Strengthening linear reformulations of pseudo-Boolean optimization problems

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comex combinatorial optimization: metaheuristics & exact methods

Definitions

Definition: Pseudo-Boolean functions

A pseudo-Boolean function is a mapping $f : \{0,1\}^n \rightarrow \mathbb{R}$.

Multilinear representation

Every pseudo-Boolean function f can be represented uniquely by a multilinear polynomial (Hammer, Rosenberg, Rudeanu [4]).

Example:

$$f(x_1, x_2, x_3) = 9x_1x_2x_3 + 8x_1x_2 - 6x_2x_3 + x_1 - 2x_2 + x_3$$

Pseudo-Boolean Optimization

Many problems formulated as optimization of a pseudo-Boolean function

Pseudo-Boolean Optimization

$$\min_{x \in \{0,1\}^n} f(x)$$

- **Optimization is $\mathcal{NP}$-hard**, even if f is quadratic (MAX-2-SAT, MAX-CUT modelled by quadratic f).
- Approaches:
  - **Linearization**: standard approach to solve non-linear optimization.
  - **Quadratization**: Much progress has been done for the quadratic case (exact algorithms, heuristics, polyhedral results...).

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Standard linearization (SL)

$$\min_{\{0,1\}^n} \sum_{S \in \mathcal{S}} a_S \prod_{k \in S} x_k,$$

$\mathcal{S} = \{S \subseteq \{1, \dots, n\} \mid a_S \neq 0\}$ (non-constant monomials)

1. Substitute monomials

$$\begin{aligned} \min \sum_{S \in \mathcal{S}} a_S z_S & \\ \text{s.t. } z_S = \prod_{k \in S} x_k, & \quad \forall S \in \mathcal{S} \\ z_S \in \{0,1\}, & \quad \forall S \in \mathcal{S} \\ x_k \in \{0,1\}, & \quad \forall k = 1, \dots, n \end{aligned}$$

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2. Linearize constraints

$$\begin{aligned} \min \sum_{S \in \mathcal{S}} & a_S z_S \\ \text{s.t. } z_S &\leq x_k, && \forall k \in S, \forall S \in \mathcal{S} \\ z_S &\geq \sum_{k \in S} x_k - (|S| - 1), && \forall S \in \mathcal{S} \\ z_S &\in \{0,1\}, && \forall S \in \mathcal{S} \\ x_k &\in \{0,1\}, && \forall k = 1, \dots, n \end{aligned}$$

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3. Linear relaxation

$$\begin{aligned} \min \ & \sum_{S \in \mathcal{S}} a_S z_S \\ \text{s.t.} \ & z_S \leq x_k, && \forall k \in S, \forall S \in \mathcal{S} \\ & z_S \geq \sum_{k \in S} x_k - (|S| - 1), && \forall S \in \mathcal{S} \\ & 0 \leq z_S \leq 1, && \forall S \in \mathcal{S} \\ & 0 \leq x_k \leq 1, && \forall k = 1, \ldots, n \end{aligned}$$

Intermediate substitutions (IS) (one monomial)

SL substitution

$$z_S = \prod_{k \in S} x_k$$

SL linearization

$$z_S \leq x_k, \qquad \forall k \in S$$
$$z_S \geq \sum_{k \in S} x_k - (|S| - 1)$$

IS substitution

$$z_S = z_A \prod_{k \in S \setminus A} x_k$$
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IS linearization

$$\begin{aligned} z_S &\leq x_k, && \forall k \in S \setminus A \\ z_S &\leq z_A, \\ z_S &\geq z_A + \sum_{k \in S \setminus A} x_k - |S \setminus A|, \\ z_A &\leq x_k, && \forall k \in A \\ z_A &\geq \sum_{k \in A} x_k - (|A| - 1). \end{aligned}$$

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Intermediate Substitutions (IS) (one monomial)

Polytope $P_{SL,1} \subseteq \mathbb{R}^{n+1}$

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z_S &\leq x_k, & \forall k \in S\\
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0 \leq x_k &\leq 1, & \forall k = 1,\dots,n\\
0 \leq z_S &\leq 1, & \forall S \in \mathcal{S}
\end{aligned}
$$

Polytope $P_{IS,1} \subseteq \mathbb{R}^{n+2}$

$$
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0 \leq x_k &\leq 1, & \forall k = 1,\dots,n\\
0 \leq z_S &\leq 1, & \forall S \in \mathcal{S}
\end{aligned}
$$

Calculating projections: Fourier-Motzkin Elimination

Notation

$\mathbb{P}_{n,S}$: *projection over the space of variables* z_S *and* $x_k, k = 1, \ldots, n$.

We calculate $\mathbb{P}_{n,S}(P_{IS,1})$ using the **Fourier-Motzkin Elimination**:

$$z_S \leq z_A \qquad z_A \leq x_k, \qquad \forall k \in A$$
$$\sum_{k \in A} x_k - (|A| - 1) \leq z_A \qquad z_A \leq z_S - \sum_{k \in S \setminus A} x_k + |S \setminus A|.$$

We also take into account the inequalities of $P_{IS,1}$ that do not involve z_A

$$z_S \leq x_k, \forall k \in S \setminus A$$

Single monomials

Theorem

$$\mathbb{P}_{n,S}(P_{IS,1}) = P_{SL,1}$$

Theorem holds for *disjoint* several monomials:
$z_S = \prod_{k \in S} x_k$, $z_T = \prod_{k \in T} x_k$, take $A \subseteq S$, $B \subseteq T$.

$$z_S = z_A^S \prod_{k \in S \setminus A} x_k \qquad z_T = z_B^T \prod_{k \in T \setminus B} x_k$$

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Linearize, and apply Fourier-Motzkin as before (constraints never contain at the same time z_A^S and z_B^T).

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Several monomials with common intersection

What happens with *non-disjoint* monomials? $A \subseteq S \cap T$, $(|A| \geq 2)$.

$$z_S = z_A \prod_{k \in S \setminus A} x_k$$

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Several monomials with common intersection

Theorem

$$\mathbb{P}_{n,S,T}(P_{IS}) \subset P_{SL}$$

Proof:

1. Fourier-Motzkin gives:

$$\mathbf{z_S} \leq \mathbf{z_T} - \sum_{\mathbf{k} \in \mathbf{T} \setminus \mathbf{A}} \mathbf{x_k} + |\mathbf{T} \setminus \mathbf{A}|, \tag{1}$$

$$\mathbf{z_T} \leq \mathbf{z_S} - \sum_{\mathbf{k} \in \mathbf{S} \setminus \mathbf{A}} \mathbf{x_k} + |\mathbf{S} \setminus \mathbf{A}|, \tag{2}$$

2. $\mathbb{P}_{n,S,T}(P_{IS}) = P_{SL} \cap \{(x_k, z_S, z_T) \mid (1), (2) \text{ are satisfied}\}$
3. Point $x_k = 1$ for $k \notin A$, $x_k = \frac{1}{2}$ for $k \in A$, $z_S = 0$, $z_T = \frac{1}{2}$, is in P_{SL} but does not satisfy (2).

Larger subset substitutions are better

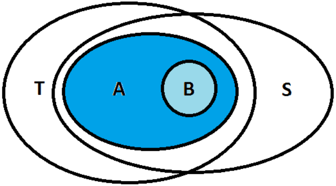
Consider $B \subset A \subseteq S \cap T$, $|B| \geq 2$.

1. Take the first cut for both subsets:
   $z_S \leq z_T - \sum_{k \in T \setminus A} x_k + |T \setminus A|$,
   $z_S \leq z_T - \sum_{k \in T \setminus B} x_k + |T \setminus B|$,
2.

$$z_S \leq z_T - \sum_{k \in T \setminus A} x_k + |T \setminus A| \leq$$

$$\leq z_T - \sum_{k \in T \setminus A} x_k + |T \setminus A| - \sum_{k \in A \setminus B} x_k + |A \setminus B| =$$

$$= z_T - \sum_{k \in T \setminus B} x_k + |T \setminus B|.$$



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Theorem

$$\mathbb{P}_{n,S,T}(P_{IS}^{A}) \subset \mathbb{P}_{n,S,T}(P_{IS}^{B}).$$

(Point $x_k = 1$ for $k \notin A$, $x_k = \frac{1}{2}$ for $k \in A \backslash B, k \in B$, $z_T = 0$, $z_S = \frac{1}{2}$ satisfies cut for B but not for A.)

Definition: 2-link inequalities

$$z_S \leq z_T - \sum_{k \in T \setminus S} x_k + |T \setminus S|$$

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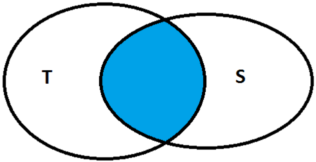

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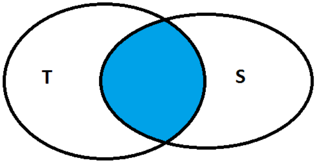

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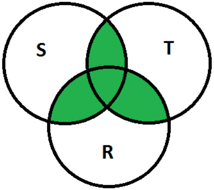
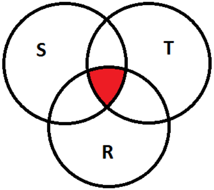
Corollary

Consider three monomials R, S, T, with intersections $R \cap S = A$, $S \cap T = B$, $R \cap T = C$, $(|A|, |B|, |C| \geq 2)$. Then it is better to do intermediate substitutions of the two-by-two intersections, than a single intermediate substitution of the common intersection $A \cap B \cap C$.





Improving the SL formulation: 2-links

SL relaxation with 2-links

$$
\begin{aligned}
\min \sum_{S \in \mathcal{S}} a_S z_S & & \\
\text{s.t. } z_S &\le x_k, & \forall k \in S, \forall S \in \mathcal{S} \\
z_S &\ge \sum_{k \in S} x_k - (|S| - 1), & \forall S \in \mathcal{S} \\
\mathbf{z_S} &\le \mathbf{z_T} - \sum_{\mathbf{k} \in \mathbf{T} \setminus \mathbf{S}} \mathbf{x_k} + |\mathbf{T} \setminus \mathbf{S}| & \forall \mathbf{S}, \mathbf{T}, |\mathbf{S} \cap \mathbf{T}| \ge \mathbf{2} \\
\mathbf{z_T} &\le \mathbf{z_S} - \sum_{\mathbf{k} \in \mathbf{S} \setminus \mathbf{T}} \mathbf{x_k} + |\mathbf{S} \setminus \mathbf{T}| & \forall \mathbf{S}, \mathbf{T}, |\mathbf{S} \cap \mathbf{T}| \ge \mathbf{2} \\
0 &\le z_S \le 1, & \forall S \in \mathcal{S} \\
0 &\le x_k \le 1 & \forall k = 1, \ldots, n
\end{aligned}
$$

How strong are the 2-links?

Standard linearization polytope:

$$P_{SL}^{conv} = \text{conv}\{(x, y_S) \in \{0,1\}^{n+|\mathcal{S}|} \mid y_S = \prod_{i \in S} x_i, \forall S \in \mathcal{S}\}$$
$$= \text{conv}\{(x, y_S) \in \{0,1\}^{n+|\mathcal{S}|} \mid y_S \leq x_i, y_S \geq \sum_{i \in S} x_i - (|S| - 1), \forall S \in \mathcal{S}\},$$

with linear **relaxation**

$$P_{SL} = \{(x, y_S) \in [0,1]^{n+|\mathcal{S}|} \mid y_S \leq x_i, y_S \geq \sum_{i \in S} x_i - (|S| - 1), \forall S \in \mathcal{S}\}$$

- **Question 1**: Are the 2-links *facet-defining* for P_{SL}^{conv}?

How strong are the 2-links?

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- **Question 1**: Are the 2-links *facet-defining* for P_{SL}^{conv}?
- **Question 2**: Is there some case for which we obtain the *convex hull* P_{SL}^{conv} when adding the 2-links to P_{SL}?

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Facet-defining cuts (2 monomials)

Theorem: 2-term objective function

The 2-links are facet-defining for $P^{conv}_{SL,2}$:

$$z_S \leq z_T - \sum_{k \in T \setminus S} x_k + |T \setminus S|$$

$$z_T \leq z_S - \sum_{k \in S \setminus T} x_k + |S \setminus T|$$

Facet-defining cuts (2 monomials)

Special forms of the cuts in some cases:

1. If $S \subseteq T$,

$$z_S \leq z_T - \sum_{k \in T \setminus S} x_k + |T \setminus S|$$

$$z_T \leq z_S$$

2. If $T = \emptyset$ (and setting by definition $z_\emptyset = 1$),

$$z_S \leq 1$$

$$1 \leq z_S - \sum_{i \in S} x_i + |S|$$

Conjecture on the convex hull (2 monomials)

Conjecture

Consider a pseudo-Boolean function consisting of two terms, its standard linearization polytope $P_{SL,2}^{conv}$ and its linear relaxation $P_{SL,2}$. Then,

$$P_{SL,2}^{conv} = P_{SL,2} \cap \{(x, y_S, y_T) \in [0,1]^{n+2} \mid \text{ 2-links are satisfied}\}.$$

Facet-defining cuts (nested monomials)

Theorem: Nested sequence of terms

Consider a pseudo-Boolean function $f(x) = \sum_{l \in L} a_{S^{(l)}} \prod_{i \in S^{(l)}} x_i$, such that $S^{(1)} \subseteq S^{(2)} \subseteq \cdots \subseteq S^{(|L|)}$, and its standard linearization polytope $P^{conv}_{SL,nest}$. The 2-links

$$z_{S^{(l)}} \leq z_{S^{(l+1)}} - \sum_{k \in S^{(l+1)} \setminus S^{(l)}} x_k + |S^{(l+1)} \setminus S^{(l)}|$$

$$z_{S^{(l+1)}} \leq z_{S^{(l)}},$$

are facet-defining for $P^{conv}_{SL,nest}$ for two consecutive monomials in the nest (and cuts are redundant for non-consecutive monomials).

Conjectures for m monomials

Conjecture: facet-defining

The 2-links are facet-defining for the case of m monomials.

Convex-hull for the general case

The 2-links and standard linearization inequalities are **not** enough to define the convex hull P_{SL}^{conv} (otherwise we could solve an $\mathcal{NP}$-hard problem efficiently...).

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- $m = 3$, set of 3 monomials for which there exists an objective function which has a fractional optimal solution on $P_{SL} \cap \{2\text{-links}\}$: $\{x_1x_2x_4, x_1x_3x_4, x_1x_2x_3\}$

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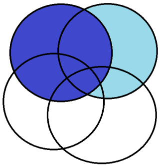
The difficulty of describing the general convex hull



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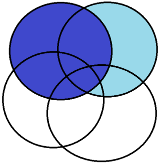

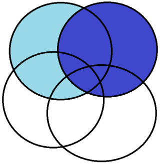

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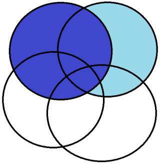

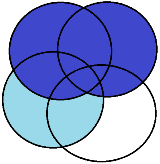

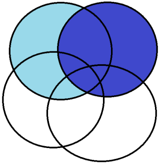

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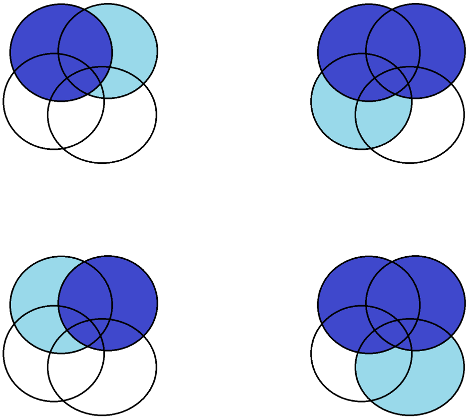

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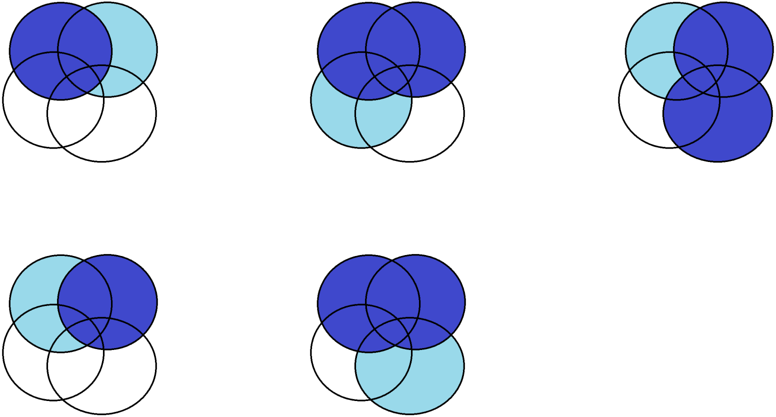

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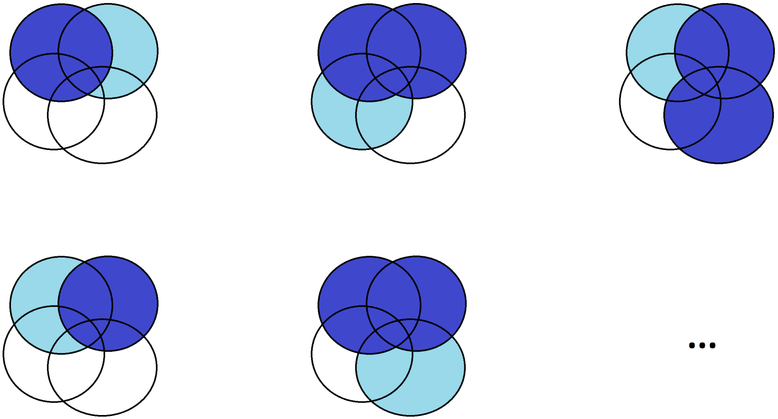

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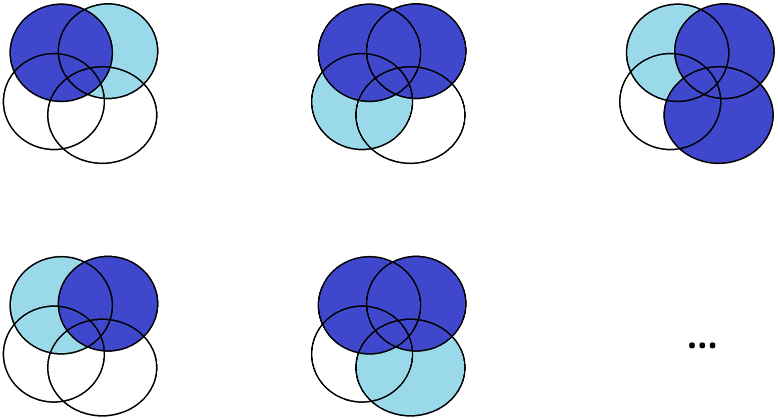

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A short summary and some ideas

- We have obtained interesting cuts for P_{SL} by applying intermediate substitutions for subsets of size ≥ 2.
- We could apply iteratively these intermediate substitutions, the last substitution step has only quadratic constraints

$$z_{ij} = x_i x_j,$$

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Some references I

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