STRESS CONSTRAINED TOPOLOGY OPTIMIZATION FOR ADDITIVE MANUFACTURING: SPECIFIC CHARACTER AND SOLUTION ASPECTS

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OUTLINE

- Introduction & Motivation

- Topology problem formulation
  - Problem statement

- Specific character of stress constrained design
  - Energy vs von Mises stress
  - Local stress constraints
  - Unequal stress limits
  - Fatigue constraints

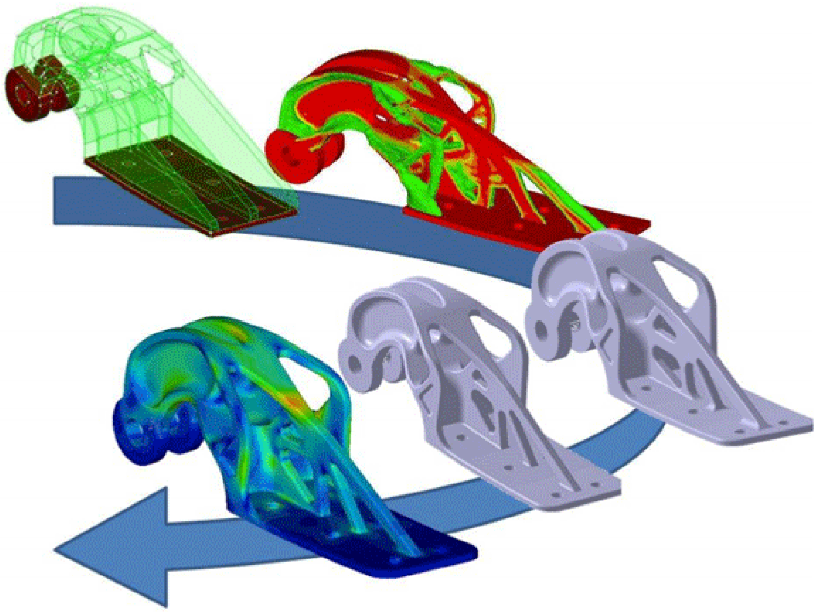
- Large scale optimization
  - Sensitivity analysis
  - Dual optimization algorithms

- Conclusion & Perspectives

INTRODUCTION & MOTIVATION

MOTIVATION

- TOPOLOGY OPTIMIZATION: a creative design tool



- ADDITIVE MANUFACTURING new way of making things

Courtesy of ALTAIR and AIRBUS

INTRODUCTION

- Topology optimization is mostly based on compliance design formulation
- Many aerospace and mechanical components are designed with respect to strength or fatigue constraints
- Need for efficient approaches to handle efficiently stress constrained problems
- Extending the scope of stress constrained topology optimization to cope with:
  - Fatigue constraints
  - Industrial applications → Large scale problems

INTRODUCTION

- This paper
  - Draws a state-of-the-art of topology optimization of continuum structures with stress constraints
  - Illustrates the specific character of maximum strength with respect to compliance design when considering
    - Several load cases
    - Different stress limits in tension and compression
  - Extends the scope of stress constrained topology optimization to unequal stress constraints, fatigue problems..
  - Draws the challenges to tackle large scale optimization problems related to local constraints

TOPOLOGY OPTIMIZATION FORMULATION

TOPOLOGY OPTIMIZATION PROBLEM

- Optimal material distribution within a given domain

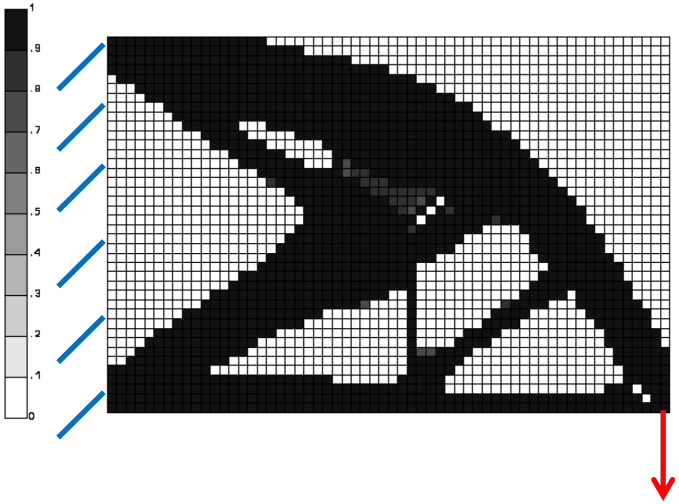
- Discretization of displacements and density distribution using FEM

$$KU = F$$

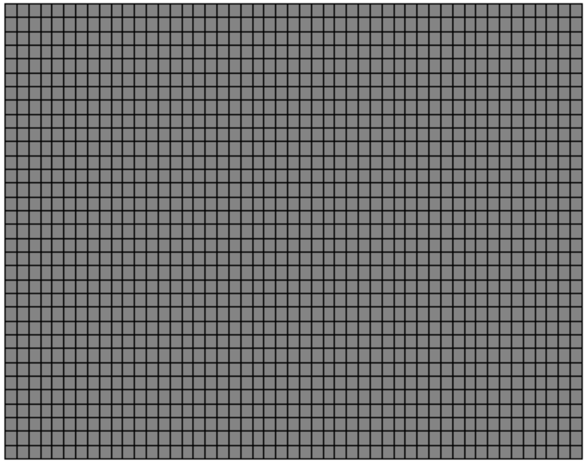
- Interpolation of material properties between void and solid and penalize intermediate densities (SIMP model)

$$E_j(x_j) = E_{min} + x_j^p \left(E_0 - E_{min}\right)$$

- Solve optimization problem using efficient MP optimizers with continuous variables



Treillis de Michell avec filtrage (L.Adam)



TOPOLOGY OPTIMIZATION

- Density filter:

$$\tilde{x}_e = \frac{1}{\sum_N H_{ej}} \sum_N H_{ej} x_j,$$

$$H_{ej} = \sum_N \max(0, r_{min} - \mathrm{dist}(e,j)),$$

- Implementation : Topology optimization tool in MATLAB based 88-line code by Andreassen et al. (2011)

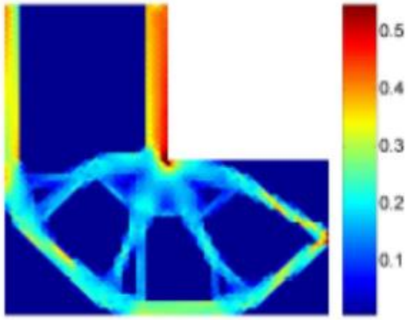
- MMA solver by Svanberg (1987)

$$\begin{array}{lll}
\min & f_0(\mathbf{x}) + z + \sum_{j=1}^{m}(c_j y_j + \frac{1}{2} d_j y_j^2) & \\
\text{s.t.:} & f_j(\mathbf{x}) - a_j z - y_j \leq 0 & j = 1 \ldots m \\
 & \underline{x}_i \;\leq\; x_i \;\leq \overline{x}_i & i = 1 \ldots n \\
 & y_j \geq 0 & j = 1 \ldots m \\
 & z \geq 0 &
\end{array}$$

TOPOLOGY OPTIMIZATION PROBLEM

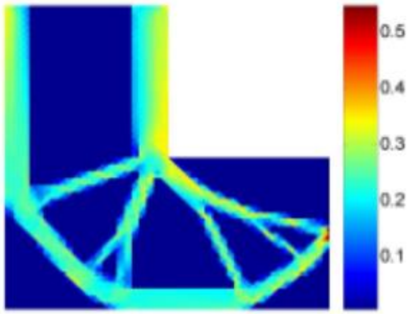
- Compliance design
  - Usual approach
  - Unable to capture the specific character of stress constraints

$$\min_{0<x\leq 1} \quad \boldsymbol{F}^T\boldsymbol{U}$$
$$\text{s.t.} \quad V = \sum_{e=1} v_e x_e \leq \bar{V}$$



- Stress constrained design
  - Technical difficulties to be solved
  - Define appropriate failure criterion
  - Computational effort compared to compliance design

$$\min_{0<x\leq 1} \quad \max_e \|\sigma_e(\boldsymbol{x})\|$$
$$\text{s.t.} \quad V = \sum_{e=1} v_e x_e \leq \bar{V}$$



[Duysinx et Bruggi (2012)]

TOPOLOGY OPTIMIZATION

- Challenges of stress constraints in topology optimization
  - Definition of relevant stress criteria at microscopic level
    - Microscopic stress should be considered

$$\sigma_{ij}^{M} = E_{ijkl}\,\epsilon_{kl}^{M} \quad \Rightarrow \quad <\sigma_{ij}^{e}> = \frac{\sigma_{ij}^{e,M}}{x_e^q}$$

  - Stress singularity phenomenon:
    - ε-relaxation (Chang and Guo, 1992)
    - q-p relaxation (Bruggi, 2008)

$$<\sigma_{ij}^{e}> = \frac{x_e^p}{x_e^q} E_{ijkl}^{0}\varepsilon_{kl}^{0} = x_e^{p-q} E_{ijkl}^{0}\varepsilon_{kl}^{0} \qquad q < p \quad \text{and} \quad q \nearrow p$$

  - Large scale optimization problem
    - Local constraints
    - Aggregation of constraints: P-norm $\left[\sum_{e=1}^{N} (<||\sigma_e||>)^P \right]^{1/P}$

SPECIFIC CHARACTER OF STRESS CONSTRAINTS

SPECIFIC CHARACTER OF STRESS CONSTRAINTS

- Bound of integrated von Mises stress by compliance Bendsoe, Diaz and Kikuchi (1993)

$$\int_{\Omega} \sigma_{VM}^2 d\Omega \leq \frac{3E}{4(1+\nu)} \int_{\Omega} \sigma : C\sigma d\Omega \leq \frac{3E}{4(1+\nu)} F^T U$$

- For single load case and minimum compliance with volume constraint :
  - Minimizing strain energy bounds almost everywhere the von Mises stress
  - Relation between energy minimization and fully stressed design nearly every where in the material
  - Compliance design is efficient to predict optimal structural lay-out

SPECIFIC CHARACTER OF STRESS CONSTRAINTS

- Local strain energy can be written as (Timoshenko and Goodier, 1970)

$$u = \frac{1}{2}\frac{\sigma_{oct}^2}{\chi} + \frac{3}{4}\frac{\tau_{oct}^2}{G}$$

  - with

$$\sigma_{oct} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \qquad \chi = \frac{E}{3(1-\nu)}$$

$$\tau_{oct} = .\frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \qquad G = \frac{E}{2(1+\nu)}$$

- Minimizing von Mises stress does not control compressibility energy!!!
  - Tri-axiality is important.
- Stiffness and strength designs can be different when
  - Several load cases
  - Several materials
  - Different stress limits in tension and compression
  - Geometrical constraints (perimeter, manufacturing constraints...)

NUMERICAL APPLICATIONS: 3-BAR TRUSS

- Famous benchmark problem with 3 independent load cases

  F_1 = 40 N

  F_2 = 30 N

  F_3 = 20 N

- Material and geometrical data

  L=1 m

  W = 2.5 m

  E = 100 N/m²

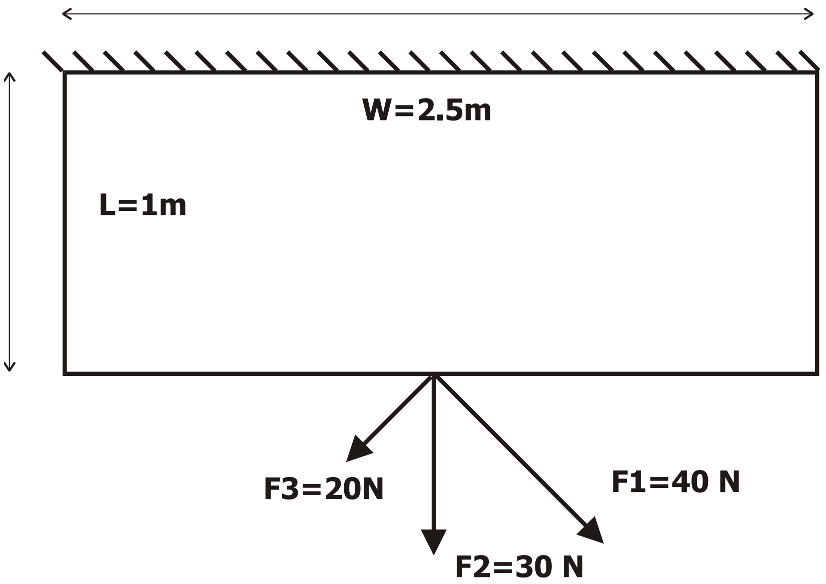
  ν = 0.3

  σ_l = 150 N/m²

  Vmax = 25%

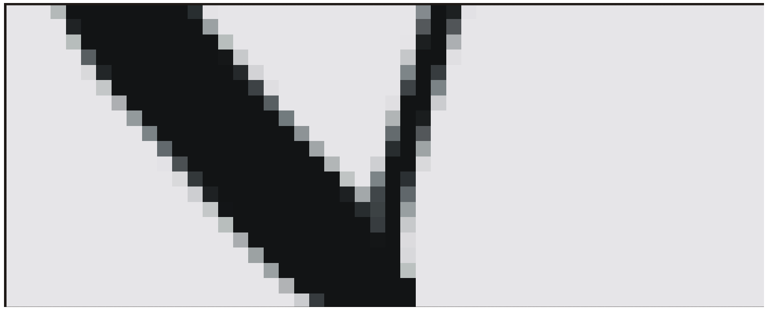
- Finite Element mesh

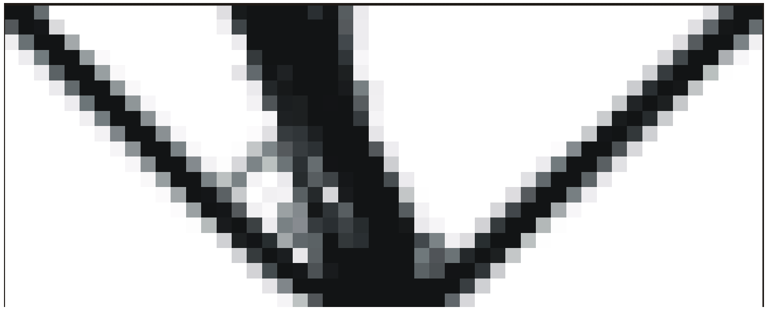
  50 x 20 finite elements



- Design variables: 1000
- Load cases: 3
- Stress constraints: 3000

NUMERICAL APPLICATIONS: 3-BAR TRUSS





- Minimum compliance design

  Compliance (1,2,3) = 73.3 Nm

  Max von Mises:
  1) 229 N/m²
  2) 571 N/m²
  3) 555 N/m²

  Volume = 25%

- Stress constrained design

  Compliance
  1) 91.2 Nm
  2) 45.6 Nm
  3) 45.0 Nm

  Max Von Mises (1,2,3)= 150N/m²

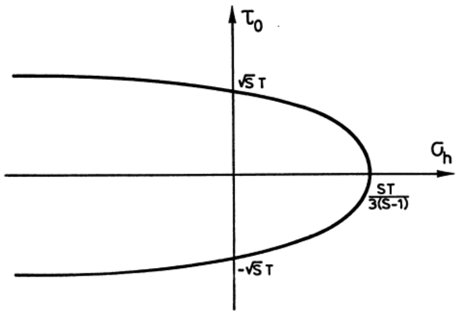
  Volume = 26.4 %

Unequal stress limits in tension and compression

- Extending Von Mises criterion to other failure criteria to cope with unequal stress limits behaviors (T $\neq$ C, s=C/T)

- Raghava criterion (parabolic criterion from Tsai-Wu criterion family)

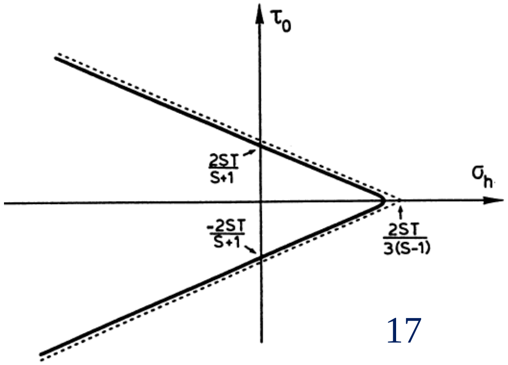
$$\sigma_{RAG}^{eq} = \frac{J_1(s-1) + \sqrt{J_1^2(s-1)^2 + 12 J_{2D} s}}{2s} \leq T$$



- Ishai criterion (hyperbolic criterion from Prager-Drucker family)

$$\sigma_{ISH}^{eq} = \frac{(s+1)\sqrt{3J_{2D}} + (s-1)J_1}{2s} \leq T$$

  - with $J_1 = \sigma_{ii}$ $\quad J_{2D} = 0.5 s_{ij} s_{ij}$



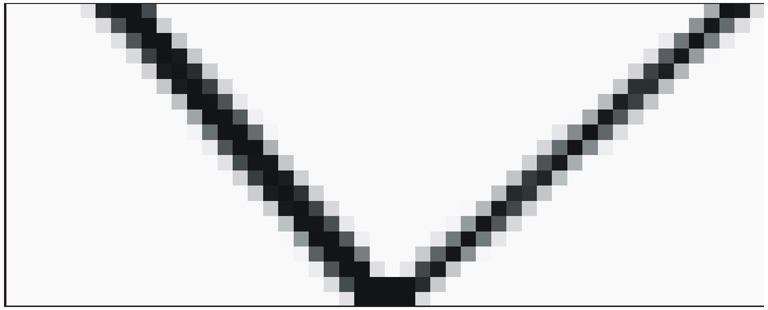
NUMERICAL APPLICATIONS: 3-BAR TRUSS

- High compressive strength ($s=C/T=3$):

  ($C=450$ N/m², $T=150$ N/m²)

  Volume = 25.6 %

  Compliance (1,2,3): 92.8, 47,3, 46,0 N*m



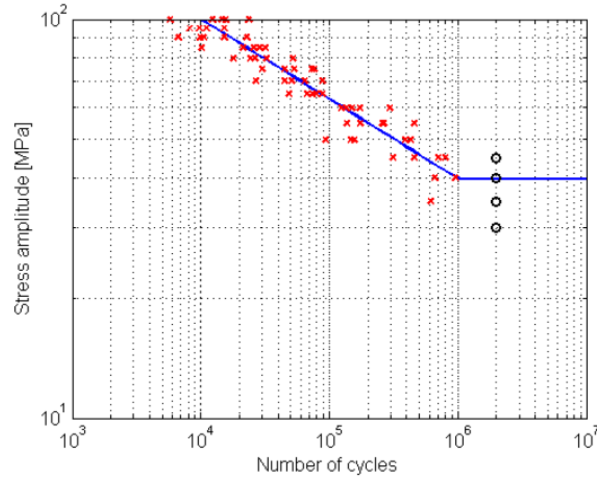
- High tensile strength ($s=C/T=1/3$):

  ($C=150$ N/m², $T=450$ N/m²):

  Volume = 12.4 %

FATIGUE (UNI AXILAL CASE)

- Wöhler's curve : fundamental work
  - Reduction of the amplitude of stress with the number of cycles

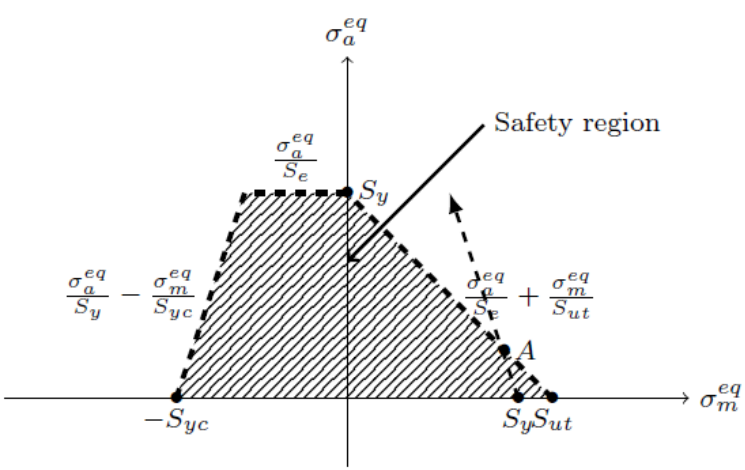


- Goodman diagram:
  - Influence of mean and alternate stress components
  - Line of equal failure probability for a certain number of cycles

$$\sigma(t) \;=\; \sigma_m + \sigma_a \,\sin(\omega\, t)$$

  - Amplitude / mean stress

$$\sigma_a = \frac{\sigma_{max} - \sigma_{min}}{2} \qquad \sigma_m = \frac{\sigma_{max} + \sigma_{min}}{2}$$



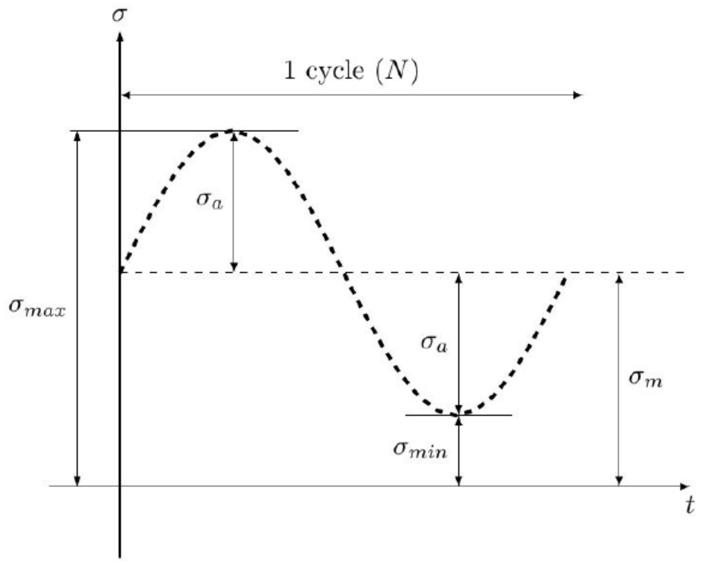
MULTI AXIAL FATIGUE CRITERIA

- Like in 1-D problem let's assume that the total stress is given by a certain amount of alternate component $c_a\ \sigma_a$ and a given amount of mean component $c_m\ \sigma_m$:

$$\sigma_{tot} = c_a \sigma_a + c_m \sigma_m$$

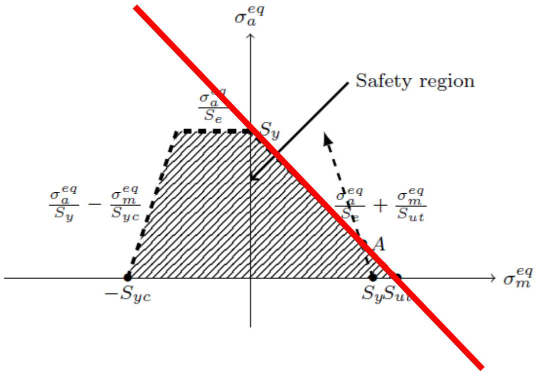
$$0 \leq c_a, c_m \leq 1$$

$$c_a + c_m = 1.$$



- In the following, let assume that alternate and mean components are defined by the same reference load case.

MULTI AXIAL FATIGUE CRITERIA: SINES



- Sines fatigue criterion:

$$\boxed{\sqrt{J_{2,a}} + \kappa \cdot \sigma_{h,m} \leq \lambda}$$

  - Where

$$\lambda = t_{-1} \qquad \kappa = \frac{6t_{-1}}{f_0} - \sqrt{6}$$

  - With t_{-1}, the fatigue limit in reverse torsion and f_0 is the fatigue in repeated bending

- For plane stress

$$J_{2,a} = \frac{1}{6}\left[(\sigma_{11,a} - \sigma_{22,a})^2 + \sigma_{22,a}^2 + \sigma_{11,a}^2 + 6\sigma_{12,a}^2\right]$$

$$\sigma_{h,m} = \frac{1}{3}(\sigma_{11,m} + \sigma_{22,m}) = \frac{J_1}{3}$$

MULTI AXIAL FATIGUE CRITERIA: CROSSLAND

- Crossland fatigue criterion is very similar to Sines criterion:

$$\boxed{\sqrt{J_{2,a}} + \kappa \cdot \sigma_{h,max} \leq \lambda}$$

- Difference lies in the fact in Crossland the hydrostatic term is evaluated on the basis of the maximum stress (not only on the mean component): $\sigma_{max} = \sigma_a + \sigma_m$:

$$\sigma_{h,max} = \sigma_{h,a} + \sigma_{h,m}$$

MULTI AXIAL FATIGUE CRITERIA: SINES

- Assuming a SIMP model, after Finite Element discretization:

$$\sigma_a^{eq} = \sqrt{3J_2(\sigma_{a,ij})} \qquad \sigma_{a,e}^{eq} = x_e^p\left(c_a\sqrt{\mathbf{U_e^T M_e^0 U_e}}\right) = x_e^p\overline{\sigma}_{a,e}^{eq}$$

$$\sigma_m^{eq} = J_1(\sigma_{m,ij}) \qquad \sigma_{m,e}^{eq} = x_e^p(c_m\mathbf{H_e^0 U_e}) = x_e^p\overline{\sigma}_{m,e}^{eq}$$

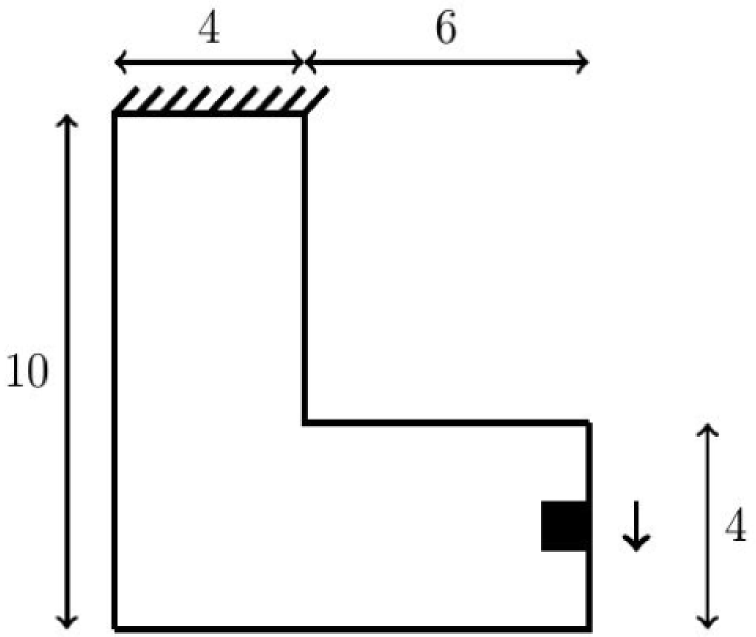
- Considering the micro stresses after applying the polarization factor

$$\langle\sigma_{ij,e}\rangle = \frac{\sigma_{ij,e}}{x_e^q}$$

- The expression Sines criterion for topology optimization reads

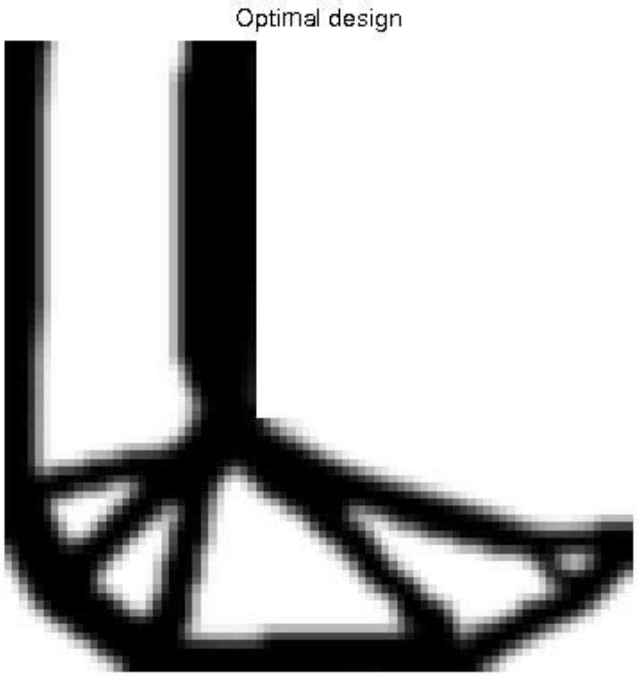
$$\boxed{\frac{x_e^{(p-q)}}{\lambda}\left[\frac{\overline{\sigma}_{a,e}^{eq}}{\sqrt{3}} + \frac{\kappa}{3}\overline{\sigma}_{m,e}^{eq}\right] \leq 1}$$

NUMERICAL APPLICATION: L-SHAPE

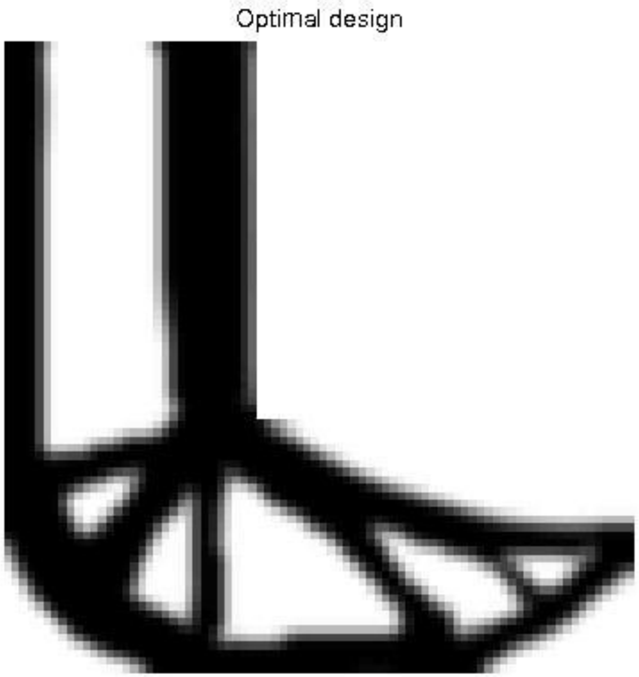


- SIMP model
  - Penalization p=3
  - q-p relaxation: q=2.6 → 2.75
- Load F=95 N
  - c_a = 0.7 and c_m = 0.3
- Material : Steel with properties from Norton (2000)
  - E = 1 Mpa (normalized), ν=0.3
  - σ_f = 580 MPa, t_{-1}= 160 MPa, f_{-1}= 260 MPa
- Compliance regularization constraint: α_c=2

NUMERICAL APPLICATION: LSHAPE

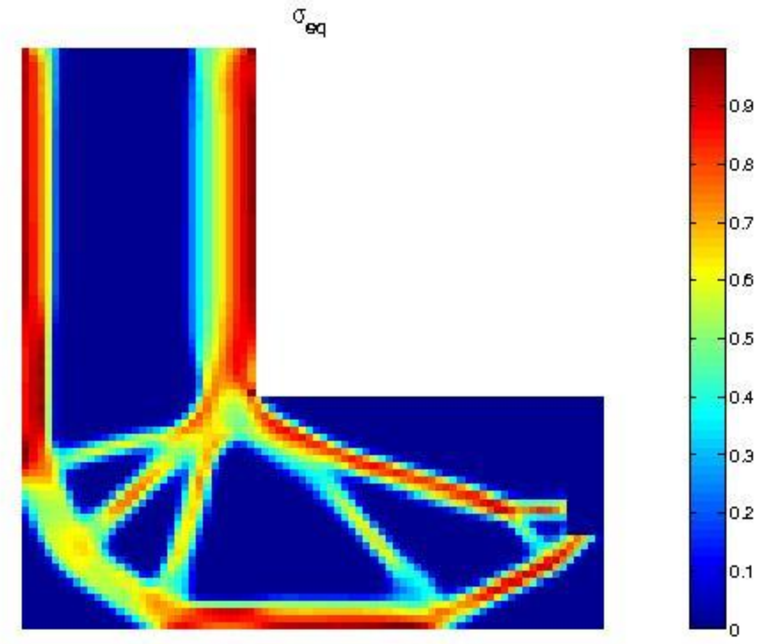


Optimal design with Sines criterion

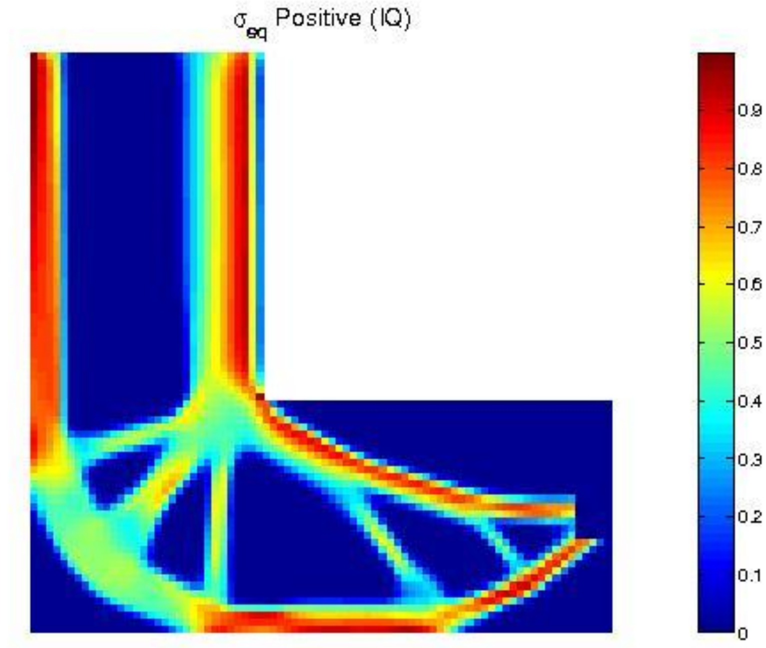


Optimal design with Crossland criterion

NUMERICAL APPLICATION: LSHAPE



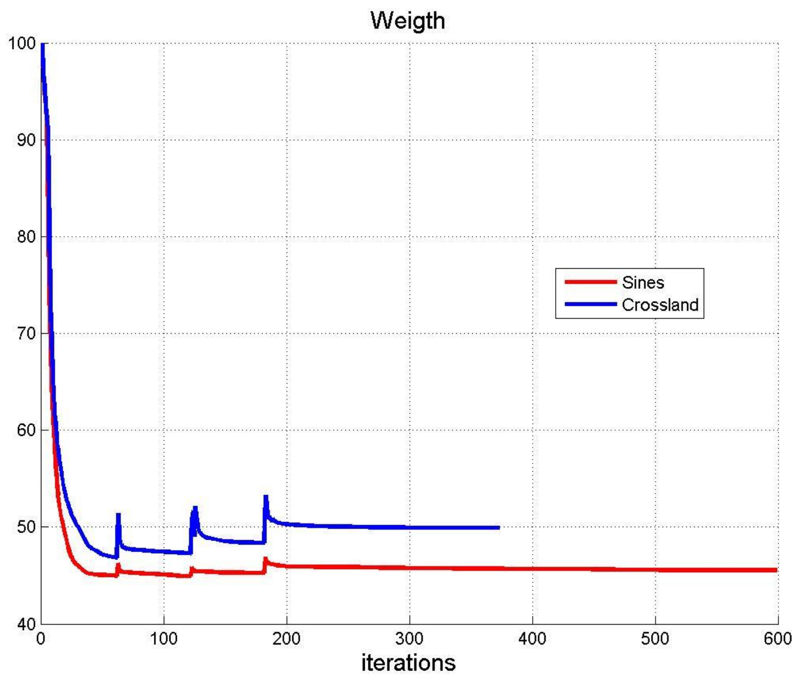
Stress map for
optimal design with Sines criterion



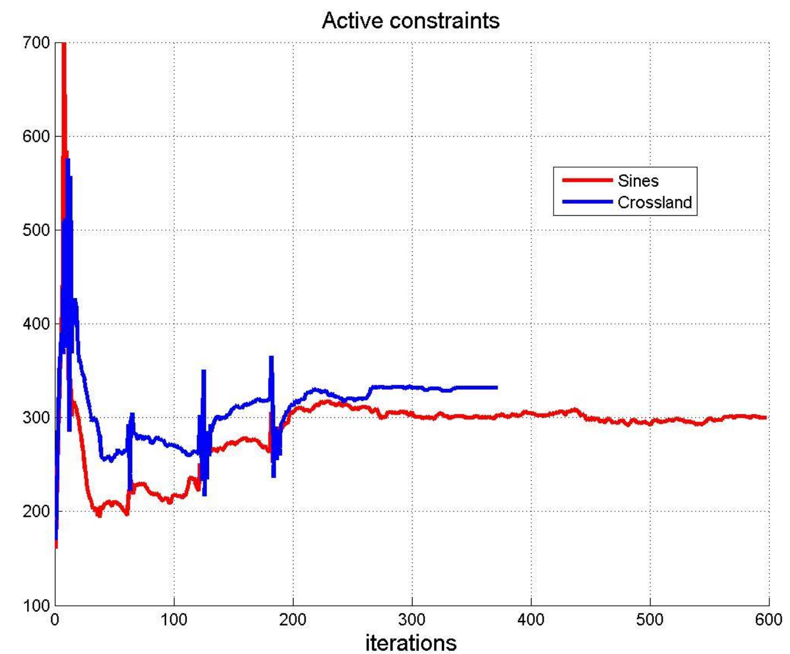
Stress map for
optimal design with Crossland criterion

NUMERICAL APPLICATION: LSHAPE

Problem	N	$\mathcal{W}/\mathcal{W}_0$	$\mathcal{C}/\mathcal{C}_0$	N_a^f
MWS	4096	0.4553	2	299
MWC	4096	0.4991	1.97	332



Evolution of the objective function volume



Evolution of the number of active constraints

SOLVING LARGE SCALE OPTIMIZATION PROBLEMS

SOLVING LARGE SCALE OPTIMIZATION

- Classical strategy: solve optimization sequential convex programming
  - Generate first order approximation sub-problems: CONLIN (Fleury, 1985) or MMA (Svanberg, 1987) or GCMMA approximation (Bruyneel et al., 2002)
  - Dual solver (Lagrangian maximization)

- When dealing with stress constrained design, one hits the limitation of currently available standard:
  - Number of active restrictions is more or less equal to the number of design variables
  - Sensitivity analysis become very expensive
  - Solution time of optimization algorithm becomes of the same order of magnitude as the FE computation.

Strategies to solve large scale problems

- Improve the sensitivity analysis:
  - Selection of potentially active constraints
  - Adjoin vs direct sensitivity analysis

- Introduction 'dummy' compliance constraint' to control the convergence during first steps (Bruggi & Duysinx, 2013)

- Use integrated stress constraints instead of a purely local approach
  - Lose of local control of stress constraints: results looks closer to compliance design (Duysinx & Sigmund, 1998)
  - Rather difficult to tackle with classical approximation (function not convex)

Sensitivity analysis

- Direct approach: solve n (#dv) load cases

$$\frac{\partial U}{\partial \rho_i} = K^{-1}\left(\frac{\partial F}{\partial \rho_i} - \frac{\partial K}{\partial \rho_i} U\right)$$

- Adjoin method: solve m (#constraints) load cases

$$\lambda = K^{-1}\left\{\frac{s-1}{2s}\; W^0 \;+\; \frac{s+1}{2s}\; \frac{1}{\sqrt{U^T V^0 \;\; U}}\; V^0 \;\; U\right\}$$

  - For one load case: m=#FE ~ n
  - For several load cases: m=#FE *#load cases >n

Problem formulation: compliance constraint

- Minimum volume with (fatigue stress) constraints and compliance constraint

$$
\begin{cases}
\min\limits_{x_0 \le x_e \le 1} & \mathcal{W} = \sum\limits_{N} x_e V_e \\
\text{s.t.} & \mathbf{K}(\mathbf{x})\,\mathbf{U} \;=\; \mathbf{F}, \\
 & \mathcal{C} \,/\, \mathcal{C}_L \;\le\; 1, \\
 & \dfrac{x_e^{(p-q)}}{\lambda} \left[\dfrac{\overline{\sigma}_{a,e}^{eq}}{\sqrt{3}} + \dfrac{\kappa}{3} \overline{\sigma}_{M,e}^{eq} \right] \le 1, \\
 & \qquad \text{for } e = 1, \ldots, N
\end{cases}
$$

- Compliance constraints is introduced to provide a better stability and effectiveness to the convergence (Bruggi & Duysinx, 2012)

$$\mathcal{C}_L = \alpha_c \mathcal{C}_0$$

Integrated (aggregated) stress constraint

- Use aggregate restriction of relaxed stress constraints (Duysinx & Sigmund, 1998)
  - q-norm

$$\left[\sum_{e=1}^{N}\left(\max\left\{0,\frac{1}{T}\ \frac{({}^{*}\sigma^{eq})_e}{\rho_e^p}-\frac{\varepsilon}{\rho_e}+\varepsilon\right\}\right)^q\right]^{1/q}\leq 1$$

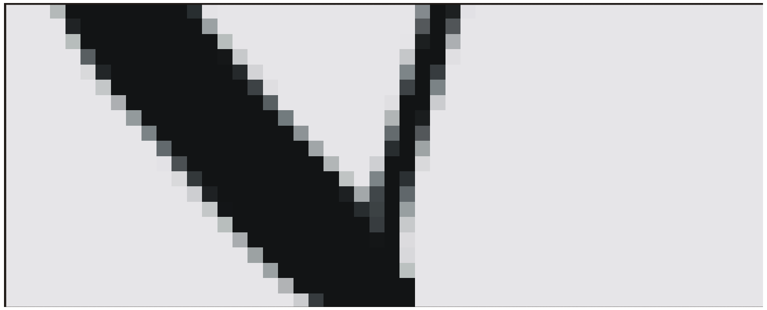
  - q-mean

$$\left[\frac{1}{N}\sum_{e=1}^{N}\left(\max\left\{0,\frac{1}{T}\ \frac{({}^{*}\sigma^{eq})_e}{\rho_e^p}-\frac{\varepsilon}{\rho_e}+\varepsilon\right\}\right)^q\right]^{1/q}\leq 1$$

- Ordering relationship

$$\left[\frac{1}{N}\sum_{e=1}^{N}\left\|\sigma_e^{*}\right\|^q\right]^{1/q}\leq\max_{e=1\ldots N}\left\|\sigma_e^{*}\right\|^q\leq\left[\sum_{e=1}^{N}\left\|\sigma_e^{*}\right\|^q\right]^{1/q}$$

NUMERICAL APPLICATIONS: 3-BAR TRUSS

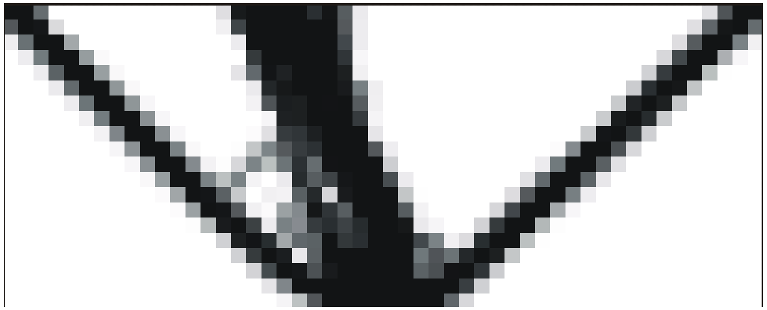


- Minimum compliance design

  Compliance (1,2,3) = 73.3 Nm

  Max von Mises:
  1) 229 N/m^2
  2) 571 N/m^2
  3) 555 N/m^2

  Volume = 25%



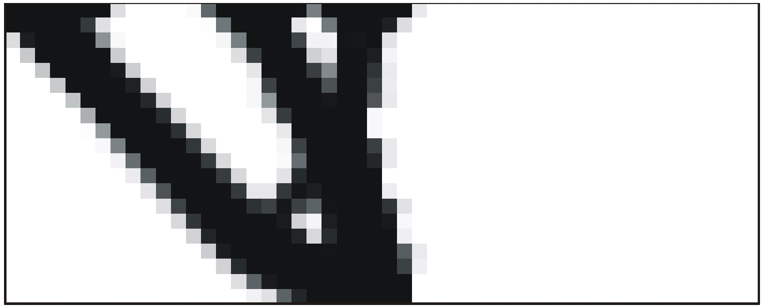
- Stress constrained design

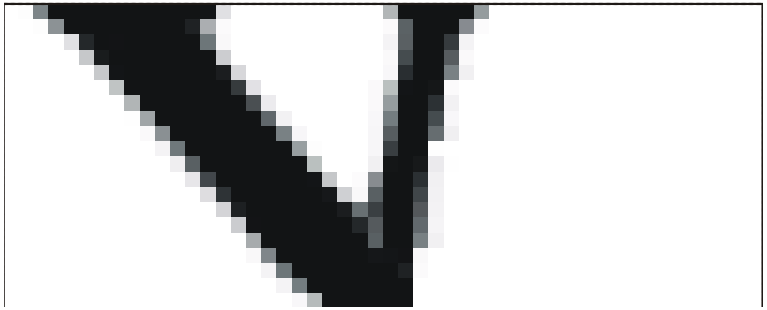
  Compliance
  1) 91.2 Nm
  2) 45.6 Nm
  3) 45.0 Nm

  Max Von Mises (1,2,3)= 150N/m^2

  Volume = 26.4 %

NUMERICAL APPLICATIONS: 3-BAR TRUSS





- q-norm of stresses (q=4):

  Bound: 500 N/m²

  Compliance: 87.3, 59.3, 67.9 Nm

  Max von Mises (local) for load case 1,2, 3 :
  230, 235, 231 N/m²

  Volume = 24.8%

- q-mean of stresses (q=4):

  Bound: 92 N/m²

  Compliance: 90.6, 50.3, 53.8 Nm

  Max von Mises (local) for load case 1,2, 3:
  237, 215, 207 N/m²

  Volume = 22.4%

Large scale optimization algorithms

- Fleury (2006) pointed out that the computation time of solution algorithm growths dramatically with the number of active constraints

- For dual maximization algorithms the explanation is rather easy. Let's consider the problem:

$$\begin{aligned} \min \quad & 1/2\, x^T x \\ s.t. \quad & C^T x \geq d \end{aligned}$$

dim x = n

dim C = nxm

- Dual function

$$\begin{aligned} \max \quad & -1/2\, \lambda^T (C^T C) \lambda + d^T \lambda \\ s.t. \quad & \lambda \geq 0 \end{aligned}$$

Large scale optimization algorithms

- Dual function maximization

$$\max \quad \ell(\lambda) = -1/2\, \lambda^T (C^T C)\lambda + d^T \lambda$$
$$s.t. \qquad \lambda \geq 0$$

- Solution algorithms

$$\nabla \ell(\lambda) = d - C^T x$$
$$\nabla^2 \ell(\lambda) = -C^T C$$

Iterative Newton scheme

$$\lambda^{(k+1)} = \lambda^{(k)} + \alpha(-C^T C)^{-1}(d - C^T \lambda^{(k)})$$

requires solving in various ways

$$(C^T C)^{-1}$$

dim C^TC = (mxm)

Large scale optimization algorithms

- Results based on numerical experiments by Fleury (2006) show that:
  - Computation time growths more or less linearly with the number n of design variables;
  - Computation time growths more or less like the power 3 of the number of active constraints.

- There is an urgent need for new solvers able to tackle huge problems with simultaneously a large number of design variables and a high number of active constraints

CONCLUSIONS
& PERSPECTIVES

CONCLUSIONS

- Additive manufacturing have put forward a revived interest for solving efficiently topology optimization problems with local constraints (e.g. stress constraints)

- Specific character of stress constraints
  - For several load cases
  - For unequal stress limits in tension and compression
  - Geometrical constraint
  - Several materials

- Extension of stress constraints to important problems for engineering applications:
  - Various failure criteria like unequal stress criteria
  - Fatigue

PERSPECTIVES

PENDING TOPICS:

- Efficient treatment of large scale optimization problems including stress constraints
  - Novel class of solution algorithms

- Accurate calculation of the stress constraints in the framework of material distribution problems:
  - Jagged / unclear boundaries
  - Stress intensity factors to take into account notches, etc.
  - Consider stress history $\sigma_i(t)$ instead of a single load case:
    - ➔ other criteria like Matake, Dang Van, Finley…
  - Consider cumulative damage Palmer Milgren

- Manufacturing constraints in order to generate designs which can be fabricated using AM

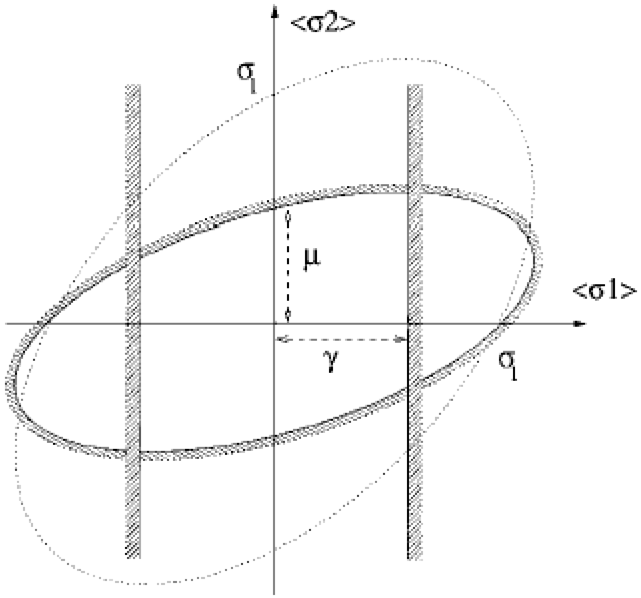
THANK YOU FOR
YOUR ATTENTION

PROBLEM FORMULATION

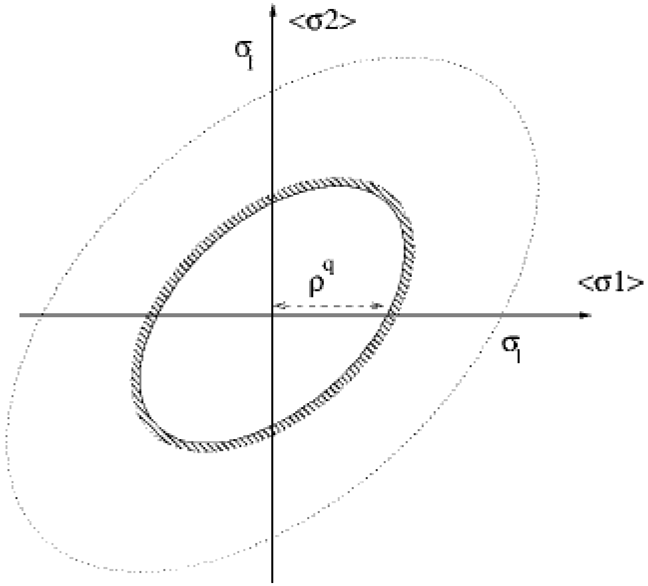
- Homogenized failure criteria predicting failure in the microstructure from macroscopic point of view:

$$\| \sigma^{eq}(\rho) \| = {}^{*}\sigma^{eq} / \rho^{p} \leq \sigma_{l}$$

- With consistency conditions requirements: p=q



Rank 2 layered material



SIMP (isotropic) material

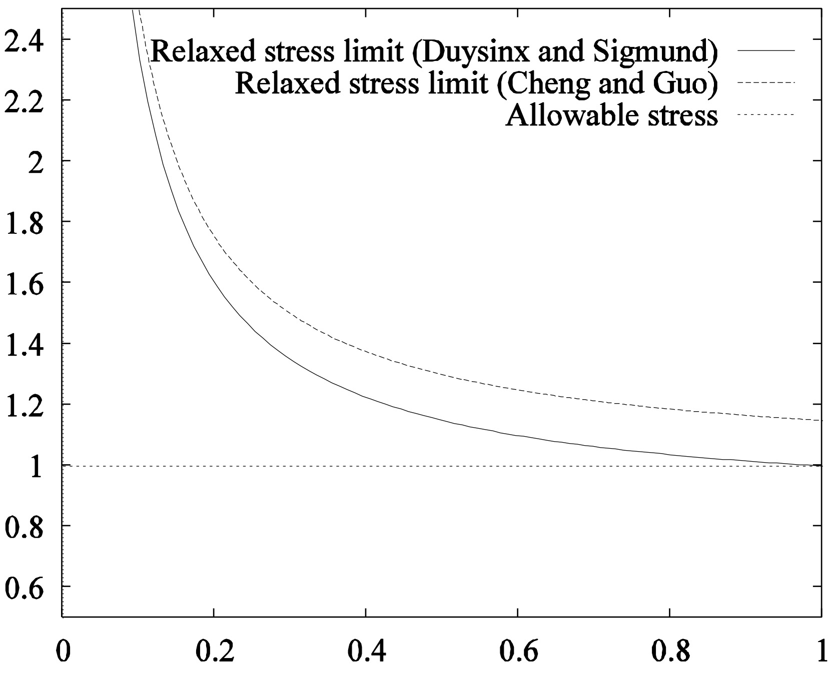
ε-relaxation: interpretation

- Relaxation of stress constraints

$$\| \sigma^{eq}(\rho) \| \leq \sigma_l \quad if \;\; \rho > 0$$

by

$$\begin{cases} \dfrac{\| \sigma^{eq}(\rho) \|}{\sigma_l} - \dfrac{\varepsilon}{\rho} + \varepsilon \leq 1 \\ \varepsilon^2 \leq \rho \end{cases}$$



- Solve a sequence of perturbated problems with a decreasing sequence of ε going to zero

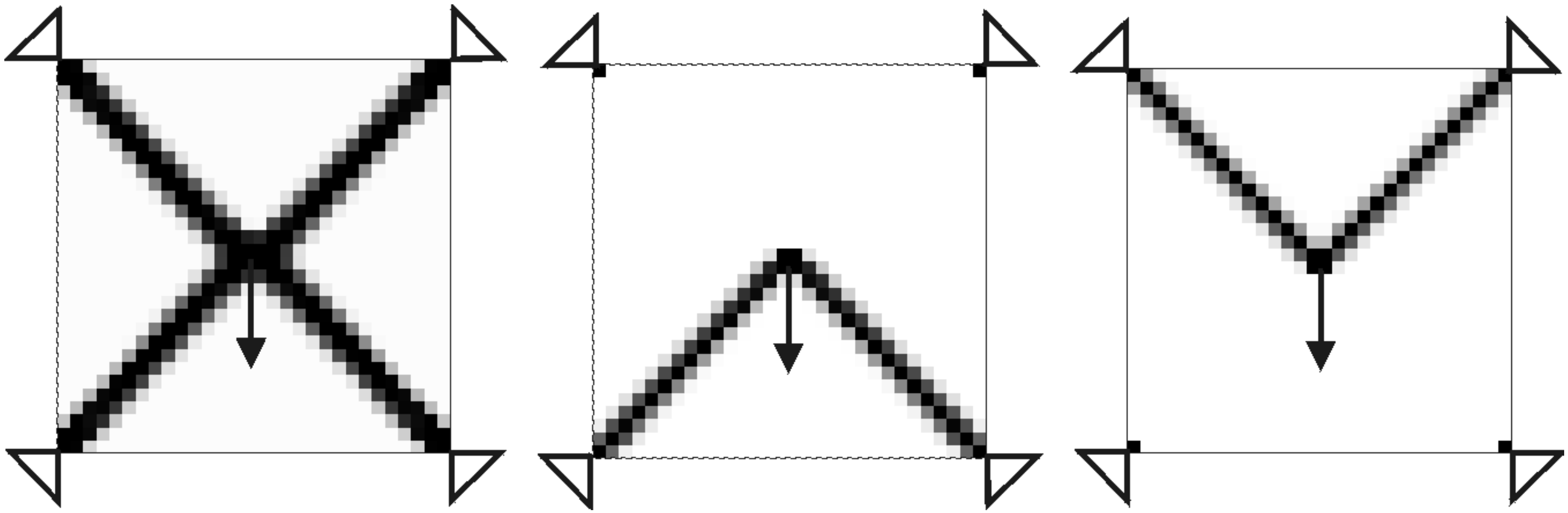
$$\min_{0 \leq \rho(x) \leq 1} \quad V = \int_\Omega \rho(x) \;\; dx$$

$$s.t. \quad \| \sigma^{eq}(\rho) \| \leq \sigma_l \; (1 - \varepsilon + \frac{\varepsilon}{\rho})$$

$$\varepsilon^2 \leq \rho$$

$$\| \sigma^{eq}(\rho) \| \leq \sigma_l \; (1 - \varepsilon + \frac{\varepsilon}{\rho})$$

NUMERICAL APPLICATIONS: 4-BAR TRUSS



Von Mises

T=C=6 N/m²

Ishai

T=6 & C=24 N/m²

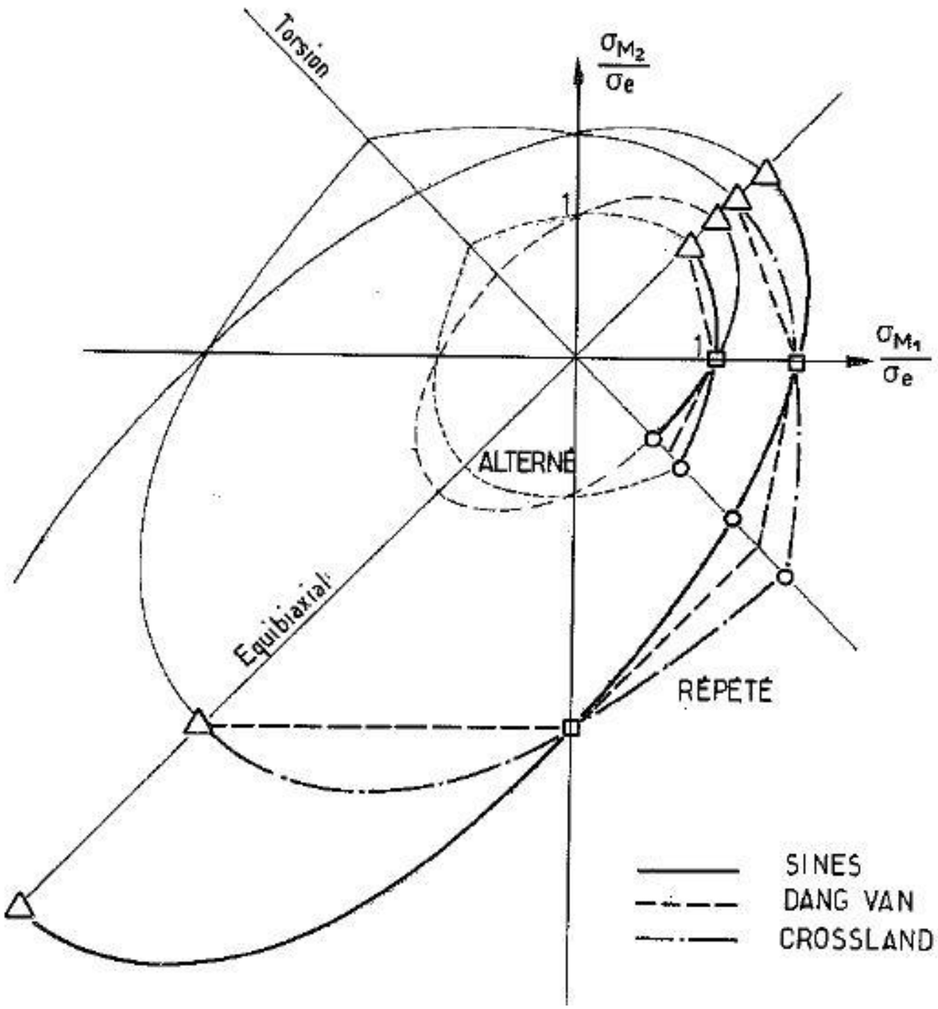
Ishai

T=24 & C=6 N/m²

E=100 N/m², ν=0.3, F =1 N, L =1 m

From Swan and Kosaka (1997)

MULTI AXIAL FATIGUE CRITERIA: CROSSLAND



MULTI AXIAL FATIGUE CRITERIA: SINES

- Assuming a SIMP model, after Finite Element discretization, one can calculate the stresses at appropriate positions (e.g. the element centroïd) using the tension matrix $\mathbf{T}_e^0$

$$\sigma_{ij} = x^p E^0_{ijkl} \varepsilon_{kl} \quad \Rightarrow \quad \boxed{\sigma_e = x_e^p \mathbf{T}_e^0 \mathbf{U}}$$

- First and second invariants can be computed by introducing the hydrostatic stress matrix $\mathbf{H}_e^0$ and the von Mises quadratic stress matrix $\mathbf{M}_e^0$:

$$\begin{aligned} J_{1,e}(\sigma_{ij}) &= x_e^p \mathbf{H}_e^0 \mathbf{U}_e \\ 3 J_{2D,e}(\sigma_{ij}) &= x_e^{2p} \mathbf{U}_e^T \mathbf{M}_e^0 \mathbf{U}_e \end{aligned}$$

- It is easy to recover the value of the alternate and mean stress components

$$\begin{aligned} \sigma_{a,e}^{eq} &= x_e^p \left(c_a \sqrt{\mathbf{U}_\mathrm{e}^\mathrm{T} \mathbf{M}_\mathrm{e}^0 \mathbf{U}_\mathrm{e}} \right) &= x_e^p \overline{\sigma}_{a,e}^{eq} \\ \sigma_{m,e}^{eq} &= x_e^p (c_m \mathbf{H}_\mathrm{e}^0 \mathbf{U}_\mathrm{e}) &= x_e^p \overline{\sigma}_{m,e}^{eq} \end{aligned}$$

MULTI AXIAL FATIGUE CRITERIA: SINES

- For topology optimization, as suggested by Duysinx & Bendsoe (1998), one should consider the micro stresses after applying the polarization factor

$$\langle \sigma_{ij,e} \rangle = \frac{\sigma_{ij,e}}{x_e^q}$$

- Sines criterion for topology optimization writes

$$\frac{\langle \sigma_{a,e}^{eq} \rangle}{\sqrt{3}\lambda} + \kappa \frac{\langle \sigma_{m,e}^{eq} \rangle}{3\lambda} \leq 1$$

- The final expression Sines criterion for topology optimization reads

$$\boxed{\frac{x_e^{(p-q)}}{\lambda} \left[\frac{\overline{\sigma}_{a,e}^{eq}}{\sqrt{3}} + \frac{\kappa}{3} \overline{\sigma}_{m,e}^{eq} \right] \leq 1}$$

SENSITIVITY ANALYSIS

- Sensitivity analysis of fatigue stress criteria requires the sensitivity analysis of the alternate, mean, and max components.

- Deriving the expression of the criteria, it comes

$$
\begin{aligned}
\frac{\partial \langle \sigma_{a,e}^{eq} \rangle}{\partial x_k} &= \delta_{ek}(p-q) x_e^{p-q-1} \, \overline{\sigma}_{a,e}^{eq} + \frac{\partial \overline{\sigma}_{a,e}^{eq}}{\partial x_k} x_e^{p-q} \\
\frac{\partial \langle \sigma_{m,e}^{eq} \rangle}{\partial x_k} &= \delta_{ek}(p-q) x_e^{p-q-1} \, \overline{\sigma}_{m,e}^{eq} + \frac{\partial \overline{\sigma}_{m,e}^{eq}}{\partial x_k} x_e^{p-q} \\
\frac{\partial \langle \sigma_{M,e}^{eq} \rangle}{\partial x_k} &= \delta_{ek}(p-q) x_e^{p-q-1} \, \overline{\sigma}_{M,e}^{eq} + \frac{\partial \overline{\sigma}_{M,e}^{eq}}{\partial x_k} x_e^{p-q}.
\end{aligned}
$$

SENSITIVITY ANALYSIS

- Selecting the adjoin methods since we have less active stress constraints that the number of design variables, one has:

$$\frac{\partial \overline{\sigma}_{a,e}^{eq}}{\partial x_k} = -\widetilde{\boldsymbol{U}}^T \frac{\partial \boldsymbol{K}}{\partial x_k} \boldsymbol{U} \quad \text{with} \quad \boldsymbol{K}\widetilde{\boldsymbol{U}} = \left[c_a (\boldsymbol{U}^T \boldsymbol{M}_e^0 \boldsymbol{U})^{-\frac{1}{2}} \boldsymbol{M}_e^0 \boldsymbol{U} \right]$$

$$\frac{\partial \overline{\sigma}_{m,e}^{eq}}{\partial x_k} = -\widetilde{\boldsymbol{U}}^T \frac{\partial \boldsymbol{K}}{\partial x_k} \boldsymbol{U}, \quad \text{with} \quad \boldsymbol{K}\widetilde{\boldsymbol{U}} = \left[c_m \boldsymbol{H}_e^0 \right]$$

$$\frac{\partial \overline{\sigma}_{M,e}^{eq}}{\partial x_k} = -\widetilde{\boldsymbol{U}}^T \frac{\partial \boldsymbol{K}}{\partial x_k} \boldsymbol{U}, \quad \text{with} \quad \boldsymbol{K}\widetilde{\boldsymbol{U}} = \left[c_a \boldsymbol{H}_e^0 + c_m \boldsymbol{H}_e^0 \right]$$

Sensitivity analysis

- Discretized equilibrium

$$K\,U = F$$

- Sensitivity of displacement vector

$$\frac{\partial U}{\partial \rho_i} = K^{-1}\left(\frac{\partial F}{\partial \rho_i} - \frac{\partial K}{\partial \rho_i}U\right)$$

- Direct approach: solve for every design variables

$$\sigma = TU$$

- Stress constraint

$$J_1^* = 3\sigma_h^* = w^T \quad \sigma = Wq \;\; and \;\; J_{2D}^* = 1/3\left(\sigma_{VM}^*\right)^2 = 1/3\;\; U^T V U$$

$$W = \rho^p W^0 \qquad\qquad V = \rho^{2p} V^0$$

Sensitivity analysis

- Sensitivity of unequal stress constraints: Ishai

$$\| \sigma_{ISH}^{eq} \| = {}^{*}\sigma_{ISH}^{eq} / \rho^{p} = \frac{s-1}{2s} W^{0} U + \frac{s+1}{2s} \sqrt{U^{T} V^{0} U}$$

- Derivative of criteria

$$\frac{\partial \| \sigma_{ISH}^{eq} \|}{\partial \rho_i} = \left\{ \frac{s-1}{2s} W^{0} + \frac{s+1}{2s} \frac{1}{\sqrt{U^{T} V^{0} U}} V^{0} q \right\}^{T} \frac{\partial U}{\partial \rho_i}$$

- Adjoin approach (for every constraint)

$$\lambda = K^{-1} \left\{ \frac{s-1}{2s} W^{0} + \frac{s+1}{2s} \frac{1}{\sqrt{q^{T} V^{0} q}} V^{0} U \right\}$$

$$\frac{\partial \| \sigma_{ISH}^{eq} \|}{\partial \rho_i} = \lambda^{T} \left(\frac{\partial g}{\partial \rho_i} - \frac{\partial K}{\partial \rho_i} U \right)$$