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COMPARATIVE ASSESSMENT OF OLD AND NEW SUBOPTIMAL CONTROL SCHEMES ON THREE EXAMPLE PROCESSES

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Abstract: This paper is concerned with methods for suboptimal control of nonlinear systems. A new approach to the synthesis of approximate optimal controllers is presented. The new methodology is compared to various known techniques on three complex examples systems that correspond to practical processes. The strengths and weaknesses of the different methods are pointed out.

Keywords: Nonlinear control, optimal control, suboptimal control, nonlinearity

1. INTRODUCTION

Optimal control is an attractive approach to control design for general nonlinear systems. Although there is extensive literature on the theory of optimal control, it is very difficult, if not impossible, to obtain the optimal feedback law for real world processes. Therefore many methods to calculate approximations of the optimal feedback law have been proposed over the last four decades.

The first class of approximate techniques consists of approaches that seek control laws that approximately satisfy the Hamilton-Jacobi- Bellman equation (HJBE). With power series expansion techniques truncated power series of the control law can be obtained either as a power series of the state vector of the plant (Al'Brekht, 1961; Lukes, 1969; Garrard and Jordan, 1977), or as a power series of a scalar perturbation parameter (Garrard, 1972; Nishikawa *et al.*, 1971). A different approach is taken in (Beard *et al.*, 1997), where a technique is developed for successive improvement of approximations of solutions to the HJBE based on projection methods. The method of state-dependent Riccati equations parallels the technique for linear systems to find suboptimal control laws (Wernli and Cook, 1975; Cloutier *et al.*, 1996). Further methods include state and control discretization in order to use discrete dynamic programming techniques, reinforcement learning, approaches via piecewise linear control, feedback linearization, regularization and inverse optimality techniques.

Instead of searching approximate solutions for the HJBE one can numerically compute optimal trajectories for a finite number of initial conditions and then construct the optimal feedback by inter-

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polation, for example by triangulation, neural nets or Green's functions (Ito and Schroeter, 1998).

In this work, a technique is presented that belongs to the second class, i.e. to the interpolation techniques. The approximate control law is identified by a technique stemming from nonlinearity quantification techniques (Schweickhardt and Allgöwer, 2004*a*). The purpose of this work is to display the capabilities of the different methods and to evaluate the benefits of the novel approach regarding former techniques. A first comparative study of suboptimal schemes was achieved in (Beeler *et al.*, 2000). It involves, however, less methods and considers test problems that present only polynomial nonlinearities.

This paper is organized as follows. In Section 2 the control problem is stated. In Section 3 we briefly review some known suboptimal control methods. In Section 3.5 we introduce a new class of methods based on nonlinearity measures. In Section 4 we present and discuss the simulation results for three example systems that correspond to real world applications. The paper ends with conclusions in Section 5.

2. PROBLEM STATEMENT

For a nonlinear plant $\dot{x} = f(x,u)$ with the state $x \in R^n$ and the control input $u \in R^p$ the control law $u = k(x)$ is sought that minimizes the performance functional $J_{x_0}(u) = \int_0^\infty G(x,u)\mathrm{d}t$ for any initial condition x_0 in a specified region of R^n. We assume that the problem is well-posed in the sense that for each x_0 an optimal control trajectory exists and is unique. Then, the optimal control policy depends on the current state of the plant $x(t)$ only and can be expressed as a function of the state $u(t) = k(x(t))$. We furthermore assume that this resulting control law $k(x)$ is continuous in x, although this may be a significant restriction.

3. CONSIDERED METHODS FOR NONLINEAR SUBOPTIMAL CONTROL

3.1 Methods based on power series expansions of the HJBE

These methods construct a control law as a power series of the plant's state vector. In Al'Brekht's and Lukes' approach (Al'Brekht, 1961; Lukes, 1969), the solution of the HJBE (i.e the cost function), the optimal control $u^*(x)$ as well as the system function driven by the optimal control $f^*(x) = f(x, u^*(x))$ are expanded in power series of the states. Substituting these expansions into the HJBE and setting the sums of coefficients of like powers to zero yields a set of equations that contains one algebraic Riccati equation (ARE) and many linear equations. The ARE gives the second order term of the cost function and the first order term of the optimal control law, which corresponds to the LQR controller of the linearized problem at the origin. The linear equations render higher order terms. Garrard and Jordan (1977) simplified this technique in case of input-affine systems and quadratic performance indices.

Other approaches introduce a scalar perturbation parameter ε into the system equations and construct a control law as a power series of ε (Garrard, 1972; Nishikawa *et al.*, 1971). The term corresponding to ε^0 requires the solution of an ARE and is again equivalent to the LQR controller of the linearized system. The second order term is easily computed by simple matrix operations. Higher order terms are, however, difficult to compute unless the system equation is polynomial. In this case these methods become equivalent to that of Al'Brekht and Lukes.

All these power series expansions are asymptotically stable in a neighborhood of the origin whatever the order of truncation.

3.2 Methods based on the state-dependent Riccati equation

These methods are a heuristic generalization of the LQR theory. In case the system equation and performance index are factorized to a linear-like structure with state-dependent coefficients,

$$\dot{x} = A(x)x + B(x)u \qquad \text{and}$$
$$J_{x_0}(u) = \int_0^\infty (x^T Q(x)x + u^T R(x)u)dt,$$

a state-feedback control can be obtained by $u(x) = -R^{-1}(x)B^T(x)P(x)x$, where $P(x)$ is the unique symmetric positive-definite solution of the state-dependent Riccati equation (SDRE)

$$A(x)^T P(x) + P(x)A(x) + Q(x) \\ -P(x)B(x)R(x)^{-1}B(x)^T P(x) = 0.$$

Two options are found in literature that use the SDRE approach. For the first method, a perturbation parameter ε is introduced into the system equation and the solution $P(x)$ is constructed as a power series in ε (e.g. (Wernli and Cook, 1975)). The second option, the Frozen Riccati Equation (FRE) method, is found in many recent applications (Cloutier *et al.*, 1996). For a fixed sampling interval, the SDRE is solved numerically at each sampling time, where x is taken to be equal to the current state of the process. The solvability of the ARE at each current state assures the local stability of the resulting controller. As the online computational effort considerably dominates the off-line computational effort, the FRE method is, in fact, more of an online control scheme (like model-predictive control) than a method to compute the optimal control law.

3.3 Successive Galerkin approximation

Beard (Beard *et al.*, 1998) transforms the solution of the HJBE into an iterative process that starts from an initial stabilizing control law the performance of which is improved at each iteration. The difficulty of the solution of the HJBE is reduced to the solution at each iteration of the so-called generalized Hamilton-Jacobi-Bellman equation (GHJBE), which is a first order linear partial differential equation (with non-constant coefficients). A technique based on Galerkin's spectral method is used to solve this equation. Its solution is expanded in a set of basis functions with unknown coefficients and plugged into the GHJBE. The coefficients of the expansion are computed from a set of linear equations that are obtained by setting the error projected onto the basis functions to zero. Some conditions assuring a uniform convergence toward the optimal control are derived in (Beard *et al.*, 1997).

3.4 Interpolation via Green's functions

The idea of the interpolation techniques is to avoid the solution of the HJBE by computing optimal trajectories for several initial states and constructing afterwards a feedback control based on these data. Ito and Schroeter propose an interpolation technique based on Green's functions (Ito and Schroeter, 1998). First, assume the optimal state trajectories $x^*_{x_0}(\cdot)$ and control trajectories $u^*_{x_0}(\cdot)$ are computed for all x_0 in a finite set of initial conditions X_0. Then, a finite set Σ of M couples $\{x^{(i)}, \bar{u}(x^{(i)})\}$ is built up, where $\bar{u}(x^*_{x_0}(t)) = u^*_{x_0}(t)$ is the optimal control corresponding to the particular state $x^*_{x_0}(t)$ at time t. The approximate feedback is defined by

$$u(\cdot) = \sum_{i=1}^{M} G(\cdot, x^{(i)})\eta_i, \qquad (1)$$

where the Green's functions are of the type $G(x, y) = |x-y|^\alpha$ with $1 \leq \alpha \leq 4$. The coefficients $\{\eta_i\}_{i=1}^{M}$ are chosen so that the feedback (1) corresponds to the optimal control at the collocation points $u(x) = \bar{u}(x) \quad \text{for } \{x, \bar{u}(x)\} \in \Sigma$.

3.5 New interpolation methods based on nonlinearity measures

The technique described in this section is based on a generalization of the *closed-loop optimal control law nonlinearity measure* (Schweickhardt and Allgöwer, 2004*b*). This measure represents the relative prediction error of the best linear approximation of the optimal controller. The approximation procedure can be generalized to a more general class of approximating control structures by defining the *closed-loop optimal control law suitability measure for a controller structure* $K_d(x)$

$$\phi^{X_0}_{K_d} \stackrel{\text{def}}{=} \inf_{d \in R^D} \sup_{x_0 \in X_0} \frac{\left\| u^*_{x_0} - K_d\left(x^*_{x_0}\right)\right\|_{L_2}}{\left\| u^*_{x_0}\right\|_{L_2}},$$

with $u^*_{x_0}$ and $x^*_{x_0}$ being the optimal control and state trajectories for the initial condition $x_0 \in X_0$. If ϕ^{X_0} is small, the controller for which the minimum is achieved represents a good approximation to the behavior of the optimal controller for the considered optimal trajectories. The measure is bounded above by one. The suboptimal controller $K_d(x)$ is parameterized by D real parameters. It can be shown that the minimization problem for d has a solution (that does not need to be unique). Furthermore, this optimization problem is convex, if the parameterization of the suboptimal control is linear. One first advantage to this interpolation method is the contribution of the entire trajectories to the controller synthesis and not only of a finite number of points as in the previous method.

Three possible structures for K_d will be used in the sequel. First, a power series expansion in the state x

$$K_d(x) = d_0 + D_1 x + \left(\sum_{i=1}^{n} D_{2,i} x_i\right) x + \ldots (2)$$

is a possible linear parameterization, where d contains the parameters $d_0 \in R^m$, $D_1 \in R^{m \times n}$, $D_2 \in R^{m \times n \times n}$ and so on. Unlike the controllers of Section 3.1, the law (2) is not local optimal, but represents a better approximation on a larger domain about the origin.

A second possibility is the truncated Fourier series

$$K_d(x) = \sum_{i_1=-N_1}^{N_1} \cdots \sum_{i_n=-N_n}^{N_n} c_{i_1,\ldots i_n} e^{i\pi \sum_{k=1}^{n} i_k \frac{x_k}{l_k}}, \qquad (3)$$

where N_k is the order of approximation in the k-th dimension of the state space and l_k denotes the maximal deviation from steady state in the same direction. The complex coefficients c have to be chosen so that $c_{i_1,\ldots,i_j,\ldots,i_n} = \bar{c}_{i_1,\ldots,-i_j,\ldots,i_n}$ for any j in order to guarantee the control variable to be real valued. If one of the design parameters N_k is set to 0, then the k-th direction is represented by a constant only and the control law is independent of the corresponding state variable. This fact can be used to synthesize partial state feedback controllers.

The third suggested structure assumes an input-affine system $\dot{x} = f(x)+g(x)u(x)$ and a index that is quadratic in the control $J_{x_0}(u) = \int_0^\infty (l(x) + u^T Ru)dt$. Then, the optimal control is explicitly related to the cost function $V(x)$ by $u(x) = -\frac{1}{2}R^{-1}g^T(x)\frac{\partial V}{\partial x}(x)$. and the controller structure can be obtained indirectly by parameterization of the cost function $V(x)$. Since $V(x)$ is a positive definite function, it is a natural idea to expand $V(x)$ in a sum of squares (SOS)

$$V(x) = \sum_{i=1}^{N}(p_i(x))^2 = Z^T QZ, \qquad (4)$$

where $p_i(x)$ is a polynomial in x of order i, Z the vector of monomials up to a prescribed order and Q a positive semi-definite matrix.

4. COMPARATIVE ASSESSMENT

These methods were all applied on three real world problems. For sake of convenience, the following abbreviations will be used:

- LQR, Linear Quadratic Regulator,
- LUK, HJBE and power series expansion,
- SDR, SDRE and power series expansion,
- FRE, Frozen Riccati equation,
- SGA, Successive Galerkin approximation,
- GRE, Interpolation via Green's functions,
- IPS, Interpolation by Power series (2),
- IFS, Interpolation by Fourier series (3),
- SOS, Interpolation by a control structure, whose cost function is a sum of squares (4).

Each method has to compute the most efficient control law within a reasonable computation time. The closed loops corresponding to each controller are simulated from $N = 30$ random initial states in order to get representative results. The following comparison criteria will be considered:

- the average distance from optimality δ, defined by $\delta = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{J_{x_i}(\tilde{u})-J_{x_i}(u*)}{J_{x_i}(u*)}\right)$, where $J_{x_i}(\tilde{u})$ is the performance of an approximate controller for an initial state x_i and $J_{x_i}(u*)$ is the performance of the optimal controller for the same initial state,
- t_1, the time required for the computation of the approximate controller,
- t_2, the average time required by the ODE solver to compute the closed-loop trajectories.

δ is a measure of the quality of the approximation, t_1 is the off-line computation time and t_2 allows to assess the online computational effort. The computation of the optimal open-loop trajectories for a given initial state was done by solving numerically a two-point boundary value problem. Finally, the reader should take heed that the continuity assumption of the optimal control could be proven for none of the problems.

4.1 Continuous stirred tank reactor

The first problem deals with the regulation of a continuous stirred tank reactor (CSTR), for details on the model see (Chen *et al.*, 1995). The model is of fourth order with the state representing two concentrations and two temperatures. Two control variables are available: the input flow rate and the heat removed by the cooler. The performance objective considered here is to keep the product concentration as close to its optimal value as possible, so the cost functional is chosen to be quadratic $J = \int_0^\infty (\bar{x}^T \mathrm{Q}\bar{x} + \bar{u}^T \mathrm{R}\bar{u})\,\mathrm{d}t$ with $Q = \mathrm{diag}\{1, 100, 0, 0\}$ and $R = \mathrm{diag}\{0.01, 0.001\}$ and where $\bar{x}$ and $\bar{u}$ denote the deviation from steady state of the state and the control, respectively. The simulation results are summarized in Table 1. It can be observed that the LQR controller leads to poor performance, while already a second order expansion of the control law gives very satisfactory results. A disadvantage of the LUK method is however that it does not present a uniform convergence toward the optimal control, the best approximation being given by the second order in this case (see Fig. 1).

Table 1. Results for the CSTR example.

Method	δ [%]	t_1 [sec]	t_2 [sec]
LQR	124.90	14	12.09
LUK - $\mathcal{O}(\mathbf{x}^2)$	3.01	16	12.39
SDR - $\mathcal{O}(\varepsilon^0)$	4.56	66	12.83
SGA - 10 basis fct.	7.42	86	12.80
GRE - 64 int. points	1.62	163	20.67
$\mathrm{IFS}^{(1)}$ - 62 param.	5.82	10800	13.47
$\mathrm{IFS}^{(2)}$ - 34 param.	1.09	7778	13.57
$\mathrm{IPS}^{(1)}$ - 8 param.	48.48	148	12.90
$\mathrm{IPS}^{(2)}$ - 28 param.	1.13	1538	13.43
$\mathrm{IPS}^{(3)}$ - 68 param.	0.36	6830	13.42
SOS - 10 param.	0.87	465	13.65

The SDR method was limited to the zero-order controller, as higher orders were computationally not tractable. Note that an ARE had to be solved, as for the LQR method, but with a different system matrix. The FRE method failed because no state-dependent factorization could be found that is well-defined everywhere in the domain of interest of the state space. The SGA considers only 10 basis functions since a bigger basis set lead to ill-conditioned problems.

The Green's function interpolation method (GRE) is a very good compromise between off-line computation time and achieved performance. The dif-

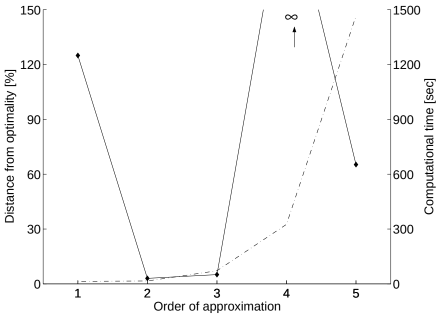

Fig. 1. Non-uniform convergence of the LUK controllers (δ: solid line ; t_1: dashed line).

ferent controller structures used by the interpolation method based on the suitability measure are detailed in Table 2. It should be mentioned

Table 2. Controller structures, ϕ being the resulting suitability measure.

	Description	ϕ
$\mathrm{IFS}^{(1)}$	$N_1 = N_2 = N_3 = N_4 = 1$ in (3)	0.3278
$\mathrm{IFS}^{(2)}$	$N_1 = N_2 = 2\,;\ N_3 = N_4 = 0$ in (3)	0.2310
$\mathrm{IPS}^{(1)}$	First order polynomial	0.6726
$\mathrm{IPS}^{(2)}$	Second order polynomial	0.1031
$\mathrm{IPS}^{(3)}$	Third order polynomial	0.0675
SOS	$V(x)$ Second order polynomial	0.1330

that the $\mathrm{IFS}^{(2)}$ is a partial state feedback, which needs the measurement of the two concentrations only. Some of these methods belong to the best in terms of performance, but in view of large computation times, only the SOS method seems to be attractive. By comparing the results, it can be seen that the suitability measure ϕ correlates with the relative performance of the methods.

4.2 Satellite

The second application considers the regulation of the altitude of a spacecraft. The plant is of sixth order and has three inputs. Since the open-loop system oscillates about the origin, the regulator has to asymptotically stabilize the body toward the origin while minimizing a given performance index. Details about the modeling and the considered performance index can be found in (Lawton and Beard, 1999). Table 3 collects the average simulation results. Already the LQR solution gives

Table 3. Results for the satellite.

Method	δ [%]	t_1 [sec]	t_2 [sec]
LQR	7.631	8.9	0.82
LUK - $\mathcal{O}(\mathbf{x}^3)$	3.395	96.0	0.83
SDR - $\mathcal{O}(\varepsilon^2)$	24.914	91.0	1.03
FRE	8.127	1.3	657.00
SGA - 30 basis fct.	8.617	1602.0	0.93
GRE - 320 int. points	3.131	494.0	3.29
IPS - 249 param.	3.348	6039.0	0.81
SOS - 21 param.	6.221	1369.0	0.87

good results. The best feasible approximation resulting from the LUK technique, given by the third order in this case, improves the LQR only slightly. The SDR methods is the worst for this example. The SGA method uses the basis functions proposed in (Lawton and Beard, 1999). In view of the required computational time, its performance is not satisfactory. The FRE method gives a similar result with a much lower off-line computational effort. The online computational effort however is considerable and might be prohibitive for some applications. All interpolation techniques except the IFS lead to good or very good results, but taking the computational effort into account, the GRE method has a clear advantage. For the IFS method, no controller was found that stabilizes the closed loop for all considered initial conditions.

4.3 Underwater robot

This last application considers the station-keeping of an underwater robotic vehicle in the horizontal plane. The vehicle dynamics are derived in (McLain and Beard, 1998). The model has six states and three control variables. The chosen performance index is quadratic with the following diagonal weighting matrices : $Q = 5000 \cdot \mathrm{diag}\{1, 1, 0, 1, 1, 1\}$ and $R = \mathrm{diag}\{1, 1, 1\}$. All simulation results are given on Table 4. Since the

Table 4. Results for the robot example.

Method	δ [%]	t_1 [sec]	t_2 [sec]
LQR	14.66	1.3	2.75
SGA - 7 basis fct.	32.06	36.2	2.02
FRE	22.30	0.4	2042.00
GRE - 384 int. points	1.91	249.5	14.60
IPS - 249 param.	8.52	11130.0	1.58

plant function is only once continuously differentiable at the origin, the LUK and SDR are restricted to the first order of approximation, i.e to the LQR controller. The SGA uses the basis set proposed in (McLain and Beard, 1998). The IFS was again not able to compute a satisfying controller within a reasonable computation time. The IPS was computed until the third order. Exemplary state trajectories are depicted in Fig. 2. It can be seen that higher order controllers assure a better damping. This system is still a challenge for control engineers. In fact, only the GRE synthesizes a very effective controller, but with a complex structure, as indicates the amount of online computational time t_2.

5. CONCLUSION

Different approaches towards the synthesis of approximate optimal control laws were investigated in this work. The examples show that none of them is an ideal one. The local LQR controller

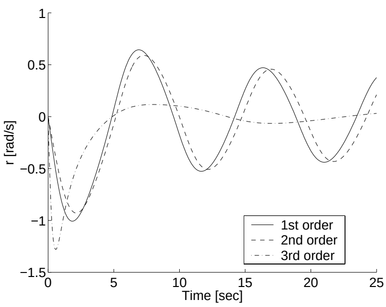

Fig. 2. Exemplary state trajectories of the underwater robot, with a IPS-based controller.

sometimes gives very good results, but sometimes also fails. The same is true for higher order local approximations (LUK), although they can be able to improve the LQR results significantly. The explicit state-dependent Riccati method (SDR) and the successive Galerkin method (SGA) rendered rather unsatisfactory results for the considered examples. Just those methods, for which only little theoretical results are available, turn out to be useful in practice. This is true for the interpolation and the frozen Riccati equation (FRE) approach. The GRE technique requires the computation of many open-loop trajectories and results in a complex control law, but a close approximation of the optimal control is obtained. For each example system there is at least one of the interpolation methods based on the suitability measure that achieves very good performance. Furthermore, these methods are easily implementable, give a measure of the quality of the approximation and can synthesize partial state feedback as well as static output feedback. However, the extensive computation time prevents these methods from being the first choice. The FRE approach requires many online computations and can be far from optimality, but it is very practical in applications. Of all methods in the test it was the one with the highest online and the lowest off-line computational burden. So examples two and three of this study favor the "MPC-way of thinking", where one also tries to reduce off-line computational cost and memory requirements for data storage at the cost of more involved online computations.

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