Nonlinear System Identification in Structural Dynamics: A Short (and Biased) History

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ABSTRACT

Although linear systems are now very well-understood in the context of structural dynamics, this is not at all the case for nonlinear systems. In particular, despite huge advances in the mathematical theory of nonlinear dynamical systems, progress in the difficult field of nonlinear system identification (learning of models from measured input/output data) has largely remained fairly slow and incremental. This is not to say that there have been no major successes in the field in its comparatively short history, and the objective of this paper is to present some of the highlights, as perceived by the authors, and to pave the way for a second paper discussing new developments and possibilities for the future.

INTRODUCTION

This paper is the first of a pair looking at the development of techniques for Nonlinear System Identification (NSI) in structural dynamics. The object of this paper is to illustrate the history of NSI by choosing a number of techniques which may be considered milestones in its development. The second paper will concentrate more on the current status of the field of NSI and on possible future directions [40]. The paper is not at all intended to be a complete history. There is a huge body of work accumulated in NSI and it is simply not possible to do justice to it in an article this short, so in choosing the techniques described here, the authors have shamelessly concentrated on those they are most familiar with, and inevitably this means, those they have worked on personally.

Before proceeding to specific techniques of NSI, it is worth pointing out that arguably the first method in NSI, is still very much in use today. Probably the simplest means of testing for linearity in a situation where the system is subjected to random excitation is to estimate the coherence function γ^2 [52],

$$\gamma^2(\omega) = \frac{|H(\omega)|^2 S_{xx}(\omega)}{S_{yy}(\omega)} \tag{1}$$

which will be unity for all accessible ω if and only if the system is linear and noise-free. $H(\omega)$ is the system transfer function and $S_{xx}(\omega)$ (resp $S_{yy}(\omega)$) is the input (resp. output) auto-spectrum. The approach was first developed in Wiener's pioneering work on spectral estimation in the 1930s [53]. The coherence can be simply calculated using FFT methods and an implementation is provided on almost all commercial spectrum analysers; however, it does not distinguish between the cases of a nonlinear system and noisy signals. Despite it's limitations, because of its simplicity and sensitivity, the coherence set the standard for other NSI methods to follow.

The layout of the paper is simple, each of the following sections concentrates on a specific NSI technique of interest. First, linearisation is covered, then: the Hilbert transform, the NARMAX model, functional series and finally, restoring force surfaces. The paper closes with a short set of conclusions.

LINEARISATION

The analysis of linear systems is very well understood. In the context of structural dynamics, modal analysis has proved to be very powerful approach to the analysis of linear systems [30]. In the simplest terms, modal analysis can be thought of as the extraction of system parameters from system FRFs. It is so effective in that restricted area that one might be tempted to apply the procedures of modal analysis directly to nonlinear systems without modification. In this situation, the curve-fitting algorithms used will associate a linear system with each FRF - in some sense the linear system which explains it best. In the case of a SDOF system, one might find the linear FRF,

$$H_{\text{eq}}(\omega) = \frac{1}{-m_{\text{eq}}\omega^2 + ic_{\text{eq}}\omega + k_{\text{eq}}} \tag{2}$$

which approximates most closely that of the nonlinear system. In the time-domain this implies a best linear model of the form,

$$m_{\text{eq}}\ddot{y} + c_{\text{eq}}\dot{y} + k_{\text{eq}}y = x(t) \tag{3}$$

and such a model is called a *linearisation*. As the nonlinear system FRF will usually change its shape as the level of excitation is changed, any linearisation is only valid for a given level. Also, because the form of the FRF is a function of the *type* of excitation used [75], different forcing types of nominally the same amplitude will require different linearisations. Only linearisations based on random excitation will be discussed here; these are arguably more fundamental because random excitation is the only excitation which generates nonlinear system FRFs which *look* like linear system FRFs [75].

The basic theory does not actually proceed via the FRFs, one operates directly on the equations of motion. The technique - *equivalent* or more accurately *statistical linearisation* dates back to the fundamental work of Caughey [17]. There are a number of variants of the procedure and an excellent recent discussion can be found in [21]. The following discussion is limited to SDOF systems; however, this is not a fundamental restriction of the method.

Given a general SDOF nonlinear system,

$$m\ddot{y} + f(y, \dot{y}) = x(t) \tag{4}$$

one seeks an equivalent linear system of the form (3). As the excitation is random, a sensible strategy is to minimise the expected value of the squared difference between the nonlinear force and that of the linear system, i.e. find the c_{eq} and k_{eq} which minimise,

$$J(y, c_{\text{eq}}, k_{\text{eq}}) = E[(f(y, \dot{y}) - c_{\text{eq}}\dot{y} - k_{\text{eq}}y)^2] \tag{5}$$

(it will be assumed that the apparent mass is unchanged i.e. $m_{\text{eq}} = m$). Using largely elementary calculus, the values of c_{eq} and k_{eq} which minimise (5) are found to be,

$$c_{\text{eq}} = \frac{E[\dot{y}f(y, \dot{y})]}{E[\dot{y}^2]} \tag{6}$$

and,

$$k_{\text{eq}} = \frac{E[yf(y, \dot{y})]}{E[y^2]} \tag{7}$$

and all that remains is to evaluate the expectations. Unfortunately this turns out to be nontrivial. The expectation of a function of random variables like $f(y, \dot{y})$ is given by,

$$E[f(y, \dot{y})] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dy d\dot{y}\; p(y, \dot{y}) f(y, \dot{y}) \tag{8}$$

where $p(y, \dot{y})$ is the joint Probability Density Function (PDF) for the processes y and $\dot{y}$. The problem is that the PDF of the response is not known for general nonlinear systems, estimating it presents formidable problems of its own. One solution to this problem is to approximate $p(y, \dot{y})$ by $p_{\text{eq}}(y, \dot{y})$ - the PDF of the equivalent linear system (3); this still requires a little thought. The fact that comes to the rescue is a basic theorem of random vibrations of *linear* systems [27], namely: if the excitation to a linear system is a zero-mean Gaussian signal, then so is the response, so,

$$p_{\text{eq}}(y_{\text{eq}}, \dot{y}_{\text{eq}}) = p_{\text{eq}}(y_{\text{eq}}) p_{\text{eq}}(\dot{y}_{\text{eq}}) = \frac{1}{\sqrt{2\pi}\sigma_{y_{\text{eq}}}\sigma_{\dot{y}_{\text{eq}}}} \exp\left(-\frac{y_{\text{eq}}^2}{2\sigma_{y_{\text{eq}}}^2} - \frac{\dot{y}_{\text{eq}}^2}{2\sigma_{\dot{y}_{\text{eq}}}^2}\right) \tag{9}$$

It may now appear that the problem has been reduced to the evaluation of integrals; unfortunately, things are not quite that simple. It remains to estimate the variances in the integrals. Now standard theory (see [52]) gives,

$$\sigma_{y_{\text{eq}}}^2 = \int_{-\infty}^{\infty} d\omega |H_{\text{eq}}(\omega)|^2 S_{xx}(\omega) = \int_{-\infty}^{\infty} d\omega \frac{S_{xx}(\omega)}{(k_{\text{eq}} - m\omega^2)^2 + c_{\text{eq}}^2\omega^2} \tag{10}$$

and,

$$\sigma^2_{\dot{y}_{\rm eq}} = \int_{-\infty}^{\infty} d\omega \frac{\omega^2 S_{xx}(\omega)}{(k_{\rm eq} - m\omega^2)^2 + c^2_{\rm eq}\omega^2} \tag{11}$$

and here lies the problem. Equation (7) ultimately expresses $k_{\rm eq}$ in terms of the variance $\sigma^2_{\dot{y}_{\rm eq}}$ and (11) expresses $\sigma^2_{\dot{y}_{\rm eq}}$ in terms of $k_{\rm eq}$. The result is a rather nasty pair of coupled nonlinear algebraic equations which must be solved for $k_{\rm eq}$. The same is of course true of $c_{\rm eq}$. However, the problem has been reduced to one of numerical analysis and appropriate solution schemes are available. In the case of the Duffing oscillator, for which,

$$m\ddot{y} + c\dot{y} + ky + k_3 y^3 = x(t) \tag{12}$$

A closed form approximation can be found [75]. Figure 1 shows the FRF of the linearised form of (12) for three increasing levels of random excitation. The natural frequency shifts up as the level increases and the cubic term begins to increase the apparent linear stiffness.

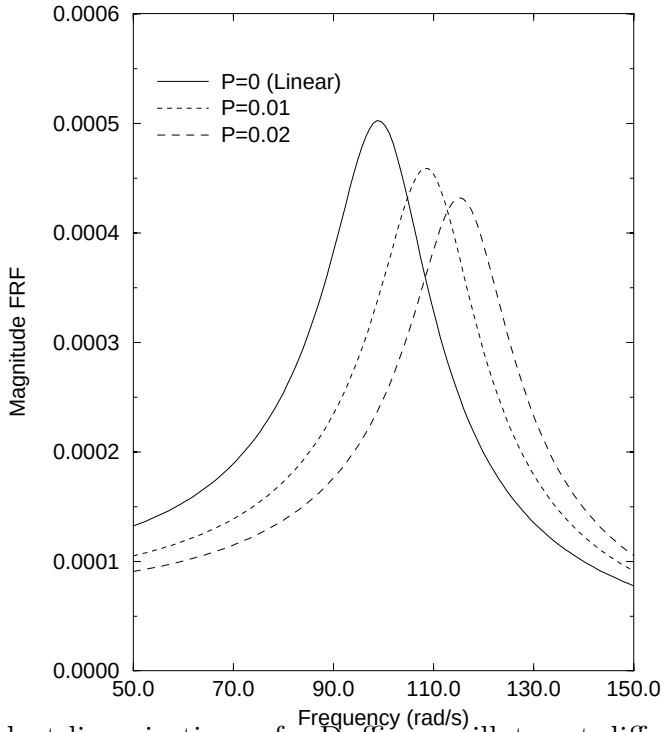

Figure 1: Equivalent linearisations of a Duffing oscillator at different excitation levels.

The problem with using (6) and (7) as the basis for an experimental method is that they require one to know what $f(y, \dot{y})$ is. In practice it will be useful to extract a linear model without knowing the details of the nonlinearity. Hagedorn and Wallaschek [36] have developed an effective experimental procedure for doing precisely this. The results are,

$$c_{\rm eq} = \frac{E[x\dot{y}]}{E[\dot{y}^2]} \tag{13}$$

and,

$$k_{\text{eq}} = \frac{E[xy] + E[\dot{y}^2]}{E[y^2]} \tag{14}$$

and it follows that the equivalent stiffnesses and dampings can be obtained experimentally if the signals $x(t)$, $y(t)$ and $\dot{y}(t)$ are measured. In fact, the experimental approach to linearisation is superior in the sense that the equivalent damping and stiffness are unbiased. The theoretical procedure yields biased values simply because the statistics of the linearised process are used in the calculation in place of the true statistics of the nonlinear process.

THE HILBERT TRANSFORM

A more sophisticated diagnostic tool than the coherence described in the introduction is provided by the Hilbert transform, which was first used for this purpose by Simon and Tomlinson [59].

The Hilbert transform diagnoses nonlinearity on the basis of measured FRF data. The Hilbert transform map on a given FRF $G(\omega)$ is,

$$\mathcal{H}[G(\omega)] = \tilde{G}(\omega) = -\frac{1}{i\pi}\int_{-\infty}^{\infty} d\Omega \frac{G(\Omega)}{\Omega - \omega} \tag{15}$$

This mapping reduces to the identity on the FRFs of linear systems. (Actually, the transform reduces to the identity for *causal* systems; however, for a large class of systems, this is the same thing.) Suppose $G(\omega)$ is decomposed so,

$$G(\omega) = G^{+}(\omega) + G^{-}(\omega) \tag{16}$$

where $G^{+}(\omega)$ (respectively $G^{-}(\omega)$) has poles only in the upper (respectively lower) half of the complex ω-plane. Then,

$$\mathcal{H}[G^{\pm}(\omega)] = \pm G^{\pm}(\omega) \tag{17}$$

The HT distortion is,

$$\Delta G(\omega) = \mathcal{H}[G(\omega)] - G(\omega) = -2G^{-}(\omega) \tag{18}$$

The form of the distortion can provide information about the *type* of nonlinearity present in the structure [75]. The major problem in using the Hilbert transform occurs when non-baseband or band-limited data is employed. The Hilbert transform can be recast in a slightly different form to that described above,

$$\Re\tilde{G}(\omega) = -\frac{2}{\pi}\int_{0}^{\infty} d\Omega \, \frac{\Im G(\Omega)\, \Omega}{\Omega^2 - \omega^2} \tag{19}$$

$$\Im\tilde{G}(\omega) = \frac{2\omega}{\pi}\int_{0}^{\infty} d\Omega \, \frac{\Re G(\Omega)}{\Omega^2 - \omega^2} \tag{20}$$

If zoomed data from $(\omega_{min}, \omega_{max})$ is measured, data is missing from the intervals $(0, \omega_{min})$ and (ω_{max}, ∞).

The problem is usually overcome by adding correction terms to the Hilbert transform evaluated [1, 31, 59]. An alternative approach to the HT exploits the pole-zero form of the FRF. A general FRF may be expanded into a rational polynomial representation,

$$G(\omega) = \frac{Q(\omega)}{P(\omega)} \tag{21}$$

Once the RP model G_{RP} is established, it can be converted into a pole-zero form,

$$G_{RP}(\omega) = \frac{\prod_{i=1}^{n_Q} (\omega - q_i)}{\prod_{i=1}^{n_P} (\omega - p_i)} \tag{22}$$

Long division and partial-fraction analysis produce the decomposition (16),

$$G^+_{RP}(\omega) = \sum_{i=1}^{N_+} \frac{C_i^+}{\omega - p_i^+} \qquad G^-_{RP}(\omega) = \sum_{i=1}^{N_-} \frac{C_i^-}{\omega - p_i^-} \tag{23}$$

Once this decomposition is established, the Hilbert transform follows.

Consider the Duffing oscillator,

$$\ddot{y} + 20\dot{y} + 10000y + 5 \times 10^9 y^3 = X \sin(\omega t) \tag{24}$$

Data were generated from 0 to 38.4 Hz, and the data were truncated by removing data above and below the range 9.25-32.95 Hz.

Figure 2 shows the Hilbert transforms of the FRF calculated by the RP method on the truncated data and by a standard numerical method *which used the full range of the data.*

(Note the characteristic distortion of the Nyquist plot for a hardening cubic stiffness nonlinearity, the curve rotates clockwise.)

The pole-zero decomposition method can also be used to compute analytical expressions for the Hilbert transform as in [41].

The Hilbert transform has also been used as a direct method of nonparametric identification. The FREEVIB and FORCEVIB approaches developed by Feldman [32, 33], can be used to construct the nonlinear damping or stiffness curves for a large class of nonlinear systems. The method works by extracting the instantaneous phase and frequency curves from experimental data. Alternative approaches have been constructed to yield the same information, in particular the method based on the Gabor transform in Brancaleoni *et al* [15] and the wavelet approach of Staszewski [61]. All of these approaches except FORCEVIB extract the information from the free decay response of the systems.

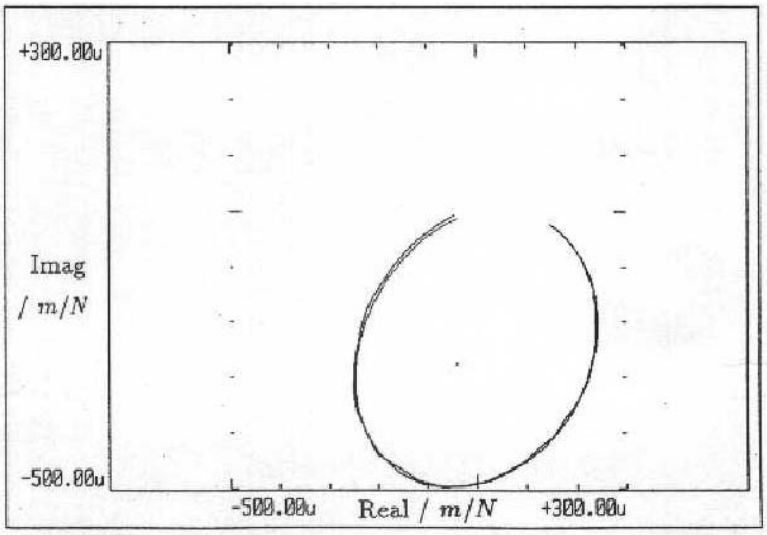

Figure 2: Comparison of Hilbert Transforms from RP approach and standard integral.

THE NARMAX MODEL

Suppose one is interested in the SDOF linear system,

$$m\ddot{y} + c\dot{y} + ky = x(t) \tag{25}$$

This can be converted by a process of discrete approximation to the discrete-time form,

$$y_i = a_1 y_{i-1} + a_2 y_{i-2} + b_1 x_{i-1} \tag{26}$$

where a_1, a_2 and b_1 are constants and functions of the original constants m, c, k and the sampling interval $\Delta t = t_{i+1} - t_i$ where the t_i are the sampling instants. In a more general form,

$$y_i = F(y_{i-1}, y_{i-2}; x_{i-2}) \tag{27}$$

This is an ARX model i.e. *Auto-Regressive with eXogenous inputs*. The advantage of adopting this form is that only the two states x and y need be measured in order to estimate all the model parameters a_1,a_2 and b_1 in (26) and thus identify the system. It is a simple matter to show that a general MDOF linear system has a discrete-time representation,

$$y_i = \sum_{j=1}^{n_y} a_j y_{i-j} + \sum_{j=1}^{n_x} b_j x_{i-j} \tag{28}$$

or,

$$y_i = F(y_{i-1}, \ldots, y_{i-n_y}; x_{i-1}, \ldots, x_{i-n_x}) \tag{29}$$

As before, all parameters $a_1, \ldots, a_{n_y}, b_1, \ldots, b_{n_x}$ can be estimated using measurements of the x and y data only.

The extension to nonlinear systems is straightforward. Consider the Duffing oscillator represented by,

$$m\ddot{y} + c\dot{y} + ky + k_3 y^3 = x(t) \tag{30}$$

One can pass to the discrete-time representation,

$$y_i = a_1 y_{i-1} + a_2 y_{i-2} + b_1 x_{i-1} + c y_{i-1}^3 \tag{31}$$

The model (31) is now termed a NARX (Nonlinear ARX) model. The regression function $y_i = F(y_{i-1}, y_{i-2}; x_{i-2})$ is now nonlinear; it contains a cubic term. If *all* terms of order three or less were included in the model structure i.e. $(y_{i-2})^2 x_{i-1}$ etc. a much more general model would be obtained,

$$y_i = F^{(3)}(y_{i-1}, y_{i-2}; x_{i-2}) \tag{32}$$

(the superscript denotes the highest order product terms) which would be sufficiently general to represent the behaviour of any dynamical systems with nonlinearities up to third order i.e. containing terms of the form $\dot{y}^3$, $\dot{y}^2 y$ etc.

The most general polynomial NARX model (including products of order $\leq n_p$) is denoted by,

$$y_i = F^{(n_p)}(y_{i-1}, \ldots, y_{i-n_y}; x_{i-1}, \ldots, x_{i-n_x}) \tag{33}$$

It has been proved by Leontaritis and Billings [43, 44], that under very mild assumptions, any input-output process has a representation by a model of the form (33). If the system nonlinearities are polynomial in nature, this model will represent the system well for all levels of excitation. If the system nonlinearities are not polynomial, they can be approximated arbitrarily accurately by polynomials over a given range of their arguments (Weierstrass approximation theorem, [58]). This means that the system can be accurately modelled by taking the order n_p high enough. However, the model would be input-sensitive as the polynomial approximation required would depend on the data. This problem can be removed by including non-polynomial terms in the NARX model as described in Billings and Chen [12]. The NARX model can even be cast as a Neural Network (Billings *et al* [13]).

The preceding analysis unrealistically assumes that the measured data is free of noise. As shown below, if the system is nonlinear the noise process can be very complex; multiplicative noise terms with the input and output are not uncommon, but can be easily accommodated in the discrete-time models as described in Leontaritis and Billings [43, 44], Korenburg *et al* [42] and Chen *et al* [20].

Suppose the measured output has the form,

$$y(t) = y_c(t) + \zeta(t) \tag{34}$$

where $y_c(t)$ is the 'clean' output from the system. If the underlying system is the Duffing oscillator of equation (15), the equation satisfied by the measured data is now,

$$m\ddot{y} + c\dot{y} + ky + k_3y^3 - m\ddot{\zeta} - c\dot{\zeta} - k\zeta - k_3\zeta^3 - 3y^2\zeta + 3y\zeta^2 = x(t) \tag{35}$$

and the corresponding discrete-time equation will contain terms of the form ζ_{i-1}, ζ_{i-2}, $\zeta_{i-1}y_{i-1}^2$ etc. Note that even simple additive noise on the output introduces cross-product terms if the system is nonlinear. Although these terms all correspond to unmeasurable states they must be included in the model. If they are ignored the parameter estimates will generally be biased. The system model (33) is therefore extended again by the addition of a *noise model* and takes the form,

$$y_i = F^{(3)}(y_{i-1}, y_{i-2}; x_{i-2}; \zeta_{i-1}, \zeta_{i-2}) + \zeta_i \tag{36}$$

This type of model is referred to as NARMAX (Nonlinear Auto-Regressive Moving-Average with eXogenous inputs).

Finally, the term 'moving-average' requires some explanation. Generally, for a linear system a moving-average model for the noise process takes the form,

$$\zeta_i = e_i + c_1e_{i-1} + c_2e_{i-2} + \dots \tag{37}$$

i.e. the system noise is assumed to be the result of passing a zero-mean white noise sequence $\{e_i\}$ through a digital filter with coefficients $c_1, c_2, \dots$. The terminology comes from the literature of time series analysis. Equation (36) requires a generalisation of this concept to the nonlinear case. This is incorporated in the NARMAX model which takes the final general form,

$$y_i = F^{(n_p)}(y_{i-1}, \dots, y_{i-n_y}; x_{i-1}, \dots, x_{i-n_x}; e_{i-1}, \dots, e_{i-n_e}) + e_i \tag{38}$$

In this form the noise sequence or *residual sequence* e_i is now zero-mean white noise. This allows the model to accomodate a wide class of possibly nonlinear noise terms.

The input and output variables x_i and y_i are usually physical quantities like force and displacement response respectively. An interesting alternative approach to this was followed by Thouverez and Jezequel [65], who fitted a NARMAX model using modal coordinates.

Having obtained a NARMAX model for a system, the next stage in the identification procedure is to determine if the structure is correct and the parameter estimates are unbiased. It is important to know if the model has successfully captured the system dynamics so that it will provide good predictions of the system output for different input excitations, or if it has simply fitted the model to the data; in which case it will be of little use since it will only be applicable to one data set. Simple validity tests arise from comparing the model predicted

data to the measured data, Billings *et al* extended the tests to include stringent correlation tests [11].

FUNCTIONAL SERIES AND HIGHER-ORDER FRFs

For a general linear system, the input-output map can be expressed by *Duhamel's integral*,

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \tag{39}$$

Equation (39) is manifestly linear and therefore cannot hold for arbitrary nonlinear systems. However, it admits a generalisation. The extended form of equation (39) was obtained by Volterra [66]. It takes the form of an infinite series,

$$y(t) = y_1(t) + y_2(t) + y_3(t) + \dots \tag{40}$$

where,

$$y_1(t) = \int_{-\infty}^{+\infty} d\tau h_1(\tau)x(t-\tau) \tag{41}$$

$$y_2(t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\tau_1 d\tau_2 h_2(\tau_1,\tau_2)x(t-\tau_1)x(t-\tau_2) \tag{42}$$

$$y_3(t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\tau_1 d\tau_2 d\tau_3 h_2(\tau_1,\tau_2,\tau_3)x(t-\tau_1)x(t-\tau_2)x(t-\tau_3) \tag{43}$$

The form of the general term is obvious from the above. The functions $h_1(\tau)$, $h_2(\tau_1,\tau_2)$, $h_3(\tau_1,\tau_2,\tau_3)$, $\dots$ $h_n(\tau_1,\dots,\tau_n)$, $\dots$ are generalisations of the linear impulse response function and are usually referred to as *Volterra kernels*. The use of the Volterra series in dynamics stems from the work of the MIT group on functional series [14, 16, 34], although the most widely-cited source is the seminal paper of Barrett [5], in which the series was applied to nonlinear differential equations. One can think of the series as a generalisation of the Taylor series from functions to functionals. The expression (39) simply represents the lowest order truncation which is of course exact only for linear systems. There are many practical issues associated with the existence and convergence of the Volterra series and a good summary can be found in [54].

There exists a dual frequency-domain representation for nonlinear systems. The *higher order FRF's* or *Volterra kernel transforms* $H_n(\omega_1,\dots,\omega_n)$, $n = 1,\dots,\infty$ are defined as the multi-dimensional Fourier transforms of the kernels, i.e.,

$$H_n(\omega_1,\dots,\omega_n) = \int_{-\infty}^{+\infty}\dots\int_{-\infty}^{+\infty} d\tau_1\dots d\tau_n h_n(\tau_1,\dots,\tau_n)e^{-i(\omega_1\tau_1+\dots+\omega_n\tau_n)} \tag{44}$$

It is a straightforward matter to obtain the frequency-domain dual of the expression (40),

$$Y(\omega) = Y_1(\omega) + Y_2(\omega) + Y_3(\omega) + \ldots \tag{45}$$

where,

$$Y_1(\omega) = H_1(\omega)X(\omega) \tag{46}$$

$$Y_2(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega_1 H_2(\omega_1, \omega - \omega_1)X(\omega_1)X(\omega - \omega_1) \tag{47}$$

One use of the Volterra series is the construction of analytic approximations to various quantities of interest in experimental structural analysis. Approximations to the FRFs of SDOF and MDOF systems with cubic nonlinearities and excited by Gaussian white noise can be found in the work of Worden and Manson [73, 74]. The approximations derived are of interest from the point of view that the FRFs constructed have all their poles in the upper-half of the complex frequency plane. This goes some way to explaining why the FRFs of randomly excited nonlinear systems appear to be invariant under the Hilbert transform.

There are various methods of determining the higher order FRF's for a system. If one has measured input and output time data, it is possible to evaluate the FRF's by carrying out many multi-dimensional Fast Fourier Transforms and averaging the results, in much the same way as one would evaluate a standard linear transfer function (Schetzen [56], Gifford and Tomlinson [35]). However, this approach requires that the input be a Gaussian white noise sequence; also, the computational burden of carrying out multi-dimensional FFT's makes evaluation of FRF's higher than second order prohibitive by this method. Wray and Green proposed an interesting method of extracting the Volterra kernels for a system by fitting the time-domain response using a Time-Delay Neural Network (TDNN) and computing the kernels from the network weights [76]. Alternatively, it is possible to estimate the higher order FRF's efficiently by harmonic testing of a system as in Storer and Tomlinson [62]. The higher order FRF's can also be obtained by impulse testing (Liu *et al* [45]).

More recent work on identification of nonlinear systems via the Volterra kernels and kernel transforms can be found in Khan and Vyas [38], Chatterjee and Vyas [19] and Tawfiq and Vihn [64].

If one knows the equation of motion of a system, an alternative approach can be used which yields exact expressions for the higher order FRF's. The method of harmonic probing was introduced by Bedrossan and Rice specifically for systems with continuous-time equations of motion [6]. The method was extended to discrete-time systems by Billings and Tsang [9, 10]. An alternative, recursive approach to probing is presented in Peyton Jones and Billings [55].

For the Duffing oscillator in equation (12), the first three higher-order FRFs are found to be,

$$H_1(\Omega) = \frac{1}{-m\Omega^2 ic\Omega + k} \tag{48}$$

$$H_2(\omega_1, \omega_2) = -\frac{k_2}{2} H_1(\omega_1) H_1(\omega_2) H_1(\omega_1 + \omega_2) \tag{49}$$

$$H_3(\omega_1, \omega_2, \omega_3) = -\frac{1}{6} H_1(\omega_1 + \omega_2 + \omega_3).$$

$$\{4k_2 \left(H_1(\omega_1) H_2(\omega_2, \omega_3) + H_1(\omega_2) H_2(\omega_3, \omega_1) + H_1(\omega_3) H_2(\omega_1, \omega_2)\right) + k_3 H_1(\omega_1) H_1(\omega_2) H_1(\omega_3)\} \tag{50}$$

The method of Wray and Green described earlier which extracted Volterra kernels from TDNN neural networks [76], proved to be inextendible to the NARX case. However, Chance *et al* [18], showed that it was possible to extract kernel transforms or HFRFs by fitting Multi-Layer Perceptron neural networks and then using harmonic probing. Ideas from machine learning theory have proved useful in other respects for Volterra series approximation. Kernel methods and ideas based on Reproducing Kernel Hilbert Spaces (RKHS) have proved fruitful in the work of Dodd & Harris [24] and Dodd and Harrison [25, 26]. One particular result of interest is a method for estimating the entire Volterra series without truncation (Wan [67]).

In order to illustrate the first two higher-order FRFs, the parameters from the system in equation (24) have been substituted in equations (48) and (49).

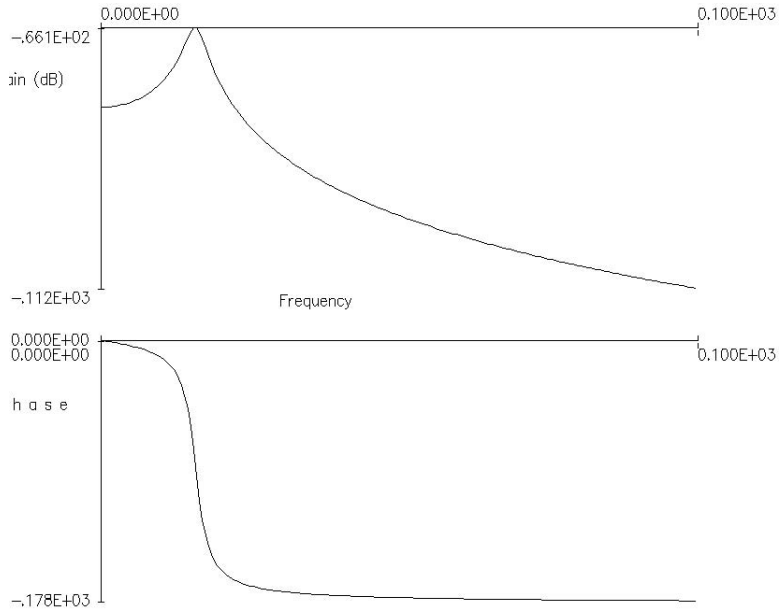

Figure 3: Estimated and exact H1s for the Duffing oscillator.

The first order FRF has been used in system identification for some time; the well-established technique of modal analysis (Ewins [30]) is based on the extraction of linear system parameters by curve-fitting to the FRF. Gifford and Tomlinson [35], showed that the technique extends naturally to nonlinear systems; nonlinear parameters are extracted by fitting surfaces or hypersurfaces to the higher order FRF's. This work was further extended by Storer and Tomlinson [62], who demonstrated that it is sufficient to curve-fit to the parts ot the FRF's

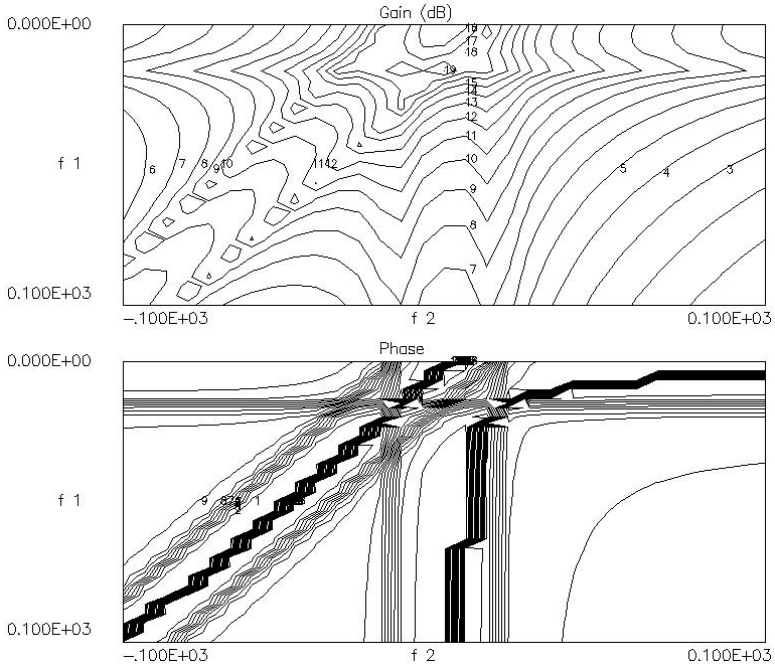

Figure 4: Estimated and exact H2s for the Duffing oscillator.

above the diagonal frequency subspaces. This allowed a significantly simpler experimental procedure based on harmonic testing.

Considering Figure 3, the interpretation of the H_1 curve is well-known. The peak in the curve corresponds to the resonant frequency at which high amplitudes of vibration are expected.

Interpretation of the second order FRF is also straightforward. The magnitude and phase of H_2 for the Duffing system above are given in Figure 4 as contour maps over the $(f_1, f_2) = (\frac{\omega_1}{2\pi}, \frac{\omega_2}{2\pi})$ plane. The frequency ranges for the plot are the same as for H_1 in Figure 3. A number of ridges are observed. These are in direct correspondence with the peak in H_1 as follows. According to equation (65), H_2 is a constant multiple of $H_1(\omega_1)H_1(\omega_2)H_1(\omega_1+\omega_2)$. As a consequence H_2 possesses local maxima at positions where the H_1 factors have local maxima. Consequently there are two ridges in the H_2 surface corresponding to the lines $\omega_1 = \omega_r = 2\pi f_r$ and $\omega_2 = \omega_r$. These are along lines parallel to the frequency axes. In addition, H_2 has local maxima generated by the $H_1(\omega_1+\omega_2)$ factor along the line $\omega_1+\omega_2 = \omega_r$. This ridge has an important interpretation; it indicates that one can expect a maximum in the second order output $y_2(t)$ if the system is excited by two sinusoids whose sum frequency is the linear resonant frequency. This shows clearly why estimation of a transfer function by linear methods is inadequate for nonlinear systems; such a transfer function would usually indicate a maximum in the output for a harmonic excitation close to the linear resonant frequency.

RESTORING FORCE SURFACES

The simple procedure described in this section allows a direct nonparametric identification for SDOF nonlinear systems. The only *a priori* information required is an estimate of the

system mass. The basic procedures were introduced by Masri and Caughey [46], although the approach described here resembles more the variant developed independently by Crawley and Aubert [22], and Crawley and O'Donnell [23] and referred to by them as 'force-state mapping'. A recent variation on the theme is the 'local' approach of Duym and Schoukens [29] which fits a piecewise linear restoring force surface.

The starting point is the equation of motion as specified by Newton's second law,

$$m\ddot{y} + f(y, \dot{y}) = x(t) \tag{51}$$

where m is the mass (or an effective mass) of the system and $f(y, \dot{y})$ is the internal restoring force which acts to return the system to equilibrium when disturbed. The function f can be a quite general function of position $y(t)$ and velocity $\dot{y}(t)$. Because f is assumed to be dependent *only* on y and $\dot{y}$ it can be represented by a surface over the phase-plane i.e. the $(y, \dot{y})$ plane. A trivial re-arrangement of equation (51) gives

$$f(y(t), \dot{y}(t)) = x(t) - m\ddot{y}(t) \tag{52}$$

Now, if the mass m is known and the excitation $x(t)$ and acceleration $\ddot{y}(t)$ are measured, all the quantities on the RHS of this equation are known and hence so is f. As usual, measurement of a time signal entails sampling it at regularly spaced intervals Δt. If $t_i = (i-1)\Delta t$ denotes the i^{th} sampling instant, then at t_i, equation (130) gives

$$f_i = f(y_i, \dot{y}_i) = x_i - m\ddot{y}_i \tag{53}$$

where $x_i = x(t_i)$ and $\ddot{y}_i = \ddot{y}(t_i)$ and hence f_i are known at each sampling instant. If the velocities $\dot{y}_i$ and displacements y_i are also known (i.e. from direct measurement or from numerical integration of the sampled acceleration data), at each instant $i = 1, \ldots, N$ a triplet $(y_i, \dot{y}_i, f_i)$ is specified. The first two values indicate a point in the phase plane, the third gives the height of the restoring force surface above that point. Given this scattering of force values above the phase plane there are a number of methods of interpolating a continuous surface on a regular grid for plotting purposes [75]. There are a couple of other issues of signal processing here. In the first case, direct sampling of the displacement, velocity and acceleration data requires considerable instrumentation, it is more economical to measure one and estimate the remaining states by numerical differentiation or integration; the problems which arise in this strategy are addressed in Worden [69]. This problem was neatly avoided in Shin and Hammond [57], where the authors adopted a state-space embedding approach and fitted a force surface of the form $f_i = f(y_i, y_{i-1})$. A second issue is the choice of excitation signal to give uniform coverage of the phase-plane; this is addressed in Worden [69] and Duym and Schoukens [28].

Once the surface is obtained, Masri and Caughey [46] construct a parametric model of the restoring force in the form of a double Chebyshev series,

$$f(y, \dot{y}) = \sum_{i=0}^{m}\sum_{j=0}^{n} C_{ij} T_i(y) T_j(\dot{y}) \tag{54}$$

where $T_i(y)$ is the Chebyshev polynomial of order i. It has since been established (Al-Hadid and Wright [2], that a straightforward polynomial expansion of the form

$$f(y,\dot{y}) = \sum_{i=0}^{m}\sum_{j=0}^{n} C_{ij} y^i \dot{y}^j \tag{55}$$

is superior in terms of ease, speed and accuracy of estimation. The only advantage of the Chebyshev form of the expansion is that the coefficients can be estimated independently of each other due to the fact that the polynomials are orthogonal [46]. Note that the identification precedes the fitting of the model; the method is truly nonparametric.

For illustration, the restoring force surface and the associated contour map for an automotive shock absorber, obtained by experiment [63] are given in Figure 5, they both show a very clear bilinear characteristic. Note that if a parametric representation of the internal force had been obtained, say a least-squares polynomial, it would have been impossible to infer the bilinear characteristic from the coefficients alone; it is the direct visualisation of the nonlinearity which makes the force surface so useful.

There exist in the literature a number of examples of the application of force surface techniques to experimental SDOF systems; One of the first is that of Crawley and O'Donnell [23], which includes a study of space-structure joints. Worden and Tomlinson consider an impacting cantilever beam in [71], as in the more recent study by Kerschen *et al* [39]. Meskell *et al* [50] applied the approach to a nonlinear fluid-loading example. Beligarde and Campanile consider an automotive shock absorber in [7]. The experimental study of Hunter *et al* [37], is also of interest in that it contains a frequency domain formulation of the method.

The identification procedure of Masri and Caughey was shown to extend to MDOF systems in Masri *et al* [47]. Although in principle, arbitrarily complex nonlinear systems could be identified; in practice, the computational burden was considerable. Attempts to obtain a practical implementation of the procedure were made by Worden and Tomlinson [68], and Al-Hadid and Wright [2]. The main difficulty being that the identification procedure is carried out in modal coordinates, the intention being to simplify matters by diagonalising the underlying linear system. The procedure therefore requires *a priori* estimates of the modal matrix and mass matrix [4]. Although the linear parts of the restoring forces are simplified by this procedure, each component of the nonlinear restoring force *vector* remains a function of *all* the coordinates. Al-Hadid and Wright showed that unless a time-consuming iterative version of the procedure was adopted, any model parameters would be biased. A further problem is that restoring force surfaces can no longer be obtained before the parameter estimation stage.

However, research continued and in Al-Hadid and Wright [2], a useful form of the identification procedure was obtained by utilising a physical coordinate representation for the nonlinear forces while retaining a modal coordinate approach to the underlying linear system. This can be contrasted with the later work of Masri *et al* [48, 49], where physical coordinates are used for the linear identification and modal coordinates for the nonlinear. In subsequent papers by Al-Hadid and Wright [3, 4], experimental results are presented for a MDOF system and a powerful technique for obtaining estimates of the mass and modal

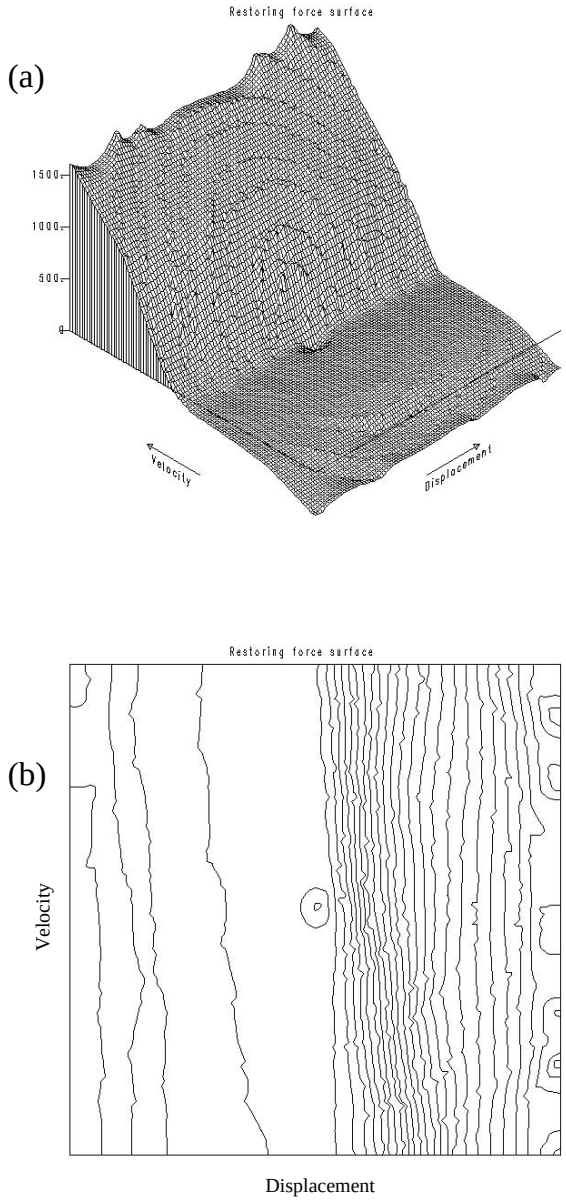

Figure 5: Experimental restoring force surface for an automotive shock absorber.

mass matrices appears.

In Worden *et al* [72], an alternative approach to MDOF system identification was described. Rather than make any use of modal coordinates, a physical coordinate system based on a lumped parameter representation of the system is adopted. Although this is now a direct parameter estimation (DPE) scheme similar to that adopted by in [48, 49] for linear systems, it was shown that all system parameters could be obtained if the system is excited at a single point. Also, restoring forces are shown to be a useful by-product. An alternative approach to DPE is described in Mohammad *et al* [51].

In an earlier formulation of a DPE scheme, Yang and Ibrahim [77], observed that if the

unforced equations of motion are considered, anoverall scale can be fixed by a knowledge of the total system mass and all system parameters can be obtained from measurements of the free oscillations.

CONCLUSIONS

The techniques described in this paper (along with all those omitted for reasons of space) represent considerable progress in the development of system identification methods for non-linear systems. They have taken the field from a state of comparative ignorance to the point were it is possible to identify nonlinear structural systems in practice. While the methods all have their individual merits, they also have individual weaknesses and it is safe to say that no single technique offers a solution to all problems. There are many outstanding problems in the field of NSI which require considerable attention and the field is very much in a state of evolution. The second paper in this short series will attempt to indicate where current progress is being made and will try to indicate where future research should be directed.

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