\beta_n}}]\left(n^\alpha + 1 + \frac{\epsilon}{2}\right). \tag{3.16}$$

3.3 A conjecture

Unfortunately, Lemma 3.4 is not enough to guarantee $v = w$ (or $V = W$). It is clear that if one could prove the existence of a constant c such that

$$\mathrm{E}[nX_{\tau^\star_n}] \leq c, \tag{3.17}$$

then one would immediately obtain from (3.7) that the asymptotic values w and W exist for the Poisson embedded problem and that they are equal to v and V, respectively. Now equation (3.17) is a statement that seems intuitively obvious and which was even taken for granted by Assaf and Samuel-Cahn (1996). We know that $\mathrm{E}[R_{\tau_n^\star}]$ is bounded and hence, in particular, $\mathrm{E}[(n-\tau_n^\star)X_{\tau_n^\star}]$ is bounded. Moreover, we know that the optimal strategy $\hat{\tau}_n$ for Moser's problem yields an asymptotic value

$$\lim_{n\to\infty} \mathrm{E}[nX_{\hat{\tau}_n}] = 2,$$

and that the correlation between a value and its rank goes to 1. Hence it seems clear that the optimal strategy for Robbins' problem should not do much worse with respect to the expected value problem than the optimal strategy for Moser's problem.

In the restricted problem, we even have more information. Indeed, recall from Chapter 1 that

$$V = \liminf U_n(\tau_n^\star)$$

where $U_n(\tau_n) = (2\mathrm{E}[nX_{\tau_n}](1+\mathrm{E}[\tau_n/n]))^{\frac{1}{2}}$, and $\tau_n^\star$ is the optimal memoryless threshold rule. Since V is bounded, we know that there exists a constant c for which the optimal memoryless threshold strategy satisfies $\liminf n\mathrm{E}[X_{\tau_n^\star}] \leq c$. From (3.16) this implies that

$$\liminf_{n\to\infty} W(n) \leq V.$$

Unfortunately these intuitions do not tell us anything definite about the value of the limit.

We have put a great deal of thought and effort into this question, and yet we were unable to close this gap. We therefore state our conclusions as conjectures.

Conjecture 1

Denote by $\tau_n^\star$ the optimal strategy for Robbins' Problem in the discrete n-arrival case, i.e. the strategy which yields the value

$$v(n) = \mathrm{E}[R_{\tau_n^\star}].$$

Then there exists a real constant c such that

$$\mathrm{E}[X_{\tau_n^\star}] < \frac{c}{n} \tag{3.18}$$

for all n sufficiently large. Hence the limiting value w exists, and satisfies

$$v = w.$$

Conjecture 2

Denote by $\sigma_n^\star$ the optimal memoryless strategy for Robbins' Problem in the discrete n-arrival case, i.e. the strategy which yields the value

$$V(n) = \mathrm{E}[R_{\sigma_n^\star}].$$

Then there exists a real constant c such that

$$\mathrm{E}[X_{\sigma_n^\star}] < \frac{c}{n} \tag{3.19}$$

for all n sufficiently large. Hence the limiting value W exists, and satisfies

$$V = W.$$

Remark 12 *Proving equation (3.17) for the memoryless threshold strategies would also prove a conjecture of Assaf and Samuel-Cahn (1996) on the optimal memoryless value $V(n)$, namely that there exists a constant C such that $n(V(n+1) - V(n)) \leq C$. To see the link between these two properties, it suffices to let $\tilde{\tau}_{n+1}$ be the rule for the $(n+1)$-size problem which uses the n-optimal memoryless strategy $\tau_n^\star$ on the observations $X_1, X_2, \ldots, X_n$ and ignores the last arrival. If we let $0 \leq a_1 \leq \ldots \leq a_{n-1} < a_n = 1$ denote the thresholds corresponding to $\tau_n^\star$, we see that $\tilde{\tau}_{n+1}$ is a memoryless threshold rule with thresholds $0 \leq b_1 \leq \ldots \leq b_{n-1} \leq b_n \leq b_{n+1} = 1$ where $b_i = a_i$ for all $i = 1, \ldots, n$. Since $\tilde{\tau}_{n+1}$ stops almost surely before the nth arrival, we have $X_{\tilde{\tau}_{n+1}} = X_{\tau_n^\star}$ and thus,*

$$V(\tilde{\tau}_{n+1}) = V(n) + \mathrm{P}[X_{\tau_n^\star} \geq X_{n+1}] = V(n) + \mathrm{E}[X_{\tau_n^\star}].$$

Since τ_{n+1} is not optimal, we know that $V(n+1) \leq V(\tilde{\tau}_{n+1})$ and hence

$$V(n+1) - V(n) \leq \mathrm{E}[X_{\tau_n^\star}] \tag{3.20}$$

From (3.20) we deduce that the unboundedness of $n(V(n+1) - V(n))$ would imply that of $\mathrm{E}[nX_{\tau_n^\star}]$, and this would yield a contradiction.

Chapter 4

Improving the bounds

This final chapter is a review of some of the research we have done on a number of specific strategies for Robbins' problem. The aim of this work was to determine whether or not the asymptotic optimal value v is strictly smaller than V, the asymptotic optimal value for memoryless thresholds. We have, unfortunately, found no strategy which yielded any significant improvement. We nonetheless give a brief overview of the ideas behind this work. We illustrate some results with numerical results. We do not go into any detail and explicit computations are omitted.

We first examine a number of strategies which do not yield any information, and we explain the reasons for this. We then study a class of strategies which yield an improvement on the known upper bounds. However, this improvement is very small and does not allow us to conclude.

4.1 Strategies with training periods

Consider the Poisson embedded Robbins' problem with fixed horizon $t > 0$. We first study strategies which are similar to the optimal strategies for the Best Choice problem (see on page 51). For this take $0 \leq t_1^\star < t$ and consider strategies of the form

$$\tau_{t_1^\star} = \inf\{k \geq 1 \text{ such that } T_k \geq t_1^\star \text{ and } X_k \text{ is of relative rank one}\}. \tag{4.1}$$

We will call the time interval $[0, t_1^\star]$ (during which no action is taken) the training period of these strategies. It serves to define a threshold depending on the history of the process.

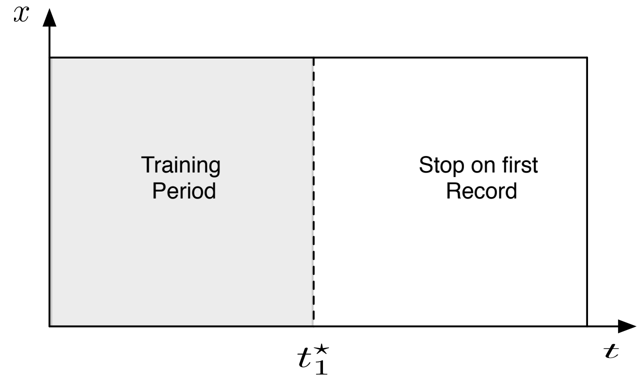

Figure 1: *Strategies with training period*

Now let τ be one such strategy. By conditioning on the number of arrivals in $[0, t_1^\star]$ we see that

$$\mathrm{P}[T_\tau > t] = \sum_{k=0}^{\infty} \mathrm{P}[T_\tau > t \mid N(t_1^\star) = k] e^{-t_1^\star} \frac{t_1^{\star k}}{k!}.$$

If $k = 0$, then there have been no arrivals during the training period, and τ will stop on the first arrival after $t_1^\star$. Hence

$$\mathrm{P}[T_\tau > t \mid N(t_1^\star) = 0] = e^{-(t-t_1^\star)}.$$

Secondly, if there have been $k \geq 1$ arrivals in $[0, t_1^\star]$, then let $X_{(1)} = \min\{X_1, \ldots, X_k\}$. If $X_{(1)} = y \in (0,1)$, then τ must stop on the first arrival after $t_1^\star$ which is smaller than y, and therefore

$$\mathrm{P}[T_\tau > t \mid X_{(1)} = y, N(t_1^\star) = k] = e^{-(t-t_1^\star)y}.$$

Hence, by conditioning on the value of $X_{(1)}$, we see that

$$\mathrm{P}[T_\tau > t] \;\; = \;\; e^{-t} + \sum_{k=1}^{\infty} \left[e^{-t_1^\star} \frac{t_1^{\star k}}{k!} \int_0^1 e^{-(t-t_1^\star)y} k (1-y)^{k-1} dy \right].$$

We can interchange the summation and the integral and, after straightforward computations, we obtain

$$\mathrm{P}[T_\tau > t] = e^{-t} + \frac{t_1^\star}{t} \left(1 - e^{-t}\right).$$

This yields

$$w(\tau) \geq \Pi(t) \mathrm{P}[T_\tau > t] = \Pi(t) \left(e^{-t} + \frac{t_1^\star}{t} \left(1 - e^{-t}\right) \right). \tag{4.2}$$

Now recall that the penalty function is linear in t (at least asymptotically). Let $\alpha : \mathbb{R} \to [0,1]$ be the proportionality coefficient of $t_1^\star$ with respect to t, i.e. the function defined by $\alpha(t) = t_1^\star / t$. With this notation we see from (4.2) that unless the proportionality coefficient $\alpha(t)$ is in the order of $1/t$, the value of strategies defined by (4.1) will not be bounded in

t. However, if $\alpha(t)$ is in the order of $1/t$ then $t_1^\star$ is negligible with respect to t when t is large and thus the training period is also negligible in comparison to the time horizon t. Hence strategy will stop on an arrival whose expected rank is asymptotically linear in t. Therefore, the value of a strategy τ defined by equation (4.1) cannot be bounded.

With this in mind, one could think of modifying the strategies defined by (4.1) by making them less demanding as one approaches the time horizon t. We considered strategies defined by a training period $[0, t_1^\star]$ in which no action is taken, a record period $[t_1^\star, t_2^\star]$ during which the strategy stops on the first record and then a 'free' period $[t_2^\star, t]$ during which the strategy stops unconditionally on the first arrival.

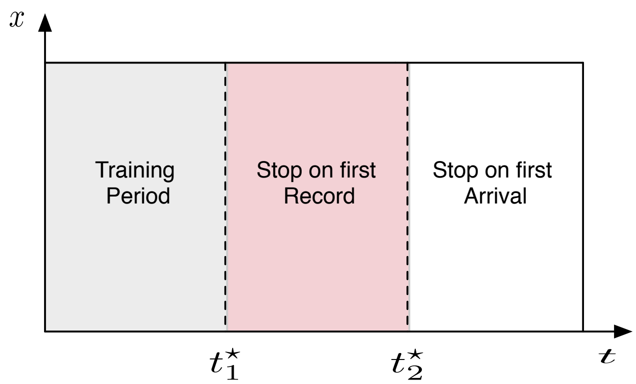

Figure 2: *Strategies with training period and free period*

Accepting arrivals unconditionally in $[t_2^\star, t]$ serves to avoid obtaining the penalty, and hence avoid the drawback of the strategies defined by (4.1). However, arguments nearly identical to those given above show that such strategies yield no improvement on the simple strategies with training period. Moreover, whatever the choice of $t_1^\star$ and $t_2^\star$, the expected loss obtained with these strategies will not be asymptotically bounded.

One could think of relaxing the conditions of (4.1) by partitioning the horizon $[0, t]$ into k intervals $[0, t_1^\star] \cup [t_1^\star, t_2^\star] \cup \ldots \cup [t_k^\star, t]$ and considering strategies which stop on the first record in $[t_1^\star, t_2^\star]$, the second best arrival in $[t_2^\star, t_3^\star]$, etc., and the kth best arrival in $[t_k^\star, t]$. From the arguments given above, we do not expect these strategies to fare any better for large t than the strategies we have already described.

A natural improvement on the strategies we have described above is to introduce a threshold which will guarantee that arrivals which have a large value will only be selected at times close to the horizon t. We therefore considered a collection of strategies, which we will call *threshold strategies with training period*. They are defined as follows. Let $g(s)$ be a threshold function (as defined on page 68) and take $t_1^\star, t_2^\star \in \mathbb{R}$ such that $0 \leq t_1^\star \leq t_2^\star \leq t$. To each pair $t_1^\star, t_2^\star$, we associate a strategy τ which stops if possible on the first record in $[t_1^\star, t_2^\star]$, and, if no arrival has been selected by time $t_2^\star$, on the first arrival (X_k, T_k) whose value satisfies $X_k \leq g(T_k)$.

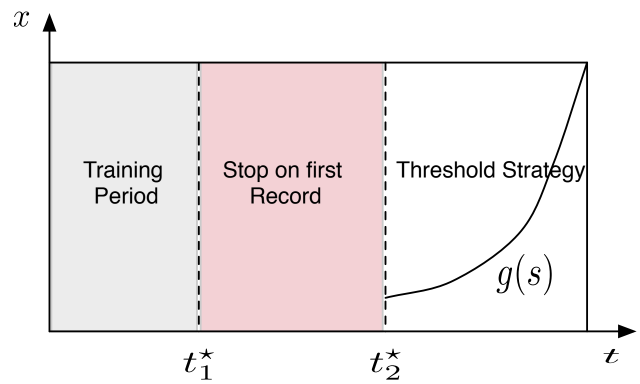

Figure 3: *Threshold strategies with training period*

Computations similar to those we have performed above allow us to obtain the distribution of any such strategy; it is given by

$$\mathrm{P}[T_\tau > s] = \begin{cases} \left(1 - \frac{t_1^\star}{s}\right)(1 - e^{-s}) & \text{if } s \in [t_1^\star, t_2^\star] \\ \\ 1 - e^{-\mu(s)} K(t_1^\star, t_2^\star) & \text{if } s \in [t_2^\star, t], \end{cases}$$

where

$$K(t_1^\star, t_2^\star) = \left(\frac{t_1^\star}{t_2^\star} + e^{-t_2^\star}\left(1 - \frac{t_1^\star}{t_2^\star}\right)\right).$$

The expected rank obtained is hard to evaluate, and makes for long computations which we do not include. These computations indicate that threshold strategies with training period do not yield an improvement on the value obtained through memoryless threshold strategies.

In order to motivate this statement, we use the estimate obtained in Section 2.4. This states

$$w(t) \leq (c+t)^{-c} \int_0^t (c+s)^c H(s,c) \mathrm{ds}, \tag{4.3}$$

where

$$H(s,c) = \int_0^{\frac{c}{c+s}} (1 + xs) \mathrm{dx} + \int_{\frac{c}{c+s}}^1 h(s,x) \mathrm{dx} + \chi(s).$$

Let us study the behavior of the difference function

$$h_\tau(s,x) = \mathrm{P}[X_\tau > x, T_\tau \leq t],$$

and that of

$$\tilde{w}_\tau(t) = (c+t)^{-c} \int_0^t (c+s)^c H_\tau(s,c) \mathrm{ds}$$

for τ a threshold strategy with training period.

A direct computation yields

$$h_\tau(t,x) = I(t,x) + J(t,x), \tag{4.4}$$

where

$$\begin{aligned} I(t,x) &= (1-x)\left(e^{-t_1^\star} - e^{-t_2^\star}\right) + \left(e^{-t_1^\star x} - e^{-t_1^\star}\right) \\ &\quad + \frac{t_1^\star}{t_2^\star}\left(e^{-t_2^\star x} - e^{-t_2^\star}\right) - x t_1^\star \int_x^1 \frac{e^{-t_1^\star y} - e^{-t_2^\star y}}{y}, \end{aligned}$$

and

$$J(t,x) = \begin{cases} K(t_1^\star, t_2^\star)\left(1 - e^{-\mu(t)} - x\displaystyle\int_{t_2^\star}^t e^{-\mu(s)} ds\right) & \text{if } 0 \le x < g(t_2^\star) \\ K(t_1^\star, t_2^\star)\left(e^{-\mu(g^{-1}(x))} - e^{-\mu(t)} - x\displaystyle\int_{g^{-1}(x)}^t e^{-\mu(s)} ds\right) & \text{if } g(t_2^\star) \le x \le 1. \end{cases}$$

Recall from Example 2 in Chapter 2, that when (4.3) is applied to the difference function computed for memoryless threshold strategies σ defined by threshold functions of the form

$$g_t(s) = \frac{c}{t-s+c},$$

then $\tilde{w}_\sigma(t)$ has a finite limit which is minimal at $c \approx 1.96$. This yields the known upper bound $w \le 2.33183$ (see equation (2.63)).

We choose to illustrate (see Figure 4) the difference functions $h_\tau(.,.)$ for threshold strategy with training period with this specific threshold function.

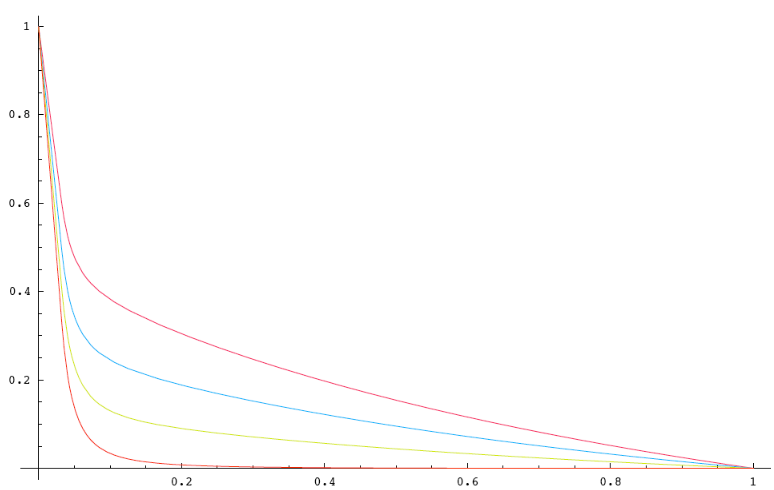

Figure 4: *Plots of the functions $h_\tau(t,x)$ for $x \in (0,1)$ computed at $t = 60$, and for $c = 2$ with, from top to bottom, $t_1^\star = 1,\ t_2^\star = 2,\ t_1^\star = 1,\ t_2^\star = 1.5$ and $t_1^\star = 1,\ t_2^\star = 1.2$. The lowest*

curve shows the difference function (2.63) computed for standard memoryless threshold strategies.

Our results indicate the following: as $t_1^\star$ and $t_2^\star$ approach zero, the difference function (4.4) of threshold strategy with training period converges to that of standard memoryless threshold rules. Minimizing the expression we obtained for $\tilde{w}_\tau(t)$ with respect to $t_1^\star$ and $t_2^\star$, we obtained that the optimal choice of $t_1^\star$ and $t_2^\star$ is given by $t_1^\star = t_2^\star = 0$, in which case $\lim_{t\to\infty} \tilde{w}_\tau(t)$ yields the known bound given above.

4.2 Mixed strategies

We know that memoryless threshold strategies perform well for Robbins' problem. Now we infer from the preceding discussion that selecting records without filtering is too coarse an approach to obtain better limiting values. This means that for a policy to yield a strict improvement on memoryless strategies, it must encompass the apparent optimality of memoryless thresholds with the information given by the relative ranks.

With this in mind, we first thought of modifying the standard threshold strategies by adding the additional requirement that for stopping at any time, the arrival must not only be under the threshold but also a record (i.e. of relative rank 1). These strategies are again too restrictive and do not yield an improvement on the standard memoryless threshold rules. We thus decided to study a whole new class of strategies, which we refer to as *k-legged mixed strategies*.

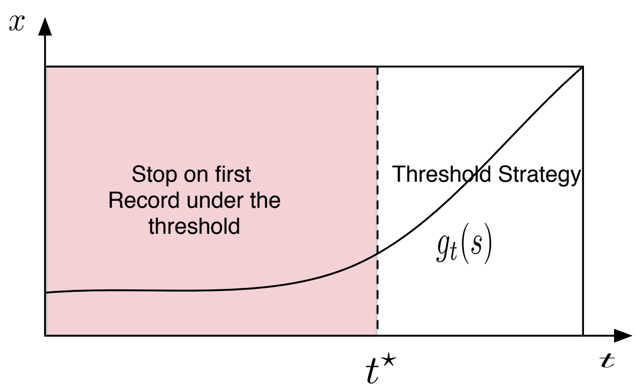

Figure 5: *One-legged mixed strategy*

Definition 4.1 *Let $g_t(s)$ be a threshold function, and choose $t_0^\star = 0 \leq t_1^\star \leq t_2^\star \leq \ldots \leq t_k^\star < t$. A k-*legged mixed strategy* $\tau \in \mathcal{T}$ is a threshold rule defined by the function $g_t(s)$ with the additional requirement that for stopping on an arrival (T_i, X_i) with $t_l^\star \leq T_i \leq t_{l+1}^\star$, $l = 0, \ldots, k$, its relative rank r_i must satisfy $r_i \leq l+1$.*

When $t_1^\star = \ldots = t_k^\star = 0$, these strategies are equivalent to standard memoryless threshold strategies.

We were unable to find a general expression for the distribution of such stopping times. We have however performed a number of numerical simulations for the cases $k = 1$, and $k = 2$. Interestingly, these simulations indicate that although there appears to be an improvement on the optimal value obtained with memoryless strategies, this improvement is very small. We do not include these computations.

As an illustration of the methods we applied, we obtain the distribution of T_τ for a one-legged mixed strategy τ. For this, fix $t^\star \in (0, t)$ and consider the corresponding one-legged mixed strategy τ.

Remark 13 *Let $g_t(.)$ be a threshold function. For convenience, we introduce the following notation. We will call record any arrival of relative rank one, and* 'record' *(with quotation marks) any arrival (T_k, X_k) with $0 \le T_k \le t^\star$ which is of relative rank one and which satisfies $X_k \le g(T_k)$.*

For $s \ge t^\star$, the distribution of T_τ is easy to obtain. It is given by

$$F_\tau(s) = 1 - (1 - F_\tau(t^\star)) \left(\frac{g_t(t^\star)}{g_t(s)} \right)^c \tag{4.5}$$

where $F_\tau(t^\star) = \mathrm{P}[T_\tau \le t^\star]$. Hence we need $\mathrm{P}[T_\tau > t]$ for $s \in [0, t^\star]$. The difficulty here lies in the fact that, at any given time, there can have been arrivals under the threshold which were not records and hence the arguments we used in Section 2.3.2 do not apply. However, conditioning on what happens close to s, we can, up to a point, bypass this difficulty. For this, fix some $\Delta s > 0$ sufficiently small. Then for any $0 \le s < t$,

$$\mathrm{P}[T_\tau \ge s + \Delta s] = \mathrm{P}[T_\tau \notin (s, s + \Delta s) \mid T_\tau \ge s]\mathrm{P}[T_\tau \ge s].$$

Now let $N(\Delta s)$ be the number of arrivals in $(s, s + \Delta s)$. Because the arrival process has independent increments, we know that the number of arrivals in $[s, s + \Delta s]$ must be independent of the history of the process before time s, so that conditioning on $N(\Delta s)$ yields

$$\begin{aligned} &\mathrm{P}[T_\tau \notin (s, s + \Delta s) \mid T_\tau \ge s] = \\ &\qquad 1 - \Delta s + \Delta s \mathrm{P}[T_\tau \notin (s, s + \Delta s) \mid T_\tau \ge s, N(\Delta s) = 1] + o(\Delta s). \end{aligned} \tag{4.6}$$

Conditionally to $N(\Delta s) = 1$, there is an arrival in $(s, s + \Delta s)$, whose value is distributed uniformly on $(0, 1)$. Hence, letting X be the value of this arrival, we get

$$\begin{aligned} &\mathrm{P}[T_\tau \notin (s, s + \Delta s) \mid T_\tau \ge s, N(\Delta s) = 1] = \\ &\qquad = \int_0^1 \mathrm{P}[X \text{ is not a `record'} \mid T_\tau \ge s, N(\Delta s) = 1, X = x] dx. \end{aligned} \tag{4.7}$$

We decompose this integral into four distinct computations.

First, if $X = x \in [0, g_t(0))$, then, conditionally to τ not having stopped before s, we know that $X = x$ will be a 'record'. Hence

$$\mathrm{P}[X \text{ is not a 'record'} \mid T_\tau \geq s, N(\Delta s) = 1, X = x] = 0 \;\text{ if } x \in [0, g_t(0)). \tag{4.8}$$

Secondly if $X = x \in (g_t(s + \Delta s), 1)$, then this arrival is not under the curve and thus cannot be a 'record'. Hence

$$\mathrm{P}[X \text{ is not a 'record'} \mid T_\tau \geq s, N(\Delta s) = 1, X = x] = 1 \;\text{ if } x \in (g_t(s + \Delta s), 1]. \tag{4.9}$$

The third case is for $X = x \in (g_t(0), g_t(s))$. For these values, the arrival will be a record if and only if there have been no arrivals under x before time s. This situation is illustrated in Figure 6.

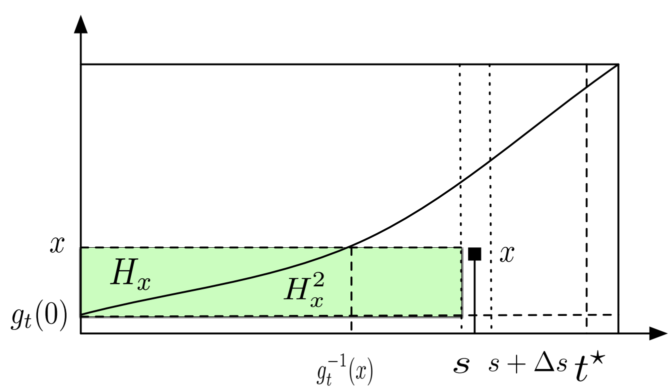

Figure 6: *There is an arrival of value x in $(s, s + \Delta s)$*

Let H_x and H_x^2 be the areas illustrated in Figure 6. Conditionally to $\tau \geq s$, we know that there have been no 'records' before s. This means that if there were arrivals in H_x^2, then there must have been smaller prior arrivals in H_x. Hence $X = x$ will not be a record if and only if there has been at least one arrival in H_x, and thus, letting $N(H_x)$ be the number of arrivals in H_x, we see that for $x \in (g_t(0), g_t(s))$,

$$\mathrm{P}[X \text{ is not a 'record'} \mid T_\tau \geq s, N(\Delta s) = 1, X = x] \;\; = 1 - \mathrm{P}[N(H_x) = 0 \mid T_\tau \geq s]. \tag{4.10}$$

The fourth and final possibility is given by $X = x \in (g_t(s), g_t(s + \Delta s))$. We do not need an explicit computation in this case because we are going to take the limit for Δs going to 0 and thus it suffices to notice that if $x \in (g_t(s), g_t(s + \Delta s))$, then $\mathrm{P}[X \text{ is not a 'record'} \mid T_\tau \geq s, N(\Delta s) = 1, X = x]$ will be a bounded integrable function of x.

Combining (4.8), (4.9) and (4.10) with (4.7) we obtain

$$\mathrm{P}[T_\tau \notin (s, s+\Delta s) \mid T_\tau \geq s, N(\Delta s) = 1]$$

$$= 1 - g_t(s+\Delta s) + \int_{g_t(0)}^{g_t(s)} (1 - \mathrm{P}[N(H_x) = 0 \mid T_\tau \geq s])dx + o(\Delta s).$$

Let us denote the function appearing in the rhs of this last equation by $1 + \chi(s, \Delta s)$. Then, applying (4.6) yields

$$\mathrm{P}[T_\tau \notin (s, s+\Delta s) \mid T_\tau \geq s] = 1 + \Delta s \chi(s, \Delta s) + o(\Delta s), \tag{4.11}$$

and hence

$$\mathrm{P}[T_\tau \geq s + \Delta s] = (1 + \Delta s \chi(s, \Delta s) + o(\Delta s))\mathrm{P}[T_\tau \geq s].$$

After straightforward manipulations, this equation yields

$$\frac{F_\tau(s+\Delta s) - F_\tau(s)}{\Delta s} = F_\tau(s)\chi_2(s, \Delta s) - \chi(s, \Delta s) + \frac{o(\Delta s)}{\Delta s}. \tag{4.12}$$

We take the limit for Δs going to 0^+ on both sides of (4.12). Since the limit of the rhs exists, that of the lhs must exist as well and we obtain an ODE on the distribution function of τ of the form

$$F'_\tau(s) = F_\tau(s)\chi(s, 0) - \chi(s, 0).$$

Now define

$$\psi(s) = g_t(0) + \int_{g_t(0)}^{g_t(s)} \mathrm{P}[N(H_x) = 0 \mid T_\tau \geq s]dx \; (= -\chi(s, 0)).$$

With this final notation, we see that the distribution function of T_τ on $[0, t^\star]$ satisfies

$$F'_\tau(s) = -\psi(s)F_\tau(s) + \psi(s) \tag{4.13}$$

with the initial conditions $F_\tau(0) = 0$. Solving (4.13) will yield the distribution of τ on $[0, t^\star]$.

Now we know that threshold functions of the form

$$g_t(s) = \frac{c}{t - s + c} \tag{4.14}$$

are close to optimality in the class of memoryless threshold rules. It follows that this choice of threshold function should also be close to optimality in the class of one-legged mixed strategies. For this reason we chose to perform our computations when the threshold function is given by (4.14). We did not try any other form of threshold functions.

Computations we do not include allowed us to obtain estimates on the value of one-legged mixed strategies for $t^\star = at$, $a \in (0,1)$. We have derived both theoretical expressions for and done simulations on these estimates but we have not computed the limiting expressions. Figure 7 is an illustration of our results for $c = 2$ and for specific choices of $a \in (0,1)$.

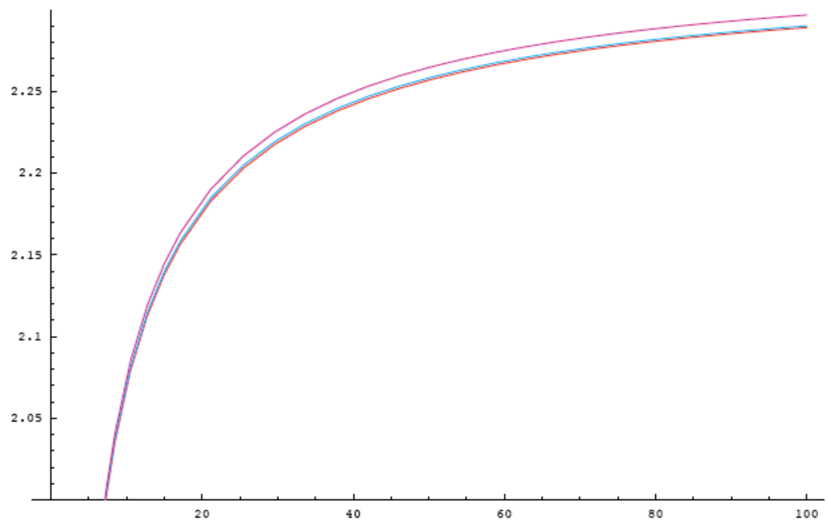

Figure 7: *Plot for $c = 2$ and $a = 0.6$, $a = 0.5$ and $a = 0$ of the estimates on the value of one-legged mixed strategies with $t \in [0, 100]$. The lowest curve represents the estimate obtained for $a = 0.5$, and the highest is that obtained for $a = 0$.*

These results indicate that one-legged mixed strategies, yield a strict improvement on the known upper bounds on w although this improvement is non-conclusive, in the sense that it does not yield a limiting value below 2.2956. Minimizing the estimates numerically, we obtain that for $a = 0.552949$, $c = 1.97605$, the limiting value is approximately equal to 2.32506. This indicates a strict but negligible improvement on the known upper bounds on v. The next table illustrates some of the numerical outputs we obtained for these strategies with $c = 2$.

t, a	0	0.1	0.2	0.4	0.6	0.8	0.9
10^5	2.33333	2.33327	2.33298	2.32945	2.32604	2.40228	2.66986
10^{10}	2.33333	2.33331	2.33302	2.32949	2.32608	2.40234	2.67001

Final comments and Conclusion

In 1990, Professor Herbert Robbins presented a classical problem of optimal stopping to experts on this subject from around the world. This was followed by the publication of three papers, namely Bruss and Ferguson (1993, 1996) and Assaf and Samuel-Cahn (1996). From that time on, despite the continued interest shown by well known probabilists[1], no new results were obtained. In a review paper (Bruss (2005)) about what is known on the problem, the author indicated that by modeling the problem in continuous time, one could obtain a differential equation on the value function. We started our research from there, with the aim of establishing this differential equation and using the continuous model to obtain information on the discrete problem. As we have stated in the Introduction, we have only been partially successful in this research.

We started our work by embedding Robbins' problem in a Poisson arrival process. For this we introduced a function $w(t)$ which is the value function of the Poisson embedded problem. We obtained the existence of optimal strategies for this problem and showed that $w(t)$ is 'well-behaved', i.e. it is uniformly continuous on its domain. However, the passage from a known to a random number of observations has its undeniable price: we were unable to show that $w(t)$ is a monotone increasing function of t. This comes as a surprise to us and is in sharp contrast with the discrete case, where the value function $v(n)$ is easily shown to be a bounded strictly increasing function of n. We nonetheless proved that the differential equation obtained in Bruss (2005) is satisfied almost everywhere by $w(t)$. This equation is a capsule containing all the information relevant to Robbins' problem in a closed form. We showed how a simple transformation of this equation yields upper bounds on the value function. However, the original equation depends on an unknown function $w(t|x)$, and we have not exploited this equation to get tight estimates on w.

We compared the discrete and the Poisson embedded problems and we showed that for all $\epsilon > 0$ no matter how small, there exists a time after which $w(t)$ will always be greater than $v - \epsilon$. This implies that if the limit w exists, it is necessarily greater than v. We then obtain an upper bound on $w(t)$ which depends only on the optimal value v and on the asymptotic behavior of the sequence $\mathrm{E}[nX_{\tau_n^\star}]$, where $\tau_n^\star$ is the optimal strategy for

[1] see Bruss (2005)

the discrete n-arrival Robbins' problem. If we were able to show that this sequence is bounded, then this would imply that the limit w exists and is equal to v.

Now both the question of the monotonicity and that of the equality between v and w share a common origin, namely that the optimal strategy in Robbins' problem is fully history dependent, and hence we have no understanding of the interplay between the value function and the optimal strategy $\tau^\star$. Although we have put considerable effort into these two claims, they share the same intrinsic difficulties.

Interestingly, these two questions also have another common characteristic: both claims are intuitively obvious! To see this, let us first consider the discrete problem. We see that it seems unreasonable to suppose that the optimal strategy would not yield values which are on average sufficiently small to ensure the boundedness of $\mathrm{E}[nX_{\tau_n^\star}]$. We know that $\mathrm{E}[R_{\tau_n^\star}]$ is bounded and hence, in particular, $\mathrm{E}[(n-\tau_n^\star)X_{\tau_n^\star}]$ is bounded. Also we know that the correlation between the values X_k and their ranks R_k goes to one as n goes to infinity (see Bruss and Ferguson (1993)). Hence an optimal strategy for the rank problem should also be a reasonable approximation of the optimal strategy for Moser's problem, for which we know that $\mathrm{E}[nX_{\tau_n^\star}] \to 2$. Such arguments are of course not sufficient to guarantee that the claim is true. Likewise, for the Poisson embedded problem with horizon t we see that as t gets large, it is intuitively clear that the task of the decision maker becomes more difficult, and thus the value he obtains should increase with t. Both problems are linked with each other to an extent which was hard to predict before studying them deeply. It seems that other new ideas would be necessary to get out of this deadlock.

Finally, memoryless strategies should attract our interest because it is those which enable to find upper bounds. We have obtained a number of results on the behavior of the value function $W(t)$ associated with such strategies, and we have seen that most of the results which hold for $w(t)$ also hold for $W(t)$.

The discussion in Chapter 4 indicates that in order to lower the known upper bounds, one must use a strategy which encompasses the apparent optimality of the memoryless threshold rules and the information based on the relative ranks. Unfortunately, such stopping rules also exemplify the pathological behavior of any history dependent strategy for Robbins' problem, and we have only obtained numerical approximations on the value function associated to such strategies. The improvements yielded by these strategies appear to be too small to give reliable directions for further research.

In this work, we have thus laid the Poisson embedded problem on a more firm basis. We have shown that $v \leq w$ so that the Poisson model is an interesting alternative setting for Robbins' problem. Further we have made it clear why we strongly conjecture that, indeed, $v = w$.

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