Toward an Optimal Design Procedure of a Nonlinear Vibration Absorber Coupled to a Duffing Oscillator

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Abstract
In the present study, the focus is set on the development of a design methodology for nonlinear vibration absorbers, termed nonlinear energy sink (NES), to optimize the vibrating level reduction on a Duffing oscillator. The idea lies in the assessment of a duality property between, on the one hand, the tuning procedure of the tuned mass damper (TMD) coupled to a linear oscillator and, on the other hand, the assumed design scheme of NES coupled to a purely nonlinear Duffing oscillator. To this end, the basics on the TMD tuning methodology are recalled and extended to the concept of frequency-energy plot (FEP) [1]. Then, based upon this concept, the development of a NES design procedure is undertaken and the related dynamics analyzed. Finally, it is shown how to design a vibration absorber to deal with a general Duffing oscillator.

1 Introduction

The tuned mass damper (TMD) is maybe the most popular device for passive vibration mitigation of a vibrating mechanical structure. Realizing that the TMD is only effective when it is precisely tuned to the frequency of a vibration mode [2, 3], a recent body of literature has addressed this limitation using a nonlinear attachment characterized by an essential nonlinearity, termed a nonlinear energy sink (NES) [1, 4–14]. The concept of essential nonlinearity is central, because it means that an NES has no preferential resonant frequency, which makes it a frequency-independent absorber. Another salient feature of an NES is its capability to realize targeted energy transfer (TET) during which energy initially induced in the primary system gets passively and irreversibly transferred to the NES. Therefore, this nonlinear device seems to be very well suited for vibration isolation of MDOF linear structures or nonlinear structures.

However, because of the strongly nonlinear character of the absorber, the seek of an optimal design procedure is a challenging problem. Only a couple of researches tried to face this issue mainly by considering numerically performed parametric studies. Moreover, all of them considered different objective functions among which one can mention the suppression of the sefl-sustained oscillations [15, 16], the amount of energy dissipated in the nonlinear absorber [17], the time required for a complete energy dissipation [18] and finally, the occurrence of quasiperiodic motions on a given frequency bandwidth [19]. Even though these parametric studies generally give good results, their related computational burden may be prohibitive and the development of other better suited techniques are worthwhile.

Therefore, based on the fairly simple TMD tuning methodology, this paper investigates the possibility to develop an NES design procedure. The idea lies in the assessment of a duality property between, on the one hand, the tuning procedure of the TMD coupled to a linear oscillator and, on the other hand, the assumed design scheme of NES coupled to a purely nonlinear Duffing oscillator. To this end, in section 1 the basics of the TMD theory are recalled and the related results extended to the concept of frequency-energy plot. Using the previous results, section 2 describes the tuning procedure of the NES on a purely nonlinear Duffing

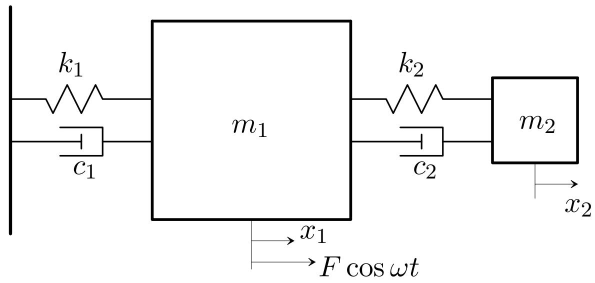

Figure 1: Linear oscillator (LO) coupled to a tuned mass damper (TMD)

oscillator. Finally, an attempt is carried out to design the best vibration absorber to couple to a general Duffing oscillator.

2 Design Methodology of a TMD coupled to a Linear Oscillator

The origin of the TMD dates back to the developments carried out by Frahm [2] on the dynamical vibration absorber coupled to a conservative linear oscillator (LO) (see figure (1) with $c_1 = c_2 = 0$). This device was efficient in a narrow frequency range centered at the natural frequency of the absorber. Later, Den Hartog [3] found that the TMD with energy dissipation mechanisms ($c_2 \neq 0$) are effective to an extended frequency range. These works have highlighted the trade-off existing during the design process between the need for good performances (attenuation efficiency, H_∞) and robustness (bandwidth, H_2). It's also worth being noted that the H_∞ and H_2 optimizations of TMD coupled to a dissipative linear oscillator ($c_1 \neq 0$) [20–23] or a MDOF linear system [24] is not straightforward and is still being investigated nowadays. Considering the forced system depicted in Figure (1), composed of a linear oscillator (of natural frequency $\omega_1 = \sqrt{k_1/m_1}$) coupled to a TMD, its equations of motion are :

$$\begin{aligned} m_1\ddot{x}_1 + c_1\dot{x}_1 + c_2(\dot{x}_1 - \dot{x}_2) + k_1x_1 + k_2(x_1 - x_2) &= (F\cos\omega t), \\ m_2\ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) &= 0. \end{aligned} \tag{1}$$

In this section the emphasis is set upon the basics of the TMD tuning procedure developed by Frahm [2] (for $c_1 = c_2 = 0$) which relies on a solid mathematical framework. If the system steady-state responses are supposed to be of the form $x_i = X_i cos(\omega t)$, than using relation (1) the displacement of the LO is expressed by :

$$X_1 = \frac{(k_2 - \omega^2 m_2)F}{(k_1 + k_2 - \omega^2 m_1)(k_2 - \omega^2 m_2) - k_2^2} \tag{2}$$

If an harmonic excitation of frequency $\omega = \omega_1$ is applied to the LO, than the vibration isolation of the LO (X_1 = 0) can be performed by realizing the condition :

$$\omega = \omega_1 = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{k_2}{m_2}} = \omega_a \tag{3}$$

Performance - Robustness Analysis

The performance of a TMD coupled to a LO (figure (1)) is now examined. To this end, the harmonic excitation ($Fcos(\omega t)$) is replaced by a direct impulsive excitation of the LO $\dot{x}_1(0) \neq 0, x_1(0) = x_2(0) = \dot{x}_2(0) = 0$. Moreover, condition (3) is supposed to be satisfied and some slight damping is introduced in both oscillators to induce energy dissipation. The initial parameter values are given in Table (1). A numerical

Parameter	Units	Value
m_1	$[kg]$	1
m_2	$[kg]$	0.05
c_1	$[Ns/m]$	0.002
c_2	$[Ns/m]$	0.002
k_1	$[N/m]$	0.5
k_2	$[N/m]$	0.025

Table 1: System parameters used for the numerical integration of equations (5).

(a) (b)

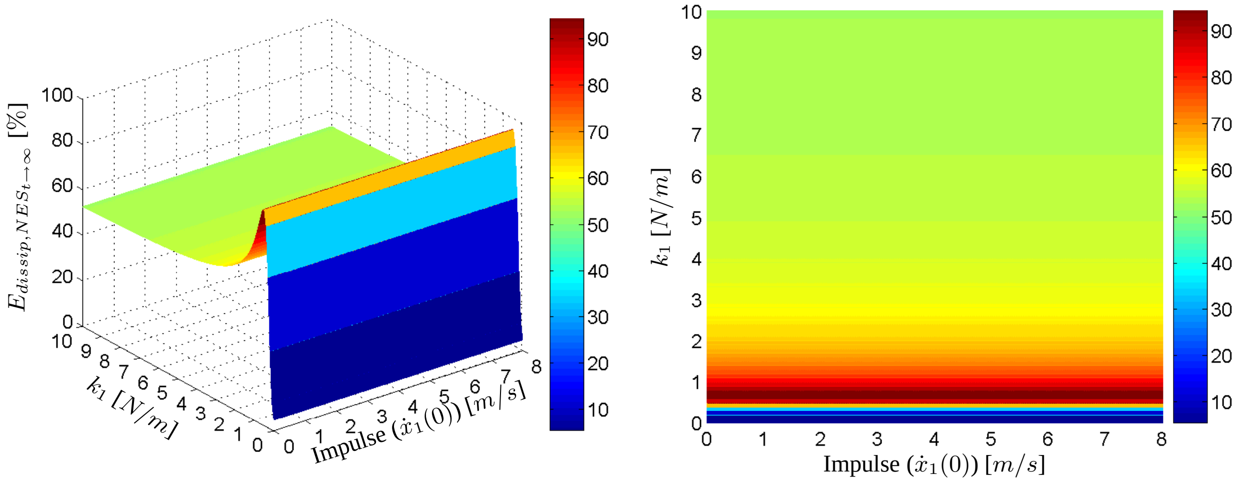

Figure 2: TMD Performance when coupled to a LO. (a) Energy dissipated in the TMD against the linear stiffness of the primary system and the impulse magnitude; (b) contour plot

integration of the equations of motion (5) is performed for varying linear stiffness k_1 and excitation amplitude $\dot{x}_1(0)$. The amount of energy dissipated in the TMD is evaluated through relations (4) corresponding to the ratio between the energy dissipated in the absorber and the energy injected in the system.

$$E_{diss,absorber,\%}(t) = 100 \frac{c_2 \int_0^t (\dot{x}_1(\tau) - \dot{x}_2(\tau))^2 d\tau}{\frac{1}{2} m_1 \dot{x}_1(0)^2} \tag{4}$$

Figure (2) depicts this quantity against the impulse magnitude $\dot{x}_1(0)$ and the linear stiffness k_1 (see Figures 2(a,b) for three-dimensional plot and a contour plot). Clearly, for $k_1 = 0.5N/m$ and whatever the magnitude of excitation $\dot{x}_1(0)$, the TMD can dissipate a large fraction (95%) of the input energy initially imparted to the LO. However, a slight mistuning in the host structure drastically reduces the TMD performances which makes this device not frequency robust. This brief reminder on the basics of the TMD design procedure gives the necessary basis to move forward to the design procedure of an NES coupled to a purely nonlinear oscillator.

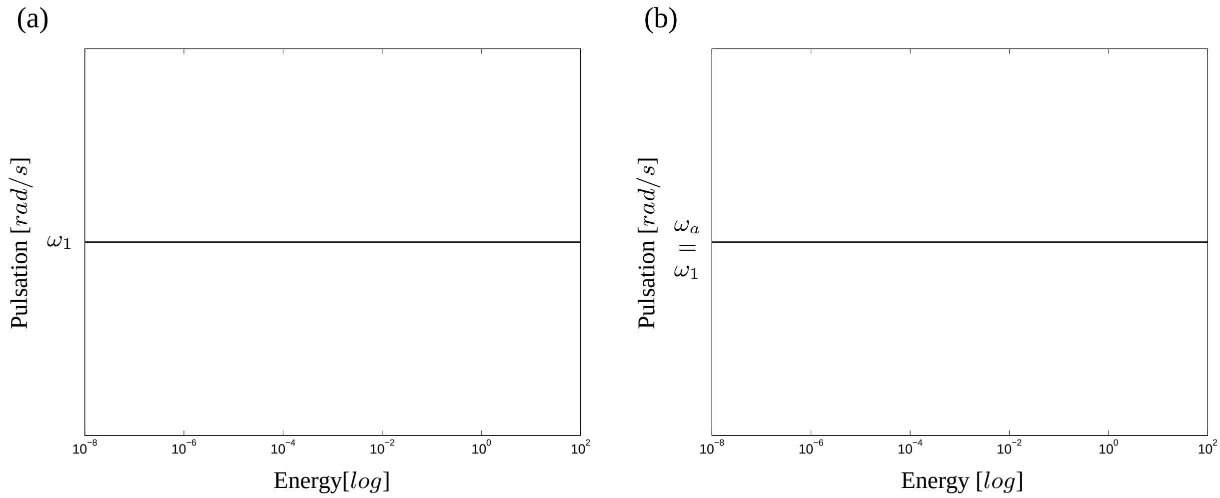

Figure 3: FEP corresponding to the LO (a) and the TMD (b) for the condition expressed in relation (3).

3 Design Methodology of a NES coupled to a Purely Nonlinear Oscillator

3.1 Frequency-Energy Plot Concept

One typical dynamical feature of nonlinear systems is the frequency-energy dependence of their oscillations. Moreover, for a given energy level injected in the system, several types of solutions may coexist. These properties make nonlinear systems more complicated to characterize than linear ones and require the development of adapted analysis tools such as the nonlinear normal modes (NNM) and the frequency-energy plot (FEP) concepts. In this study only the basics are recalled and for a complete description of these concepts the interested reader may refer to [1, 12, 25–27].

An NNM motion can be defined as a non-necessarily synchronous periodic motion of an undamped nonlinear mechanical system [25, 26]. However, as discussed in [1, 12, 25, 27], the damped dynamics can often be interpreted based on the topological structure of the NNMs of the underlying conservative system. Therefore their computation, using continuation methods [26], is a key point for characterizing the dynamical behavior of the nonlinear system under the scope.

Because of the frequency-energy dependence property of nonlinear systems, the modal curves and frequencies of NNMs vary with the amount of energy in the system. Consequently, the representation of NNMs in a frequency-energy plot (FEP) is particularly convenient [1, 12, 27]. An NNM is represented by a point in the FEP, which is drawn at a frequency corresponding to the minimal period of the periodic motion and at an energy corresponding to the conserved total energy during the motion. A branch, represented by a solid line, is a family of NNM motions possessing the same qualitative features.

3.2 Linear - Nonlinear Tuning Analogy

For linear systems, the use of a FEP is not of interest as they do not exhibit any frequency-energy dependence. However in the present study it is interesting to see how the tuning condition of a TMD on a LO (expressed in relation (3)) can be extended to the FEP concept. Figures (3 a-b) depict both the FEP of the LO and the TMD. Accordingly to linear theories, whatever the amount of energy injected in the system, both oscillators keep being tuned on the same resonant frequencies.

Now, considering a purely nonlinear Duffing oscillator (figure (4) (a)) composed of a mass m_1 and a nonlinear stiffness k_{nl_1}, its dynamical behavior is first characterized through the computation of the related FEP

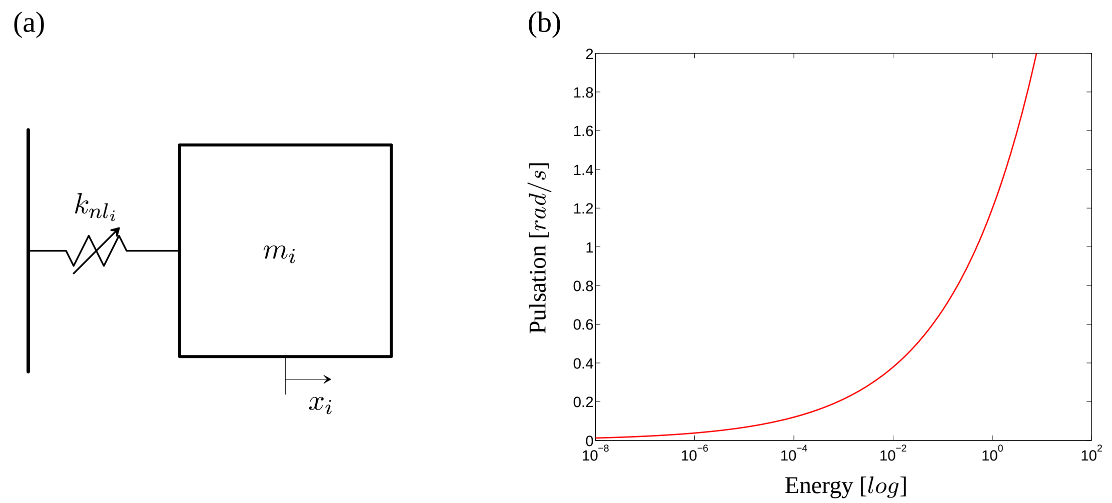

Figure 4: (a) Purely nonlinear Duffing oscillator ($\equiv$ NES); (b) related FEP for $m_1 = 1[kg]$ and $k_{nl_2} = 1[N/m^3]$.

(figure (4) (b)). The frequency-energy dependence property is highlighted and the absence of linear stiffness in the system explains the nearly zero resonant frequency at low energy levels. Regarding the vibration mitigation of this oscillator, the objective lies in determining the best absorber to couple. Accordingly to the TMD tuning procedure on a LO, the analogy for nonlinear systems should constrain both the nonlinear oscillator and its absorber to exhibit the same frequency-energy dependence (FEP). This would impose both oscillators to get the same resonant frequency for an identical energy amount. This feature can only be fulfilled if the structural configuration of the absorber is similar to the one of the primary structure. Therefore, the essentially nonlinear stiffness of the NES seems to make this device the best suited.

3.3 NES Parameters Computation

The assessment of the mass m_2 and nonlinear stiffness k_{nl_2} of the NES has to be carried out to perfectly match the Duffing oscillator FEP (figure (4) (b)).

Numerical Computation

The problem consists in addressing the following system of equations :

$$\begin{cases} m_2\ddot{x}_2 + k_{nl_2}x_2^3 & = 0, \\ \frac{1}{4}k_{nl_2}x_2^4(T_i) - E_{Duff}(\omega_i) & = 0 \quad\quad i = 1, ..., n. \end{cases} \tag{5}$$

where the first relation is the equation of motion of a conservative purely nonlinear oscillator (Figure (4) (a)), for instance, the NES and the second expresses the matching of both the Duffing oscillator and the NES frequency-energy dependence behaviors. The system can be solved using the combination of a shooting and an optimization procedure that allow the assessment of periodic solutions ($x_2(0) = x_2(T)$) and the NES parameters (m_2,k_{nl_2}), respectively. This procedure gives accurate results but the optimization efficiency is very dependent on the initial guess allocated. Another drawback lies in the prohibitive computational cost for a large number of matching points ($n \upuparrows$).

Analytical Computation

The analytical computation offers an interesting alternative to the numerical approach. Even though it is well known that for nonlinear systems no closed form solutions can be worked out, it is shown that, for this problem, the complexification averaging technique [28] or the simple harmonic balance method [29] can give very good approximations, in a straightforward manner. Here the second method is considered.

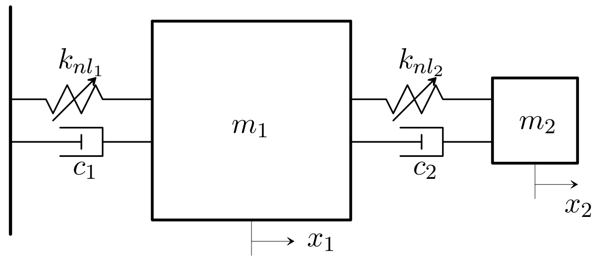

Figure 5: Purely nonlinear Duffing oscillator coupled to a nonlinear energy sink (NES).

The equation of motion of a grounded and conservative purely nonlinear Duffing oscillator is :

$$m_i\ddot{x}_i + k_{nl_i}x_i^3 = 0 \tag{6}$$

Considering the approximated motion $x_i(t) = A_i \cos \omega t$ and taking into account trigonometry transformations, the averaging over the fundamental frequency recasts equation (6) into:

$$A_i(\frac{3}{4}k_{nl_i}A_i^2 - \omega^2 m_i) = 0 \tag{7}$$

The related solutions (8) clearly show the frequency-amplitude dependence.

$$\begin{aligned} A_{i_1} &= 0 \\ A_{i_{2,3}} &= \pm\sqrt{\frac{4\omega^2 m_i}{3k_{nl_i}}} \end{aligned} \tag{8}$$

At initial time $t = 0$, the motion is $x_i(0) = A_i$ and the total energy in the system consists in the potential energy stored in the nonlinear stiffness :

$$E_i = \frac{1}{4}k_{nl_i}A_i^4 \tag{9}$$

According to previous discussions (section (3.2)), the tuning of the NES on the Duffing oscillator imposes the energy of both DOFs to be equal $E_1(\omega) = E_2(\omega)$. Introducing the frequency-amplitude relation (8) in (9), and identifying the terms with respect to the frequency, one gets the following tuning relation :

$$\frac{m_2^2}{k_{nl_2}} = \frac{m_1^2}{k_{nl_1}} \Rightarrow k_{nl_2} = \frac{m_2^2 k_{nl_1}}{m_1^2} \tag{10}$$

For this study or similar problems, the use of the numerical of analytical procedure gives identical results, however the second approach should be considered for its reduced computational time. For more complex systems, according to the accuracy required, the use of the numerical approach may be mandatory.

3.4 Performance - Robustness Analysis

In this section, the dynamics of the 2DOF system depicted in figure (5) and composed of a purely nonlinear Duffing oscillator coupled to a tuned NES is investigated. The related equations of motion are expressed by :

$$\begin{aligned} m_1\ddot{x}_1 + c_1\dot{x}_1 + c_2(\dot{x}_1 - \dot{x}_2) + k_{nl_1}x_1^3 + k_{nl_2}(x_1 - x_2)^3 &= 0, \\ m_2\ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + k_{nl_2}(x_2 - x_1)^3 &= 0. \end{aligned} \tag{11}$$

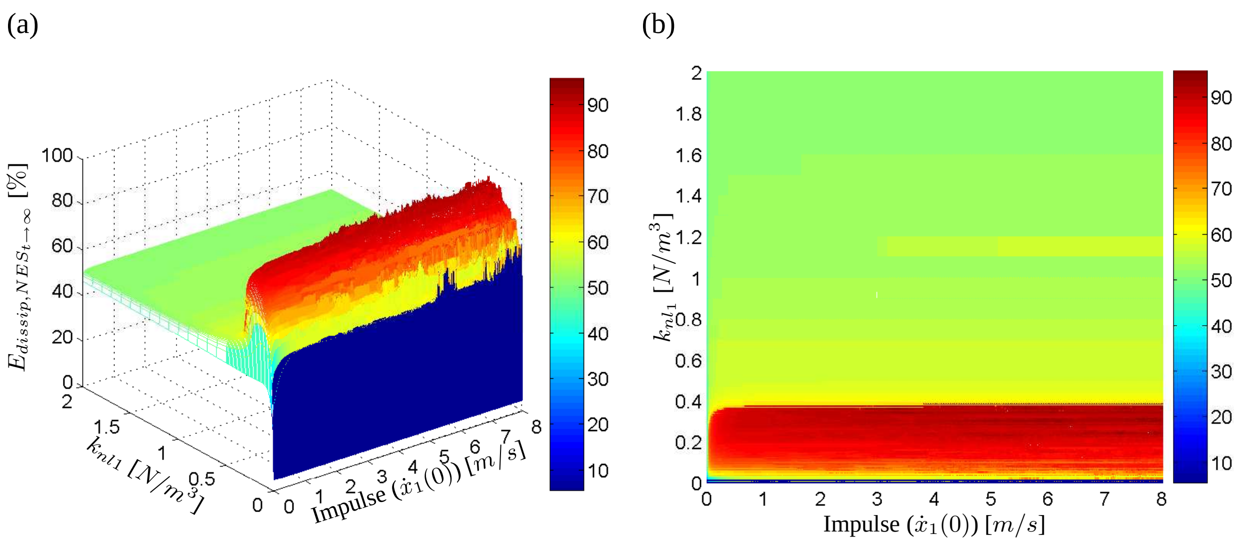

Figure 6: NES Performance when coupled to a purely nonlinear Duffing oscillator. (a) Energy dissipated in the NES against the nonlinear stiffness of the primary system and the impulse magnitude; (b) contour plot

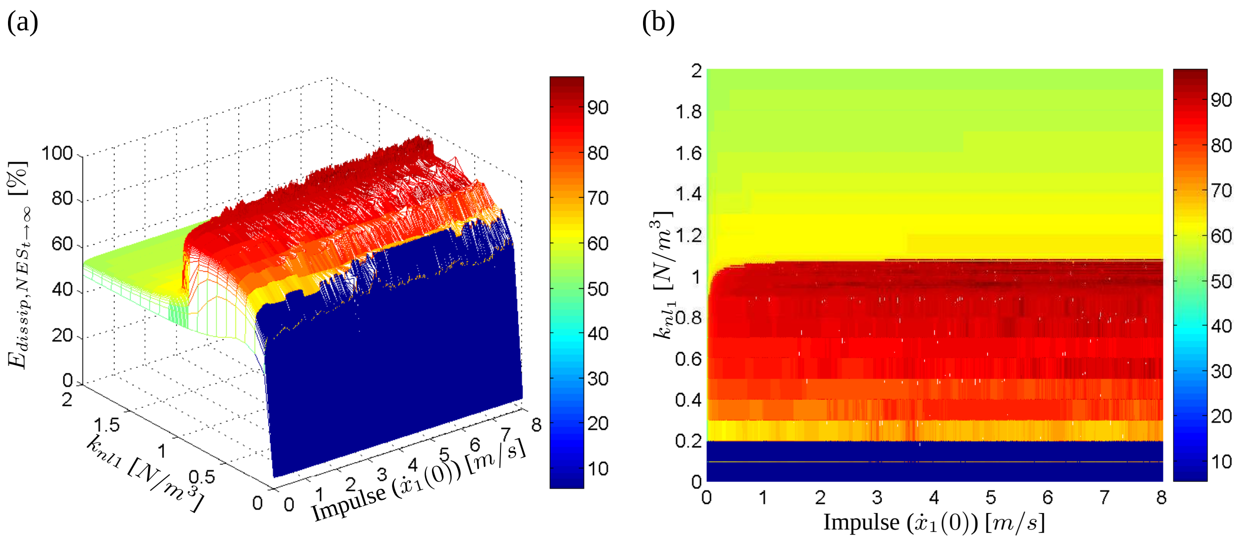

Figure 7: New NES configuration $k_{nl_2} = 0.0075[N/m^3]$. (a) Energy dissipated in the NES against the nonlinear stiffness of the primary system and the impulse magnitude; (b) contour plot

The structural configuration of the Duffing oscillator is given by $m_1 = 1[kg]$ and $k_{nl_1} = 1[N/m^3]$. Due to some practical reasons the NES mass is chosen to be small enough $m_2 = 0.05[kg]$ and considering relation (10), the nonlinear stiffness of the NES is computed $k_{nl_2} = 0.0025[N/m^3]$. Finally, some small damping $c_1 = c_2 = 0.002[Ns/m]$ is introduced in both oscillators to induce energy dissipation.

Similarly to the procedure followed in section (2), the performance of the tuned NES is now examined by numerically integrating the equations of motion (11). A three-dimensional plot of energy dissipated in the NES (relation (4)) against the nonlinear stiffness of the Duffing oscillator and the impulse magnitude is numerically computed and illustrated in figure (6).

The analysis of these results highlights three important features :

1. Similarly to the system composed of a LO coupled to a TMD, the system investigated is amplitude robust. Indeed, whatever the amplitude of excitation the level of energy dissipated in the NES keeps around 95% for a given nonlinear stiffness of the Duffing oscillator k_{nl_1}.

2. The region of high energy dissipation is not localized to a particular value of k_{nl_1} but spread over the

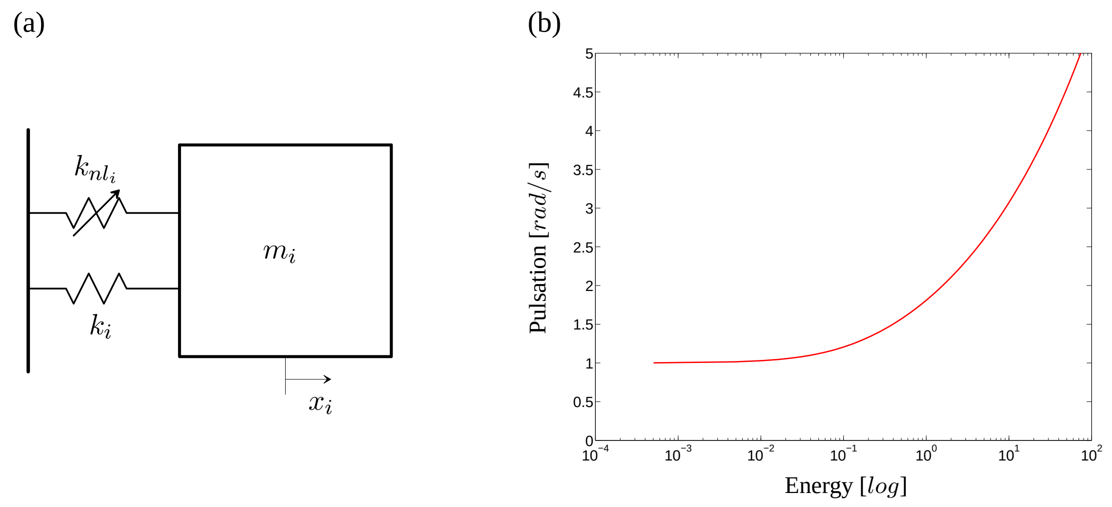

Figure 8: (a) Duffing oscillator composed of linear and nonlinear stiffness elements; (b) FEP related to this nonlinear oscillator for a particular set of parameters : $m_1 = 1[kg]$, $k_{nl_1} = 4[N/m^3]$, $k_1 = 1[N/m]$.

range $k_{nl_1} = [0.1-0.35][N/m^3]$ which makes the NES mistuning robust (or in other words, frequency robust).

3. Despite the tuning of the NES on the purely nonlinear Duffing oscillator, the nominal value of nonlinear stiffness of the primary structure, $k_{nl_1} = 1[N/m^3]$, is not included in the region of high energy dissipation level.

The last feature may be due to the assumption of zero or weak coupling between both DOFs. Indeed, the matching of both FEP does not take the coupling between the oscillators into account. Therefore, the nonlinear stiffness of the absorber, k_{nl_2}, is adjusted around its initially computed value in order to shift the high energy dissipation region around the nominal value of $k_{nl_1} = 1[N/m^3]$. The new nonlinear stiffness is fixed at $k_{nl_2} = 0.0075[N/m^3]$ and the induced improvement is depicted in Figure (7).

These results are of high interest as they show the possibility to make the NES amplitude robust. This property coupled to the frequency robustness character of the absorber makes it very efficient in mitigating nonlinear vibrating structures. Overall the objective of determining a duality property between the development of a linear absorber and a nonlinear one seems to be fulfilled.

4 Design Methodology of an Absorber Coupled to a General Nonlinear Oscillator.

The development of a TMD to mitigate a vibrating linear SDOF primary system has been investigated in section 1. Based on this methodology, the development of NES tuned on a purely nonlinear Duffing oscillator has been carried out in section 2. So far, linear and nonlinear dynamics have been clearly separated. Here, the focus is set on a the development of the best suited absorber to cope with the vibration mitigation of a general Duffing oscillator composed of both linear and nonlinear stiffness elements (figure (8) (a)).

Because of its nonlinear character, this oscillator (depicted in figure (8) (a)) presents a frequency-energy dependence illustrated by its related FEP in figure (8) (b). At low energy level, the dynamics is governed by the underlying linear system and the oscillator exhibits a constant resonant frequency at $\omega = 1[rad/s]$. For an increasing amount of energy injected in the system, the nonlinear character is predominant and the fundamental resonant frequency varies.

The vibration isolation of such a structure cannot be realized only using a TMD or a NES. Indeed, the TMD would be very well suited at low energy levels (ω_{res} is constant) whereas the NES would be efficiently performing only for higher amount of energy injected (ω_{res} varies). Therefore, based upon the tuning principle established in section (3.2), the better suited absorber should be the mirror of the primary structure in terms of structural components to perfectly fit the frequency-energy dependence behavior depicted in figure (8) (b).

4.1 Absorber Parameters Computation.

Similarly to section (3.3) the computation of the absorber parameters can be numerically or analytically performed.

Numerical Computation

The problem consists in addressing the system of equations (12) using the combination of a shooting and an optimization procedure. These methods allow the assessment of periodic solutions ($x_2(0) = x_2(T)$) and the absorber parameters (m_2,k_{nl_2},k_2), respectively. The advantages and drawbacks remain the same as those presented in section (3.3).

$$\begin{cases} m_2\ddot{x}_2 + k_2x_2 + k_{nl_2}x_2^3 & = 0, \\ \frac{1}{4}k_{nl_2}x_2^4(T_i) + \frac{1}{2}k_2x_2^2(T_i) - E_{Duff}(\omega_i) & = 0 \quad i = 1, ..., n. \end{cases} \tag{12}$$

Analytical Computation

The equation of motion of the Duffing oscillator is :

$$m_i\ddot{x}_i + k_ix_i + k_{nl_i}x_i^3 = 0 \tag{13}$$

Considering the approximated motion $x_i(t) = A_i \cos \omega t$ and taking into account trigonometry transformations, the averaging over the fundamental frequency yields:

$$A_i(\frac{3}{4}k_{nl_i}A_i^2 - \omega^2 m_i + k_i) = 0 \tag{14}$$

The related solutions are given by :

$$\begin{aligned} A_{i_1} &= 0 \\ A_{i_{2,3}} &= \pm\sqrt{\frac{4(\omega^2 m_i - k_i)}{3k_{nl_i}}} \end{aligned} \tag{15}$$

At initial time $t = 0$, the motion is expressed by $x_i(0) = A_i$ and the total energy in the system consists in the potential energy stored in the nonlinear stiffness. Moreover, taking into account relation (15), one gets :

$$E_i(\omega) = \frac{1}{4}k_{nl_i}A_i^4 + \frac{1}{2}k_{nl_i}A_i^2 = \frac{4}{9k_{nl_i}}\left(m_i^2\omega^4 - \frac{1}{2}k_i(m_i\omega^2 + k_i)\right) \tag{16}$$

Because the tuning condition imposes $E_1(\omega) = E_2(\omega)$, the terms identification with respect to the frequency yields the following relations :

$$\begin{aligned} \frac{m_1^2}{k_{nl_1}} &= \frac{m_2^2}{k_{nl_2}} \\ \frac{k_1m_1}{k_{nl_1}} &= \frac{k_2m_2}{k_{nl_2}} \\ \frac{k_1^2}{k_{nl_1}} &= \frac{k_2^2}{k_{nl_2}} \end{aligned} \tag{17}$$

Parameter	Units	Value
m_1	$[kg]$	1
k_{nl_1}	$[N/m^3]$	4
k_1	$[N/m]$	1
m_2	$[kg]$	0.05
k_{nl_2}	$[N/m^3]$	0.01
k_2	$[N/m]$	0.05

Table 2: Parameter values of the conservative 2DOF system.

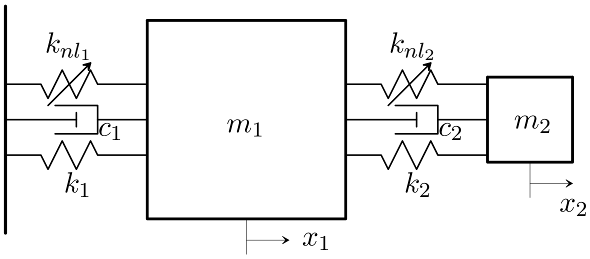

Figure 9: Duffing oscillator coupled to an absorber 'mirror'.

4.2 Performance - Robustness Analysis

The 2DOF system composed of the Duffing oscillator and the absorber is depicted in figure (9) and the related equations of motion are :

$$\begin{aligned} m_1\ddot{x}_1 + c_1\dot{x}_1 + c_2(\dot{x}_1 - \dot{x}_2) + k_1 x_1 + k_{nl_1} x_1^3 + k_{nl_2}(x_1 - x_2)^3 + k_2(x_1 - x_2) &= 0, \\ m_2\ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) + k_{nl_2}(x_2 - x_1)^3 &= 0. \end{aligned} \tag{18}$$

For given values of the primary system (m_1,k_1,k_{nl1}) the absorber parameters (m_2,k_2,k_{nl2}) can be assessed. For practical reasons, the absorber mass is imposed to be small $m_2 = 0.05[kg]$ and using relations (17), the other parameters are computed. All the parameters values are gathered in Table (2). For the coming analysis, some small damping ($c_1 = c_2 = 0.002[Ns/m]$) is introduced in both oscillators to induce energy dissipation.

Similarly to section (3.4), a three-dimensional plot of energy dissipated in the absorber (equation (4)) against the nonlinear stiffness of the Duffing oscillator and the impulse magnitude is numerically computed and illustrated in figure (10).

For small excitation amplitudes ($\dot{x}_1(0) < 0.4[m/s]$) the related amount of energy injected in the structure is very small and the nonlinear elements have no influence on the dynamics. Therefore the 2DOF system depicted in figure (9) reduces to the underlying 2DOF linear system composed of a LO coupled to a TMD and the dynamical behavior is linear. This feature explains the presence of a high energy dissipation region whatever the value of the nonlinear stiffness of the primary structure k_{nl_1}.

For increasing excitation amplitudes ($\dot{x}_1(0) > 0.4[m/s]$), the influence of the nonlinear elements on the overall dynamics cannot be neglected anymore. Progressively, the dynamical behavior is mainly governed by the nonlinearities of both oscillators and the system tends toward a purely nonlinear 2DOF system. Therefore, the conclusions established in section (3.4) can be extended to this analysis :

1. Amplitude robustness : whatever the amplitude of excitation the level of energy dissipated in the absorber keeps around 95% for a given nonlinear stiffness of the Duffing oscillator k_{nl_1}.

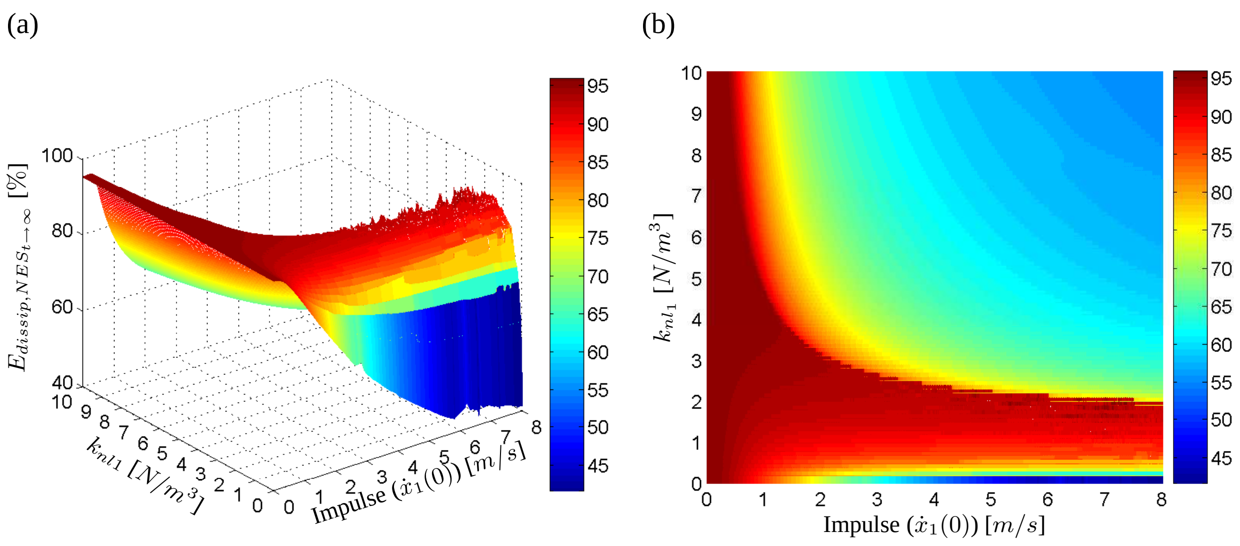

Figure 10: Absorber Performance when coupled to a Duffing oscillator. (a) Energy dissipated in the absorber against the nonlinear stiffness of the primary system and the impulse magnitude; (b) contour plot

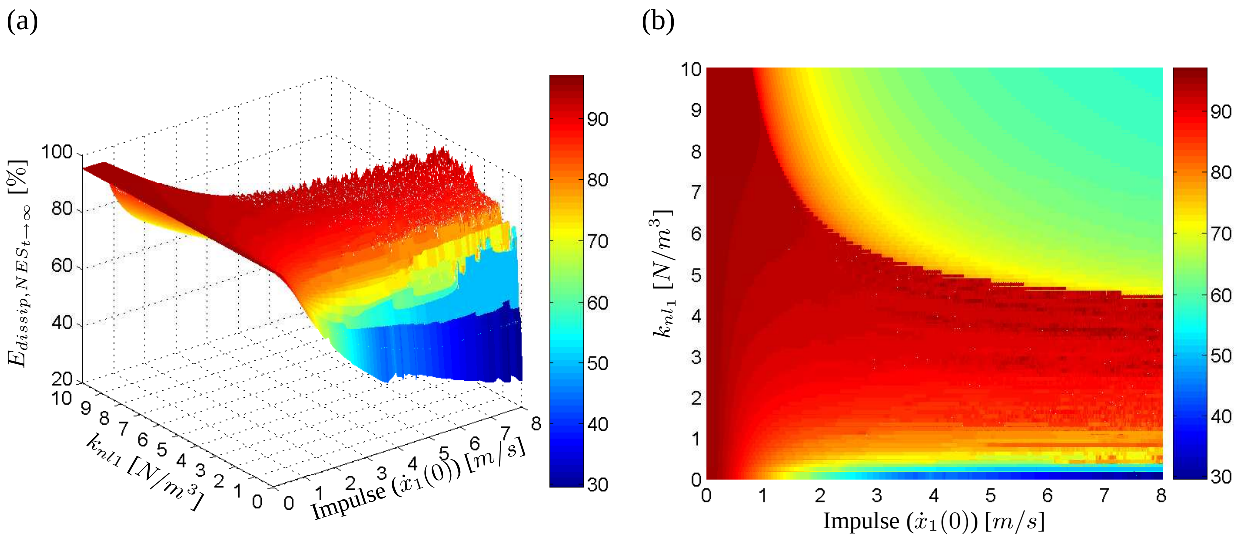

Figure 11: New absorber configuration $k_{nl_2} = 0.025[N/m^3]$. (a) Energy dissipated in the absorber against the nonlinear stiffness of the primary system and the impulse magnitude; (b) contour plot

2. Mistuning robustness (frequency robustness) : the region of high energy dissipation is not localized to a particular value of k_{nl_1} but spread over the range $k_{nl_1} = [0.8 - 2][N/m^3]$.

3. Despite the tuning of the NES on the Duffing oscillator, the nominal value of nonlinear stiffness of the primary structure, $k_{nl_1} = 4[N/m^3]$, is not included in the region of high energy dissipation level.

For the same reason as the one mentioned in section (3.4) the absorber nonlinear stiffness is adjusted to $k_{nl_2} = 0.025[N/m^3]$ and figure (11) shows improved results as the region of energy dissipation includes the nominal value of $k_{nl_1} = 4[N/m^3]$.

Finally, compared to the study carried out in [17], the dynamics created by this new absorber revealed an further improved amplitude robustness.

5 Conclusions and Future Work

In this paper, a design methodology of a nonlinear absorber was introduced. Based upon the design procedure of a TMD on a LO, a valuable dual tuning procedure of an NES on a purely nonlinear Duffing oscillator was achieved using the FEP concept. More generally, given a primary structure composed of both linear and nonlinear stiffness elements, the best absorber possible was revealed to be the mirror of the primary system from a structural viewpoint. As a final result, in addition to the frequency robustness, the new design scheme drastically improved the amplitude robustness of the absorber which makes it very efficient for passive vibration mitigation of nonlinear vibrating structures.

A complete study of this tuning procedure requires the development of complementary analyzes such as the mechanisms of energy dissipation or the effects of viscous damping. This will be investigated in subsequent works.

Moreover, among the next objectives, the design of a practical NES is of high interest for giving the possibility of performing experimental validation. Some preliminary works investigate the possibility to realize it electrically using piezoelectric patches and nonlinear shunts.

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