PROPER ORTHOGONAL MODES FOR THE CHARACTERIZATION OF NONLINEAR DYNAMIC SYSTEMS

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Abstract

Modal analysis is extensively used for the analysis and design of structures. However, a major concern for structural dynamicists is that its validity is limited to systems showing a linear behaviour. New developments have thus been proposed in order to tackle nonlinear systems among which the theory based on the nonlinear normal modes is indubitably the most appealing. In this study, a different approach is adopted and the modes extracted from the proper orthogonal decomposition are considered for the characterization of nonlinear systems.

INTRODUCTION

The proper orthogonal decomposition (POD) is a multivariate statistical method that aims at obtaining a compact representation of the data. This method may serve two purposes, namely order reduction by projecting high-dimensional data into a lower-dimensional space and feature extraction by revealing relevant but unexpected structure hidden in the data.

The key idea of the POD is to reduce a large number of interdependent variables to a much smaller number of uncorrelated variables while retaining as much as possible of the variation in the original variables. An orthogonal transformation to the basis of the eigenvectors of the sample covariance matrix is performed, and the data are projected onto the subspace spanned by the eigenvectors corresponding to the largest eigenvalues. This transformation decorrelates the signal components and maximizes variance.

The most striking property of the POD is its optimality in the sense that it minimizes the average squared distance between the original signal and its reduced linear representation. Although it is frequently applied to nonlinear problems, it should be borne in mind that the POD only gives the optimal approximating linear manifold in the configuration space represented by the data. The linear nature of the method is appealing because the theory of linear operators is available, but it also exhibits its major limitation when the data lie on a nonlinear manifold. As stated in reference [1], the POD is "*a safe haven in the intimidating world of nonlinearity; although this may not do the physical violence of linearization methods*".

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The POD, also known as the Karhunen-Loève decomposition, was proposed independently by several scientists including Karhunen, Kosambi, Loève, Obukhov and Pougachev and was originally conceived in the framework of continuous second-order processes. When restricted to a finite dimensional case, the POD is equivalent to principal component analysis (PCA). It is emphasized that the method appears in various guises in the literature and is known by other names depending on the area of application, namely PCA in the statistical literature, empirical orthogonal functions in oceanography and meteorology, and factor analysis in economics.

The first applications of the POD in nonlinear structural dynamics date back to the 1990s with the work of Cusumano and co-authors [2, 3]. It should be noted that, at the same time, FitzSimons and Rui [4] used it for modal reduction of structures for control. Since then, the POD has enjoyed some success (see e.g. references [5, 6, 7, 8, 9, 10, 11]). The reader is referred to reference [12] for an overview of the method.

The aim of this paper is to show that the modes extracted from the POD may be useful for the characterization of nonlinear systems. Specifically, their utility for model reduction and finite element model updating is highlighted.

MATHEMATICAL FORMULATION OF THE POD

Let $\theta(x,t)$ be a random field on a domain Ω. This field is first decomposed into mean $\mu(x)$ and time varying parts $\vartheta(x,t)$:

$$\theta(x,t) = \mu(x) + \vartheta(x,t) \qquad (1)$$

At time t_k, the system displays a snapshot $\vartheta^k(x) = \vartheta(x,t_k)$. The POD aims at obtaining the most characteristic structure $\varphi(x)$ of an ensemble of snapshots of the field $\vartheta(x,t)$. This is equivalent to finding the basis function $\varphi(x)$ that maximizes the ensemble average of the inner products between $\vartheta^k(x)$ and $\varphi(x)$:

$$\text{Maximize} \quad \langle|(\vartheta^k,\varphi)|^2\rangle \qquad \text{with } \|\varphi\|^2 = 1 \qquad (2)$$

where $(f,g) = \int_\Omega f(x)g(x)\ d\Omega$ is the inner product in Ω; $\langle\cdot\rangle$ denotes averaging; $\|\cdot\| = (\cdot,\cdot)^{\frac{1}{2}}$ is the norm; $|\cdot|$ is the modulus. Reference [13] shows that the maximization reduces to the following integral eigenvalue problem:

$$\int_\Omega \langle\vartheta^k(x)\vartheta^k(x')\rangle\varphi(x')\ dx' = \lambda\varphi(x) \qquad (3)$$

where $\langle \vartheta^k(x)\vartheta^k(x')\rangle$ is the averaged auto-correlation function. The solution of the optimization problem (2) is thus given by the orthogonal eigenfunctions $\varphi_i(x)$ of the integral equation (3), called the proper orthogonal modes or POD modes (POMs). The corresponding eigenvalues λ_i ($\lambda_i \geq 0$) are the proper orthogonal values (POVs). The POM associated with the greatest POV is the optimal vector to characterize the ensemble of snapshots. The POM associated with the second greatest POV is the optimal vector to characterize the ensemble of snapshots but restricted to the space orthogonal to the first POM, and so forth. The energy ϵ contained in the data is defined as the sum of the POVs, i.e., $\epsilon = \sum_j \lambda_j$, and the energy percentage captured by the ith POM is given by $\lambda_i / \sum_j \lambda_j$.

The POMs may thus be used as a basis for the decomposition of the field $\vartheta(x,t)$:

$$\vartheta(x,t) = \sum_{i=1}^{\infty} a_i(t)\varphi_i(x) \tag{4}$$

where the coefficients $a_i(t)$ are uncorrelated, i.e., $\langle a_i(t)a_j(t)\rangle = \delta_{ij}\,\lambda_i$, and are determined by $a_i(t) = (\vartheta(x,t), \varphi_i(x))$.

PRACTICAL COMPUTATION OF THE POD

In practice, the data are discretized in space and time. Accordingly, m observations of a n-dimensional vector $\mathbf{x}$ are collected, and an $(n \times m)$ response matrix is formed:

$$\mathbf{X} = [\mathbf{x}_1 \;\cdots\; \mathbf{x}_m] = \begin{bmatrix} x_{11} & \cdots & x_{1m} \\ \cdots & \cdots & \cdots \\ x_{n1} & \cdots & x_{nm} \end{bmatrix} \tag{5}$$

Since the data are now discretized and do not necessarily have a zero mean, the averaged auto-correlation function is replaced by the covariance matrix $\mathbf{S} = E[(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})^T]$, where $E[\cdot]$ is the expectation and $\boldsymbol{\mu} = E[\mathbf{x}]$ is the mean of the vector $\mathbf{x}$. The POMs and POVs are thus characterized by the eigensolutions of matrix $\mathbf{S}$. If the data have a zero mean, an estimate of the covariance matrix is merely given by the following expression:

$$\mathbf{S} = \frac{1}{m}\mathbf{X}\mathbf{X}^T \tag{6}$$

It is emphasized that the POD can also be computed through the singular value decomposition of matrix $\mathbf{X}$:

$$\mathbf{X} = \mathbf{U}\Sigma\mathbf{V}^T \tag{7}$$

where $\mathbf{U}$ is an $(m \times m)$ orthonormal matrix containing the left singular vectors; Σ is an $(m \times n)$ pseudo-diagonal and semi-positive definite matrix with diagonal entries containing the singular values σ_i and $\mathbf{V}$ is an $(n \times n)$ orthonormal matrix containing the right singular vectors.

The POMs, defined as the eigenvectors of the covariance matrix $\mathbf{S}$, are thus equal to the left singular vectors of $\mathbf{X}$. The POVs are the square of the singular values divided by the number of samples m. The main advantage in considering the SVD to compute the POD is that additional information is obtained through the matrix $\mathbf{V}$. The column $\mathbf{v}_i$ of matrix $\mathbf{V}$ contains the amplitude modulation of the corresponding POM $\mathbf{u}_i$, normalized by the singular value σ_i. This information may also provide important insight into the system dynamics.

MODEL REDUCTION OF NONLINEAR SYSTEMS

The first application of the POMs considered in this paper is model reduction of nonlinear systems. Specifically, it is shown that the modes extracted from a chaotic orbit are more representative of the system dynamics than any other set of POMs extracted from a non-chaotic response. Indeed, a chaotic orbit is assumed to cover a portion of the phase space of higher dimension, and hence of greater measure [than say a periodic orbit]. This higher dimensional data is further assumed to contain more information about the system dynamics than data of a lower-dimensional periodic orbit.

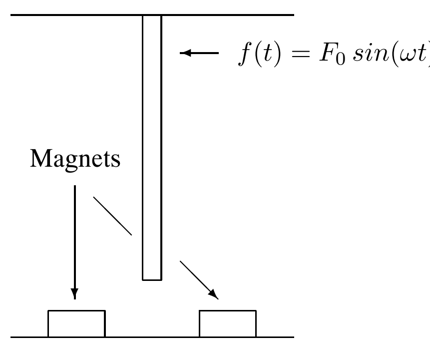


Figure 1: Clamped beam with two permanent magnets.

The procedure is illustrated using the numerical application depicted in Figure 1. It is a clamped beam ($l = 0.545m$ and $EI = 49.34\,10^{-3}\,Nm^2$) under the action of a sinusoidal force applied at a single point. Magnets are placed to the left and right of the free end to buckle the beam. The magnetic field creates forces which are assumed to be concentrated at the free end of the beam. The system is simulated with a forcing frequency equal to 5 Hz and for amplitudes F_0 going from 0 up to 3.5 N. To illustrate the varied dynamics of the system, a bifurcation diagram is presented in Figure 2. This plot represents the horizontal displacement at the end of the beam (at the moment when the sine force is equal to F_0) as a function of the amplitude F_0. For a particular amplitude, the number of points is related to the period. The initial conditions at each amplitude are equal to zero and the bifurcation diagram is obtained by checking for any periodicity within the first 800 s. In the absence of periodicity, the transients are not yet died out or the dynamics is non-periodic. It is worth pointing out that the beam can oscillate equally around either a magnet or the other depending on the initial conditions. This is the reason why points in the diagram may be associated with a positive or negative displacement. For some amplitudes, particularly between 1 and 2.5 N, the number of points is large, which may be interpreted as a non-periodic oscillation within the time allowed for the simulation.

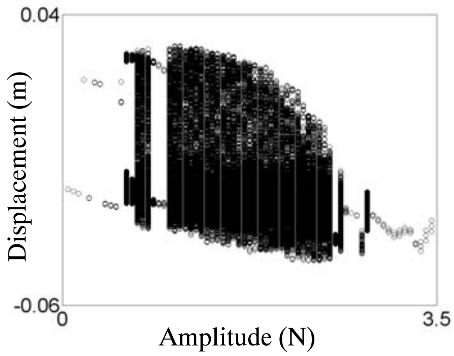


Figure 2: Bifurcation diagram of the full model.

The POMs obtained from a chaotic orbit are used to build a low-dimensional model of the beam. However, the system is chaotic for a wide range of amplitudes, and it remains to determine the value of the amplitude for which the modes are extracted. Interestingly enough, it was observed that the POMs do not vary significantly as long as the response is chaotic. The first five modes corresponding to an amplitude F_0=1 N constitute 99.99 % of the total signal energy, and were thus chosen. The bifurcation diagram of the reduced-order model is presented in Figure 3 and is in remarkable agreement with the diagram obtained from the full model.

To confirm that the POMs extracted from a chaotic orbit represent a convenient choice, the results are also compared to another reduced-order model based on the POMs of a periodic orbit, i.e., for F_0=0.01 N. In this case, only three POMs are retained in the model in accordance with the 99.99 % criterion. However, it is worth noticing that five of these POMs were also used for reduced-order modelling, and the results were not significantly improved. The reconstructed bifurcation diagram is represented in Figure 4, and it can be observed that this diagram presents severe distortions in comparison with that obtained using the full model.

FINITE ELEMENT MODEL UPDATING OF NONLINEAR SYSTEMS

Model updating involves the tuning of parameter values based on a comparison between responses in the model and experiment. The exploitation of the POMs for finite element model updating is motivated by the fact that, for non-linear systems, new features must be defined because

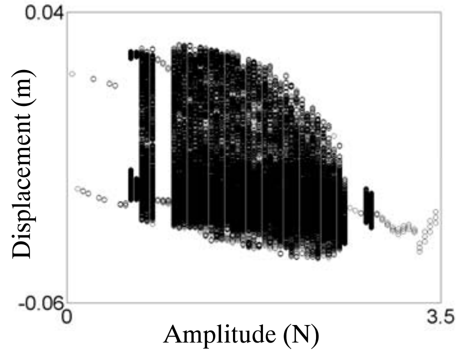


Figure 3: Bifurcation diagram of the reduced-order model (chaotic orbit).

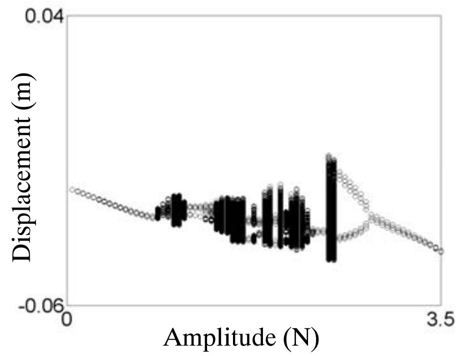


Figure 4: Bifurcation diagram of the reduced-order model (periodic orbit).

classical mode shapes are no longer effective to represent the system dynamics.

In reference [11], the proposed model updating procedure relies on the solution of an optimization problem that consists of minimizing the difference between the POMs of the measured and predicted data. In order to retain the information contained in matrix $\mathbf{V}$ (see equation 7) but without the drawback of its oscillatory nature, the Wavelet transform of matrix $\mathbf{V}$ is computed. The instantaneous frequencies and envelopes corresponding to the maxima in the time-frequency plane are then extracted and included into the objective function together with the POMs.

The investigation of the experimental structure shown in Figure 5 was performed. The structure consists of a clamped beam with a geometric non-linearity on its extremity (see reference [11] for more details). The beam was excited using a hammer impact, and transient responses were measured. The comparison between the POMs extracted from the measurements and those extracted from the finite element model shows that the predictive accuracy of this model is not satisfactory enough (see Figure 6). Several parameters need thus to be updated, namely, the rotational and translational stiffnesses at the clamped ends and at the junction, the coefficient and exponent of the non-linearity, and the damping ratio of the first four modes of the underlying linear system. The results obtained after updating are shown in Figure 7 and enable us to conclude that a model with a good predictive capability has now been obtained. It should be noted that the amplitude modulations (not represented here) of the experimental and simulated POMs are also in satisfactory agreement.

CONCLUDING REMARKS

The goal of this paper was to highlight the utility of the POD for dynamic characterization of nonlinear mechanical systems. Although the modes extracted from the decomposition do not have the theoretical foundations of the nonlinear normal modes (NNMs), they, however, provide a viable alternative for purposes such as model reduction and finite element model updating.

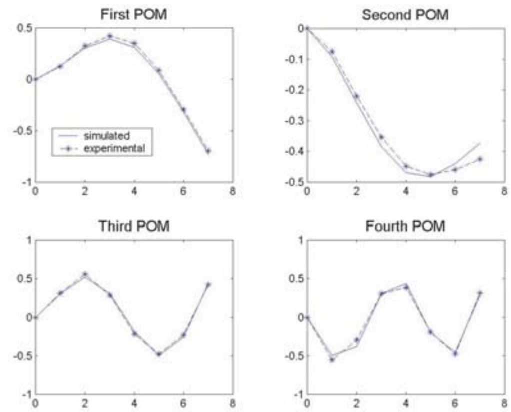
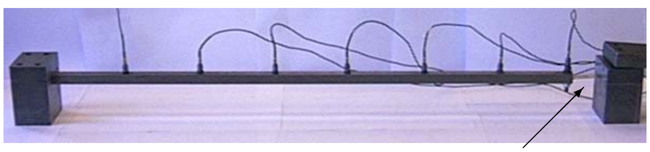


Figure 7: POMs after model updating.



Thin beam part=geometrical NL

Figure 5: Experimental set-up.

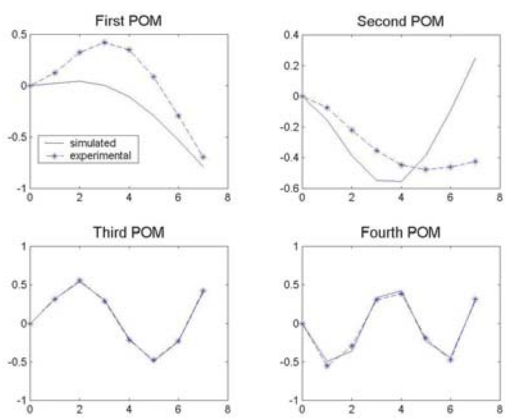


Figure 6: POMs before model updating.

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