Frequency Domain Approaches for the Identification of an Experimental Beam with a Local Non-linearity.

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ABSTRACT

Two approaches for the identification of multi-degree-of-freedom non-linear systems in the frequency domain have recently been introduced, i.e., the conditioned reverse path (CRP) method and the non-linear identification through feedback of the outputs (NIFO) method. The key idea of these methods is to eliminate the distortions caused by the presence of non-linearities in frequency response functions. In this paper, the theoretical background of the CRP and NIFO methods is briefly recalled. Then, the ability of these techniques to identify the behaviour of an experimental cantilever beam with a local geometrical non-linearity is tested. The results obtained with both methods are compared and discussed.

1 INTRODUCTION

The importance of diagnosing, identifying and modelling non-linearity has been recognised for a long time but it is only recently that non-linear theory is beginning to be applied for structural dynamic design. The identification of non-linear systems began with the study of single-degree-of-freedom (s.d.o.f.) systems. Since the reference paper of Masri et al. in 1979 [1], techniques which can consider multi-degree-of-freedom (m.d.o.f.) systems were introduced, e.g. the Hilbert transform [2], NARMAX models [3, 4] and Volterra series [5]. However, it appeared quickly that these techniques are not suitable for systems with high modal density. For a detailed review of the past years, the reader is referred to [6, 7]. Progress in the treatment of m.d.o.f. systems has been realised recently and can be attributed to a confluence of new methods of analysis and to the expansion of computer processor power.

Proper orthogonal decomposition (POD), also known as principal component analysis or Karhunen-Loève transform, has shown promise for model updating of structural parameters in m.d.o.f. non-linear systems [8, 9]. The method is based on the solution of an optimisation problem which consists in minimising the difference between the bi-orthogonal decompositions of the measured and simulated data respectively.

An other approach using a model based in modal space has been proposed by Wright et al. [10]. The basic philosophy of this approach aims at performing a multi-stage identification by considering a single mode or a group of modes instead of treating all the modes of the structure in a single step. For this purpose, it combines the resonant decay method, the force appropriation and the restoring force surface method.

The development of frequency response function-based approaches has received increasing attention in the last ten years. In this paper, it is proposed to study by means of the conditioned reverse path (CRP) [11] and non-linear identification through feedback of the outputs (NIFO) [12] methods a continuous non-linear system consisting of an experimental cantilever beam with a geometrical non-linearity.

2 CONDITIONED REVERSE PATH METHOD

The concept of a reverse path model was introduced by Bendat [13, 14] and adapted to m.d.o.f. systems by Rice and Fitzpatrick [15]. However, this technique requires excitation at every response location. The CRP formulation [11] extends the application of the reverse path algorithm to systems characterised by non-linearities away from the location of the applied force. This method has been developed by generalising the concepts introduced by Bendat [13, 14].

In the presence of non-linear forces, the classical H_1 and H_2 estimators [16] should not be used because the non-linearities corrupt the underlying linear characteristics of the response. In the CRP method, spectral conditioning techniques are exploited to remove the effects of non-linearities. Conditioned frequency responses are computed and yield the underlying linear properties without influence of non-linearities. The non-linear coefficients are identified in a second step.

The vibrations of a general non-linear system are governed by equation

$$\mathbf{M}\ddot{\mathbf{x}}(t)+\mathbf{C}\dot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)+\sum_{j=1}^{n}\mathbf{A}_j\mathbf{y}_j(t)=\mathbf{f}(t) \qquad (1)$$

where $\mathbf{y}_j(t)$ is a non-linear function vector and $\mathbf{A}_j$ contains the coefficients of the non-linear terms $\mathbf{y}_j(t)$. In the frequency domain, equation (1) becomes

$$\mathbf{B}(\omega)\mathbf{X}(\omega)+\sum_{j=1}^{n}\mathbf{A}_j\mathbf{Y}_j(\omega)=\mathbf{F}(\omega) \qquad (2)$$

where $\mathbf{X}(\omega),\mathbf{Y}_j(\omega)$ and $\mathbf{F}(\omega)$ are the Fourier transforms of $\mathbf{x}(t),\mathbf{y}_j(t)$ and $\mathbf{f}(t)$ and $\mathbf{B}(\omega)=-\omega^2\mathbf{M}+i\omega\mathbf{C}+\mathbf{K}$ is the linear dynamic stiffness matrix.

2.1 Estimation of the underlying linear system properties

The key idea of the CRP formulation is the separation of the non-linear part of the system response from the linear part and the construction of uncorrelated response components in the frequency domain.

The spectra of the measured responses $\mathbf{X}$ can be decomposed into a component which is correlated with the spectrum of the first non-linear vector $\mathbf{Y}_1$, denoted by $\mathbf{X}_{(+1)}$, through a frequency response matrix $\mathbf{L}_{1X}$, and a component which is uncorrelated with the spectrum of the non-linear vector, denoted by $\mathbf{X}_{(-1)}$. The spectral component $\mathbf{X}_{(-1:n)}$ is the component of the response uncorrelated with the spectra of all n non-linear function vectors and may be viewed as the response of the underlying linear system

$$\mathbf{X}_{(-1:n)}=\mathbf{X}-\sum_{j=1}^{n}\mathbf{X}_{(+j)}=\mathbf{X}-\sum_{j=1}^{n}\mathbf{L}_{jX}\mathbf{Y}_{j(-1:j-1)} \qquad (3)$$

It can be shown [11] that the path between $\mathbf{X}_{(-1:n)}$, i.e., the response of the underlying linear system, and $\mathbf{F}_{(-1:n)}$, i.e., the part of the force uncorrelated with the non-linear response, is the linear dynamic stiffness matrix $\mathbf{B}$

$$\mathbf{F}_{(-1:n)}(\omega)=\mathbf{B}(\omega)\mathbf{X}_{(-1:n)}(\omega) \qquad (4)$$

By transposing equation (4), pre-multiplying by the complex conjugate of $\mathbf{X}$, noted $\mathbf{X}^*$, taking the expectation and finally multiplying by $2/T$, the underlying linear system may be identified without corruption from the non-linear terms

$$\mathbf{G}_{XF(-1:n)}=\frac{2}{T}E[\mathbf{X}^*\mathbf{F}^T_{(-1:n)}]=\frac{2}{T}E[\mathbf{X}^*(\mathbf{B}\mathbf{X}_{(-1:n)})^T]$$

$$=\frac{2}{T}E[\mathbf{X}^*\mathbf{X}^T_{(-1:n)}\mathbf{B}^T]=\mathbf{G}_{XX(-1:n)}\mathbf{B}^T \qquad (5)$$

where $\mathbf{G}_{XF(-1:n)}$ and $\mathbf{G}_{XX(-1:n)}$ are conditioned power spectral density matrices.

For the dynamic compliance matrix $\mathbf{H}$,

$$H_{c2}:\mathbf{H}^T=\mathbf{G}^{-1}_{XF(-1:n)}\mathbf{G}_{XX(-1:n)} \qquad (6)$$

This expression is known as the conditioned H_{c2} estimate. If relation (4) is multiplied by the complex conjugate of $\mathbf{F}$, the conditioned H_{c1} is obtained

$$H_{c1}:\mathbf{H}^T=\mathbf{G}^{-1}_{FF(-1:n)}\mathbf{G}_{FX(-1:n)} \qquad (7)$$

2.2 Estimation of the non-linear coefficients

Once the linear dynamic compliance $\mathbf{H}$ is identified, the non-linear coefficients $\mathbf{A}_j$ may be estimated. Using the same procedure as for equation (5), the following relationship is obtained

$$\mathbf{G}_{iF(-1:i-1)}=\mathbf{G}_{iX(-1:i-1)}\mathbf{B}^T+\sum_{j=1}^{n}\mathbf{G}_{ij(-1:i-1)}\mathbf{A}_j^T \qquad (8)$$

It should be noted that $\mathbf{G}_{ij(-1:i-1)}=E\left[\mathbf{Y}^*_{i(-1:i-1)}\mathbf{Y}_j^T\right]=\mathbf{0}$ for $j<i$ since $\mathbf{Y}^*_{i(-1:i-1)}$ is uncorrelated with the spectra of the non-linear function vectors $\mathbf{Y}_1$ through $\mathbf{Y}_{i-1}$. If equation (8) is pre-multiplied by $\mathbf{G}^{-1}_{ii(-1:i-1)}$, the first term in the summation is $\mathbf{A}_i^T$. Finally,

$$\mathbf{A}_i^T=\mathbf{G}^{-1}_{ii(-1:i-1)}(\mathbf{G}_{iF(-1:i-1)}-\mathbf{G}_{iX(-1:i-1)}\mathbf{B}^T$$

$$-\sum_{j=i+1}^{n}\mathbf{G}_{ij(-1:i-1)}\mathbf{A}_j^T) \qquad (9)$$

The identification process starts with the computation of $\mathbf{A}_n$ working backwards to $\mathbf{A}_1$. At this stage, it is important to emphasise that the computed non-linear coefficients are frequency dependent. However, by taking the spectral mean, the actual value of the coefficients may be retrieved.

Conditioned power spectral density matrices [17] like $\mathbf{G}_{XF(-1:n)}$ may be obtained from

$$\mathbf{G}_{ij(-1:r)}=\mathbf{G}_{ij(-1:r-1)}-\mathbf{G}_{ir(-1:r-1)}\mathbf{L}_{rj}^T \qquad (10)$$

where

$$\mathbf{L}_{rj}^T=\mathbf{G}^{-1}_{rr(-1:r-1)}\mathbf{G}_{rj(-1:r-1)} \qquad (11)$$

2.3 Coherence functions

For linear systems, the ordinary coherence function is a means to assess the quality of transfer function estimates [16]. However, for a multiple input model with correlated inputs, the sum of ordinary coherences between the inputs and the output may be greater than unity. In reference [18], the concept of ordinary coherence function is replaced by the concept of cumulative coherence function γ^2_{Mi}

$$\gamma^2_{Mi}(\omega) = \gamma^2_{X_iF(-1:n)}(\omega) + \gamma^2_{YF}(\omega) =$$

$$\gamma^2_{X_iF(-1:n)}(\omega) + \sum_{j=1}^{n} \gamma^2_{jF(-1:j-1)}(\omega) \qquad (12)$$

- $\gamma^2_{X_iF(-1:n)}$ is the ordinary coherence function between the i^{th} element of $X_{(-1:n)}$ and excitation F

$$\gamma^2_{X_iF(-1:n)} = \frac{\left|G_{X_iF(-1:n)}\right|^2}{G_{X_iX_i(-1:n)}G_{FF}} \qquad (13)$$

and indicates the contribution from the linear spectral component of the response of the i^{th} output.

- $\gamma^2_{jF(-1:j-1)}$ is the ordinary coherence function between the conditioned spectrum $Y_{j(-1:j-1)}$ and excitation F

$$\gamma^2_{jF(-1:j-1)} = \frac{\left|G_{jF(-1:j-1)}\right|^2}{G_{jj_(-1:j-1)}G_{FF}} \qquad (14)$$

and $\sum_{j=1}^{n} \gamma^2_{jF(-1:j-1)}$ indicates the contribution from the non-linearities.

The cumulative coherence function is always between 0 and 1 and may be considered as a measure of the accuracy of the model.

3 NON-LINEAR IDENTIFICATION THROUGH FEEDBACK OF THE OUTPUTS METHOD

This technique exploits the spatial information and treats the non-linear forces as internal feedback forces in the underlying linear model of the system. The key advantage of this method lies in its ability to estimate the frequency response functions (FRFs) of the underlying linear system and the non-linear coefficients in a single step. This is carried out in a least-squares system of equations through averaging.

Let us write equation (2) in the following form

$$\mathbf{B}(\omega)\mathbf{X}(\omega) = \mathbf{F}(\omega) - \sum_{j=1}^{n} \mathbf{A}_j \mathbf{Y}_j(\omega) \qquad (15)$$

then the non-linear forces may be viewed as internal feedback forces. Pre-multiplying equation (15) by the dynamic compliance matrix $\mathbf{H}(\omega)$ yields

$$\mathbf{X}(\omega) = \mathbf{H}(\omega)\mathbf{F}(\omega) - \mathbf{H}(\omega)\sum_{j=1}^{n} \mathbf{A}_j \mathbf{Y}_j(\omega) \qquad (16)$$

and finally,

	Length (m)	Width (m)	Thickness (m)
Main beam	0.7	0.014	0.014
Thin beam	0.04	0.014	0.0005

TABLE 1: Geometrical properties of the set-up

$$\mathbf{X}(\omega) = \left[\mathbf{H}(\omega) \quad \mathbf{H}(\omega)\mathbf{A}_1 \quad ... \quad \mathbf{H}(\omega)\mathbf{A}_n\right] \begin{bmatrix} \mathbf{F}(\omega) \\ -\mathbf{Y}_1(\omega) \\ \vdots \\ -\mathbf{Y}_n(\omega) \end{bmatrix} \qquad (17)$$

If the excitation force $\mathbf{F}(\omega)$ and the system response $\mathbf{X}(\omega)$ are measured, and since the non-linear forces can be evaluated from the measured outputs (e.g. for a cubic stiffness $\mathbf{Y}_i(\omega) = \mathcal{F}[(x_m(t) - x_{m+1}(t))^3]$ where $\mathcal{F}$ stands for Fourier transform), the system described by equation (17) may be solved at each frequency. This allows us to compute the FRFs of the underlying linear system $\mathbf{H}(\omega)$ together with the non-linear coefficients $\mathbf{A}_i(\omega)$.

It is important to emphasise that equation (17) is not considered in this form to compute the parameters. A 'PSD version' of this equation obtained by using the same procedure as for equation (5) is preferred. The use of PSDs reduces the degree to which linearly correlated terms corrupt the conditioning of the data matrices. An orthogonal least-squares solution [20] is also used to reduce the level of ill-conditioning.

4 EXPERIMENTAL SET-UP

The experimental application involves a clamped beam with a thin beam part at the end of the main beam (cf. Figure 1). The geometrical properties of the set-up are listed in Table 1. This structure is similar to the benchmark proposed by the Ecole Centrale de Lyon (France) in the framework of COST Action F3 working group on "Identification of non-linear systems".

Figure 1: Experimental set-up.

Seven accelerometers which span regularly the beam are used to measure the response and in addition a displacement sensor is also located at the end of the beam, i.e., at position 7. The shaker, located at position 3, produces a white-noise sequence band-limited in the 0-500 Hz range. Different exci-

Excitation level (Nrms)	1st freq. (Hz)	2nd freq. (Hz)	3rd freq. (Hz)
1.4	30.74	139.47	390.22
2.8	30.99	139.49	390.16
5.5	31.60	139.64	390.22
11	33.13	140.29	390.17
16	34.81	140.84	390.34
22	36.80	141.80	390.54

TABLE 2: Natural frequencies (H_2 estimate)

tation levels are considered in the $1.4 - 22$ Nrms range.

Due to the thin beam part, the effect of gravity is not negligible. The static deflection of the two beams imposes a non negligible prestress in the thin beam part. In order to reduce its influence, a set-up was built in which the thin beam is vertical and the shaker excites the structure in a horizontal plane (Figure 2).

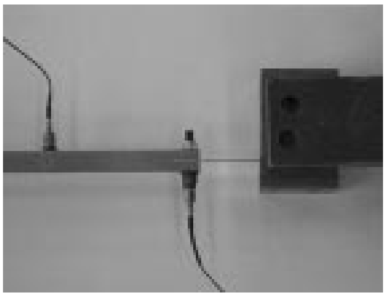

Figure 2: Experimental set-up, above view.

5 IDENTIFICATION RESULTS: H_2 ESTIMATE

In order to have some ideas about the influence of the non-linearity, the FRFs are first computed using the classical H_2 estimate. Figure 3 displays the magnitude of H_{73} for the lowest (1.4 Nrms) and highest (22 Nrms) excitation levels. The structure can be considered as linear for the lowest level. If the excitation level is increased, the thin part is excited in large deflection and a geometrical non-linearity is activated.

The natural frequencies were also estimated in the 0-500 Hz range using the least squares complex exponential (LSCE) method [16] and the results are summarised in Table 2. The first two natural frequencies are shifted towards higher frequencies when the excitation level is increased. This is due to the stiffening effect of the thin beam part. The third natural frequency does not seem to be affected by the presence of the non-linearity.

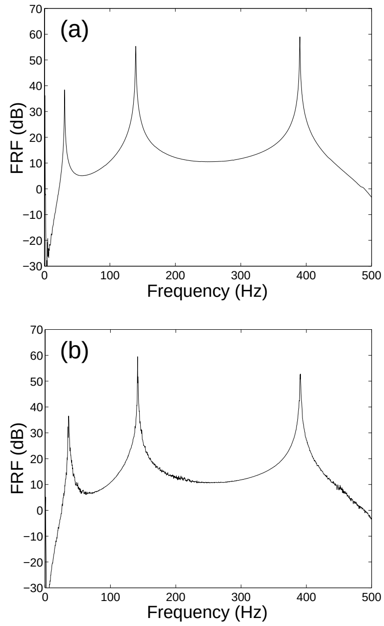

Figure 3: Magnitude of H_{73} (H_2 estimate). (a) 1.4 Nrms; (b) 22 Nrms.

6 IDENTIFICATION RESULTS: CRP METHOD

The first step in the identification procedure is the model selection. For this purpose, the cumulative coherence function γ^2_{Mi} as defined in section 2.3 is exploited. Similar work [19] was done using partial and multiple coherences defined by Bendat in [17]. To model the stiffening effect of the thin beam part, a grounded symmetrical non-linearity of type $|x|^\alpha\ sign(x)$ is introduced in the model at the end of the beam (location 7). Thus, the non-linearity is modelled as

$$f(x) = A\,|x|^\alpha\ sign(x) \qquad (18)$$

where x is the displacement at the end of the beam. Exponent α is determined by seeking the maximum value for the spectral mean of the averaged cumulative coherence of all the seven sensors

$$accuracy = \frac{1}{N}\sum_{\omega=10}^{500}\left(\frac{1}{7}\sum_{i=1}^{7}\gamma^2_{Mi}(\omega)\right) \qquad (19)$$

where N is the number of frequencies considered in the range from 10 to 500 Hz. The maximum value is found for $\alpha = 2.8$. and is equal to 0.9873.

Figure 4 represents the magnitude of H_{73} (H_{c2} estimate) for the 22 Nrms level. Figure 5 shows the FRFs obtained using the H_{c2} and H_2 estimates for the first two resonances (22 Nrms level). These FRFs are also compared with the *true* FRF, i.e., the FRF for the lowest level (1.4 Nrms level) for which the behaviour of the structure is linear. It can clearly be seen that the FRF computed by the H_{c2} estimate provides a close

Excitation level (Nrms)	1st freq. (Hz)	2nd freq. (Hz)	3rd freq. (Hz)
1.4	30.74	139.47	390.22
2.8	30.65	139.41	390.12
5.5	30.69	139.41	390.02
11	30.62	139.34	389.87
16	30.62	139.35	389.85
22	30.51	139.33	389.78

TABLE 3: Natural frequencies (H_{c2} estimate)

match to the *true* FRF while the FRF computed by the H_2 estimate is contaminated by the presence of the non-linearity. Table 3 gives the natural frequencies identified from the H_{c2} estimate for the different excitation levels. In comparison with Table 2, the frequencies are not shifted towards higher frequencies anymore and are now well estimated.

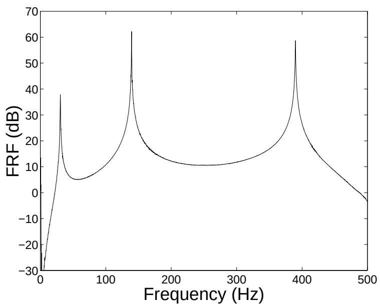

Figure 4: Magnitude of H_{73} (H_{c2} estimate), 22 Nrms.

The last step of the identification procedure is the computation of the non-linear coefficient. Figure 6 represents the real part of coefficient A and the spectral mean (10-250 Hz) of this coefficient is listed in Table 4. Aside from the 2.8 and 5.5 Nrms levels for which the non-linearity does not participate sufficiently in the system response, a stable value for the non-linear coefficient is identified. It is worthwhile noticing that the imaginary part of the coefficient, without any physical meaning, is several orders of magnitude below the real part.

7 IDENTIFICATION RESULTS: NIFO METHOD

The same model as for the CRP method is considered (see equation (18), with $\alpha = 2.8$). Figure 7 represents the FRFs obtained using the H_{c2} and H_2 estimates for the first two resonances (22 Nrms level). These FRFs are also compared with the *true* FRF. The FRF computed by the NIFO estimate closely matches the *true* FRF. However, the comparison of Figures 5 and 7 reveals that some slight distortions are introduced in the

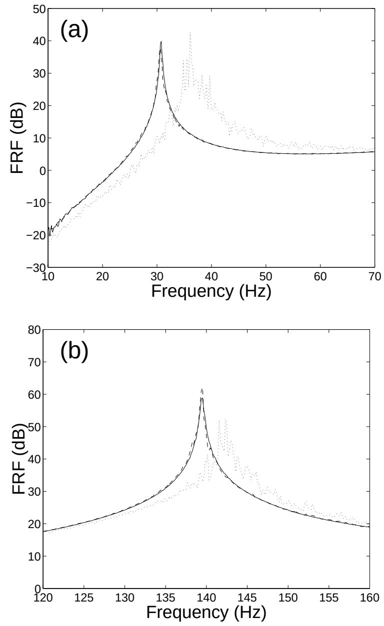

Figure 5: Magnitude of H_{73}. ——, True FRF (1.4 Nrms); $\cdots$, H_2 estimate (22 Nrms); - - -, H_{c2} estimate (22 Nrms). (a) 1st resonance; (b) 2nd resonance.

Excitation level (Nrms)	$A(N/m^{2.8})$
2.8	$2.69\ 10^9 - i\ 2.76\ 10^7$
5.5	$2.08\ 10^9 - i\ 6.07\ 10^7$
8	$1.94\ 10^9 + i\ 1.09\ 10^6$
16	$1.96\ 10^9 - i\ 6.20\ 10^6$
22	$1.96\ 10^9 + i\ 1.55\ 10^7$

TABLE 4: Spectral mean (10-250 Hz) of the non-linear coefficient (CRP)

FRF by the NIFO estimate. These distortions do not influence the natural frequencies computed from the NIFO estimate as shown in Table 5.

The real part of the non-linear coefficient is illustrated in Figure 8 and its spectral mean is listed in Table 6. The comparison between Figures 6 and 8 indicates that much larger deviations for the non-linear coefficient occur with the NIFO method. Hopefully, the spectral mean of this coefficient does not seem to be affected (Table 6).

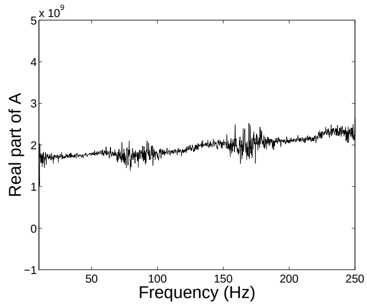

Figure 6: Real part of the non-linear coefficient (22 Nrms)

Excitation level (Nrms)	1st freq. (Hz)	2nd freq. (Hz)	3rd freq. (Hz)
1.4	30.74	139.47	390.22
2.8	30.74	139.43	390.15
5.5	30.75	139.38	390.04
11	30.75	139.26	389.92
16	30.67	139.35	389.81
22	30.63	139.33	389.77

TABLE 5: Natural frequencies (NIFO)

8 COMPARISON OF THE RESULTS AND CONCLUSIONS

The results obtained with the CRP and NIFO methods on the set-up considered in this work are excellent:

- the FRFs computed by both techniques provide a close match to the FRF of the structure considered as linear ;
- the natural frequencies estimated from the FRFs are in excellent agreement with the frequencies for the structure considered as linear. This is not the case with the H_2 estimate ;
- the spectral means of the non-linear coefficient are almost identical.

Both methods are appealing because of their ability to consider m.d.o.f. non-linear systems. The NIFO technique is attractive for its simplicity and its capability to estimate the linear and non-linear coefficients in a single step. The correlation between the linear and non-linear terms is a critical issue in NIFO and care must be taken (e.g. orthogonal least-squares) to achieve a good conditioning of the data matrices. However, from our experience, NIFO does not seem to guarantee the conditioning that is naturally present in the CRP method (see Figures 5 and 7). In addition, the CRP technique offers an efficient means to characterise the type of non-linearity through

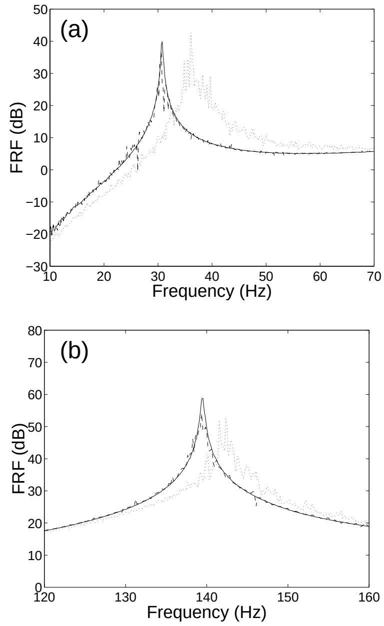

Figure 7: Magnitude of H_{73}. ——, True FRF (1.4 Nrms); $\cdots$, H_2 estimate (22 Nrms); - - -, NIFO estimate (22 Nrms). (a) 1st resonance; (b) 2nd resonance.

the use of the cumulative coherence. This latter is also useful in the sense that it may be viewed as a measure of the accuracy of the model.

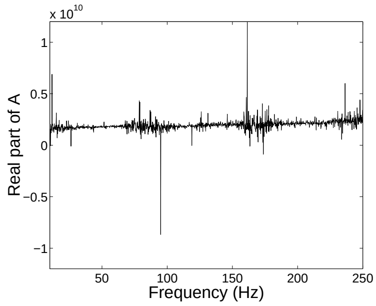

Figure 8: Real part of the non-linear coefficient (22 Nrms)

ACKNOWLEDGMENTS

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Excitation level (Nrms)	$A(N/m^{2.8})$
2.8	$2.69\ 10^9 - i\ 1.46\ 10^8$
5.5	$1.99\ 10^9 - i\ 7.35\ 10^7$
8	$1.91\ 10^9 - i\ 3.21\ 10^7$
16	$1.95\ 10^9 + i\ 1.80\ 10^6$
22	$1.96\ 10^9 + i\ 2.28\ 10^7$

TABLE 6: Spectral mean (10-250 Hz) of the non-linear coefficient (NIFO)

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