STRESS CONSTRAINED TOPOLOGY OPTIMIZATION FOR ADDITIVE MANUFACTURING: SPECIFIC CHARACTER AND SOLUTION ASPECTS

Pierre DUYSINX*, Maxime COLLET*, Simon BAUDUIN*, and Matteo BRUGGI+

* Aerospace and Mechanical Engineering Dept, University of Liege, Belgium
+ Dept of Civil and Environmental Engineering, Politecnico di Milano, Italy

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OUTLINE

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  - Local stress constraints
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INTRODUCTION & MOTIVATION
MOTIVATION

- **TOPOLOGY OPTIMIZATION:**
  a creative design tool

- **ADDITIVE MANUFACTURING:**
  new way of making things

Courtesy of ALTAIR and AIRBUS
INTRODUCTION

- Topology optimization is mostly based on compliance design formulation.

- Many aerospace and mechanical components are designed with respect to strength or fatigue constraints.

- Need for efficient approaches to handle efficiently stress constrained problems.

- Extending the scope of stress constrained topology optimization to cope with:
  - Fatigue constraints
  - Industrial applications → Large scale problems
INTRODUCTION

- This paper
  - Draws a state-of-the-art of topology optimization of continuum structures with stress constraints
  - Illustrates the specific character of maximum strength with respect to compliance design when considering
    - Several load cases
    - Different stress limits in tension and compression
  - Extends the scope of stress constrained topology optimization to unequal stress constraints, fatigue problems..
  - Draws the challenges to tackle large scale optimization problems related to local constraints
TOPOLOGY OPTIMIZATION FORMULATION
TOPOLOGY OPTIMIZATION PROBLEM

- Optimal material distribution within a given domain
- Discretization of displacements and density distribution using FEM
  \[ KU = F \]
- Interpolation of material properties between void and solid and penalize intermediate densities (SIMP model)
  \[ E_j(x_j) = E_{\text{min}} + x_j^p (E_0 - E_{\text{min}}) \]
- Solve optimization problem using efficient MP optimizers with continuous variables
TOPOLOGY OPTIMIZATION

- Density filter:

\[
\tilde{x}_e = \frac{1}{\sum_N H_{ej}} \sum_N H_{ej} x_j,
\]

\[
H_{ej} = \sum_N \max(0, r_{min} - \text{dist}(e, j)),
\]

- Implementation: Topology optimization tool in MATLAB based 88-line code by Andreassen et al. (2011)

- MMA solver by Svanberg (1987)

\[
\begin{align*}
\min & \quad f_0(x) + z + \sum_{j=1}^m (c_j y_j + \frac{1}{2} d_j y_j^2) \\
\text{s.t.:} & \quad f_j(x) - a_j z - y_j \leq 0 \quad j = 1 \ldots m \\
& \quad x_i \leq x_i \leq \bar{x}_i \quad i = 1 \ldots n \\
& \quad y_j \geq 0 \quad j = 1 \ldots m \\
& \quad z \geq 0
\end{align*}
\]
TOPOLOGY OPTIMIZATION PROBLEM

- Compliance design
  - Usual approach
  - Unable to capture the specific character of stress constraints

- Stress constrained design
  - Technical difficulties to be solved
  - Define appropriate failure criterion
  - Computational effort compared to compliance design

\[
\begin{align*}
\min_{0 < x \leq 1} & \quad F^T U \\
\text{s.t.} & \quad V = \sum_{e=1} v_e x_e \leq \tilde{V}
\end{align*}
\]

\[
\begin{align*}
\min_{0 < x \leq 1} & \quad \max_e ||\sigma_e(x)|| \\
\text{s.t.} & \quad V = \sum_{e=1} v_e x_e \leq \tilde{V}
\end{align*}
\]

[Duysinx et Bruggi (2012)]
Challenges of stress constraints in topology optimization

- Definition of relevant stress criteria at microscopic level
  - Microscopic stress should be considered
    \[ \sigma_{ij}^M = E_{ijkl} \epsilon_{kl}^M \]

  \[ < \sigma^e_{ij} > = \frac{\sigma_{ij}^e M}{x_e^q} \]

- Stress singularity phenomenon:
  - \( \varepsilon \)-relaxation (Chang and Guo, 1992)
  - q-p relaxation (Bruggi, 2008)

  \[ < \sigma^e_{ij} > = \frac{x_e^p E_{ijkl}^0 \epsilon_{kl}^0}{x_e^q} = x_e^{p-q} E_{ijkl}^0 \epsilon_{kl}^0 \]

  \[ q < p \quad \text{and} \quad q \nearrow p \]

- Large scale optimization problem
  - Local constraints
  - Aggregation of constraints: P-norm

\[ [ \sum_{e=1}^{N} < \| \sigma_e \| >^P ]^{1/P} \]
SPECIFIC CHARACTER OF STRESS CONSTRAINTS
SPECIFIC CHARACTER OF STRESS CONSTRAINTS

  \[ \int_{\Omega} \sigma_{VM}^2 d\Omega \leq \frac{3E}{4(1+\nu)} \int_{\Omega} \sigma : C\sigma d\Omega \leq \frac{3E}{4(1+\nu)} F^T U \]

- For single load case and minimum compliance with volume constraint:
  - Minimizing strain energy bounds almost everywhere the von Mises stress
  - Relation between energy minimization and fully stressed design nearly everywhere in the material
  - Compliance design is efficient to predict optimal structural lay-out
SPECIFIC CHARACTER OF STRESS CONSTRAINTS

- **Local strain energy** can be written as (Timoshenko and Goodier, 1970)

\[
 u = \frac{1}{2} \frac{\sigma_{oct}^2}{\chi} + \frac{3}{4} \frac{\tau_{oct}^2}{G}
\]

- with

\[
 \sigma_{oct} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}
\]

\[
 \chi = \frac{E}{3(1-\nu)}
\]

\[
 \tau_{oct} = \frac{1}{3} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}
\]

\[
 G = \frac{E}{2(1+\nu)}
\]

- Minimizing von Mises stress does not control compressibility energy!!!

- Tri-axiality is important.

- **Stiffness and strength designs can be different** when

  - Several load cases
  - Several materials
  - Different stress limits in tension and compression
  - Geometrical constraints (perimeter, manufacturing constraints...)

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NUMERICAL APPLICATIONS: 3-BAR TRUSS

- Famous benchmark problem with 3 independent load cases
  - $F_1 = 40 \text{ N}$
  - $F_2 = 30 \text{ N}$
  - $F_3 = 20 \text{ N}$

- Material and geometrical data
  - $L = 1 \text{ m}$
  - $W = 2.5 \text{ m}$
  - $E = 100 \text{ N/m}^2$
  - $\nu = 0.3$
  - $\sigma_l = 150 \text{ N/m}^2$
  - $V_{max} = 25\%$

- Finite Element mesh
  - 50 x 20 finite elements

- Design variables: 1000
- Load cases: 3
- Stress constraints: 3000
NUMERICAL APPLICATIONS: 3-BAR TRUSS

- Minimum compliance design
  Compliance (1,2,3) = 73.3 Nm
  Max von Mises:
  1) 229 N/m²
  2) 571 N/m²
  3) 555 N/m²
  Volume = 25%

- Stress constrained design
  Compliance
  1) 91.2 Nm
  2) 45.6 Nm
  3) 45.0 Nm
  Max Von Mises (1,2,3)= 150N/m²
  Volume = 26.4 %
Unequal stress limits in tension and compression

- Extending Von Mises criterion to other failure criteria to cope with unequal stress limits behaviors ($T \neq C$, $s=C/T$)

- **Raghava criterion** (parabolic criterion from Tsai-Wu criterion family)

  \[
  \sigma_{RAG}^{eq} = \frac{J_1(s-1) + \sqrt{J_1^2(s-1)^2 + 12J_{2D}s}}{2s} \leq T
  \]

- **Ishai criterion** (hyperbolic criterion from Prager-Drucker family)

  \[
  \sigma_{ISH}^{eq} = \frac{(s+1)\sqrt{3J_{2D}} + (s-1)J_1}{2s} \leq T
  \]
  
  - with $J_1 = \sigma_{ii}$, $J_{2D} = 0.5s_{ij}s_{ij}$
NUMERICAL APPLICATIONS: 3-BAR TRUSS

- High compressive strength ($s = C/T = 3$):
  $(C = 450 \text{ N/m}^2, \ T = 150 \text{ N/m}^2)$

  Volume = 25.6 %

  Compliance (1,2,3): 92.8, 47.3, 46.0 N*m

- High tensile strength ($s = C/T = 1/3$):
  $(C = 150 \text{ N/m}^2, \ T = 450 \text{ N/m}^2)$:

  Volume = 12.4 %
FATIGUE (UNI AXILAL CASE)

- Wöhler’s curve: fundamental work
  - Reduction of the amplitude of stress with the number of cycles

- Goodman diagram:
  - Influence of mean and alternate stress components
  - Line of equal failure probability for a certain number of cycles

\[ \sigma(t) = \sigma_m + \sigma_a \sin(\omega t) \]

- Amplitude / mean stress

\[ \sigma_a = \frac{\sigma_{\text{max}} - \sigma_{\text{min}}}{2} \]
\[ \sigma_m = \frac{\sigma_{\text{max}} + \sigma_{\text{min}}}{2} \]
MULTI AXIAL FATIGUE CRITERIA

- Like in 1-D problem let’s assume that the total stress is given by a certain amount of alternate component $c_a \sigma_a$ and a given amount of mean component $c_m \sigma_m$:

$$\sigma_{tot} = c_a \sigma_a + c_m \sigma_m$$

$$0 \leq c_a, c_m \leq 1$$

$$c_a + c_m = 1.$$ 

- In the following, let assume that alternate and mean components are defined by the same reference load case.
MULTI AXIAL FATIGUE CRITERIA: SINES

- Sines fatigue criterion:
  \[ \sqrt{J_{2,a}} + \kappa \cdot \sigma_{h,m} \leq \lambda \]
  
  - Where
    \[ \lambda = t_{-1} \quad \kappa = \frac{6t_{-1}}{f_0} - \sqrt{6} \]
  
  - With \( t_{-1} \), the fatigue limit in reverse torsion and \( f_0 \) is the fatigue in repeated bending

- For plane stress
  \[ J_{2,a} = \frac{1}{6} \left[ (\sigma_{11,a} - \sigma_{22,a})^2 + \sigma_{22,a}^2 + \sigma_{11,a}^2 + 6\sigma_{12,a}^2 \right] \]

  \[ \sigma_{h,m} = \frac{1}{3}(\sigma_{11,m} + \sigma_{22,m}) = \frac{J_1}{3} \]
Crossland fatigue criterion is very similar to Sines criterion:

\[ \sqrt{J_{2,a}} + \kappa \cdot \sigma_{h,max} \leq \lambda \]

Difference lies in the fact in Crossland the hydrostatic term is evaluated on the basis of the maximum stress (not only on the mean component): \( \sigma_{\text{max}} = \sigma_a + \sigma_m \):

\[ \sigma_{h,max} = \sigma_{h,a} + \sigma_{h,m} \]
MULTI AXIAL FATIGUE CRITERIA: SINES

- Assuming a SIMP model, after Finite Element discretization:

\[
\begin{align*}
\sigma_{a}^{eq} &= \sqrt{3J_2(\sigma_{a,ij})} \\
\sigma_{m}^{eq} &= J_1(\sigma_{m,ij})
\end{align*}
\]

\[
\sigma_{a,e}^{eq} = x_{e}^{p} \left( c_{a} \sqrt{U_{e}^{T}M_{e}^{0}U_{e}} \right) = x_{e}^{p} \bar{\sigma}_{a,e}^{eq}
\]

\[
\sigma_{m,e}^{eq} = x_{e}^{p} (c_{m} H_{e}^{0} U_{e}) = x_{e}^{p} \bar{\sigma}_{m,e}^{eq}
\]

- Considering the micro stresses after applying the polarization factor:

\[
\langle \sigma_{ij,e} \rangle = \frac{\sigma_{ij,e}}{x_{e}^{q}}
\]

- The expression Sines criterion for topology optimization reads:

\[
\frac{x_{e}^{(p-q)}}{\lambda} \left[ \frac{\sigma_{a,e}^{eq}}{\sqrt{3}} + \frac{\kappa}{3} \bar{\sigma}_{m,e}^{eq} \right] \leq 1
\]
NUMERICAL APPLICATION: L-SHAPE

- SIMP model
  - Penalization $p=3$
  - $q-p$ relaxation: $q=2.6 \rightarrow 2.75$

- Load $F=95$ N
  - $c_a = 0.7$ and $c_m = 0.3$

- Material: Steel with properties from Norton (2000)
  - $E = 1$ Mpa (normalized), $\nu=0.3$
  - $\sigma_f = 580$ MPa, $t_{-1} = 160$ MPa, $f_{-1} = 260$ MPa

- Compliance regularization constraint: $\alpha_c = 2$
NUMERICAL APPLICATION: LSHAPE

Optimal design with Sines criterion

Optimal design with Crossland criterion
NUMERICAL APPLICATION: LSHAPE

Stress map for optimal design with Sines criterion

Stress map for optimal design with Crossland criterion
NUMERICAL APPLICATION: LSHAPE

<table>
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<tr>
<th>Problem</th>
<th>N</th>
<th>$\mathcal{W}/\mathcal{W}_0$</th>
<th>$C/C_0$</th>
<th>$N_a^f$</th>
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<td>MWS</td>
<td>4096</td>
<td>0.4553</td>
<td>2</td>
<td>299</td>
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<tr>
<td>MWC</td>
<td>4096</td>
<td>0.4991</td>
<td>1.97</td>
<td>332</td>
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</tbody>
</table>

Evolution of the objective function volume

Evolution of the number of active constraints
SOLVING LARGE SCALE OPTIMIZATION PROBLEMS
SOLVING LARGE SCALE OPTIMIZATION

- Classical strategy: solve optimization sequential convex programming
  - Generate first order approximation sub-problems: CONLIN (Fleury, 1985) or MMA (Svanberg, 1987) or GCMMA approximation (Bruyneel et al., 2002)
  - Dual solver (Lagrangian maximization)

- When dealing with stress constrained design, one hits the limitation of currently available standard:
  - Number of active restrictions is more or less equal to the number of design variables
  - Sensitivity analysis become very expensive
  - Solution time of optimization algorithm becomes of the same order of magnitude as the FE computation.
Strategies to solve large scale problems

- Improve the sensitivity analysis:
  - Selection of potentially active constraints
  - Adjoin vs direct sensitivity analysis

- Introduction ‘dummy’ compliance constraint’ to control the convergence during first steps (Bruggi & Duysinx, 2013)

- Use integrated stress constraints instead of a purely local approach
  - Lose of local control of stress constraints: results looks closer to compliance design (Duysinx & Sigmund, 1998)
  - Rather difficult to tackle with classical approximation (function not convex)
Sensitivity analysis

- **Direct approach:** solve n (#dv) load cases
  \[
  \frac{\partial U}{\partial \rho_i} = K^{-1} \left( \frac{\partial F}{\partial \rho_i} - \frac{\partial K}{\partial \rho_i} U \right)
  \]

- **Adjoin method:** solve m (#constraints) load cases
  \[
  \lambda = K^{-1} \left\{ \frac{s-1}{2s} W^0 + \frac{s+1}{2s} \frac{1}{\sqrt{U^T V^0 U}} V^0 U \right\}
  \]
  - For one load case: \( m = \#\text{FE} \sim n \)
  - For several load cases: \( m = \#\text{FE} \times \#\text{load cases} > n \)
Problem formulation: compliance constraint

- Minimum volume with (fatigue stress) constraints and compliance constraint

\[
\begin{aligned}
\min_{x_0 \leq x_e \leq 1} \quad & W = \sum_{N} x_e V_e \\
\text{s.t.} \quad & K(x) U = F, \\
& C / C_L \leq 1, \\
& \frac{x_e^{(p-q)}}{\lambda} \left[ \frac{\sigma_{a,e}^{eq}}{\sqrt{3}} + \frac{\kappa_{e}^{eq} \sigma_{M,e}}{3} \right] \leq 1, \\
& \text{for } e = 1, \ldots, N
\end{aligned}
\]

- Compliance constraints is introduced to provide a better stability and effectiveness to the convergence (Bruggi & Duysinx, 2012)

\[
C_L = \alpha_c C_0
\]
Integrated (aggregated) stress constraint

- Use aggregate restriction of relaxed stress constraints (Duysinx & Sigmund, 1998)
  - q-norm
    \[
    \left( \sum_{e=1}^{N} \left( \max \left\{ 0, \frac{1}{T} \left( \frac{\sigma^{eq}}{\rho_e} - \frac{\varepsilon}{\rho_e} + \varepsilon \right) \right\} \right)^q \right)^{1/q} \leq 1
    \]
  - q-mean
    \[
    \left( \frac{1}{N} \sum_{e=1}^{N} \left( \max \left\{ 0, \frac{1}{T} \left( \frac{\sigma^{eq}}{\rho_e} - \frac{\varepsilon}{\rho_e} + \varepsilon \right) \right\} \right)^q \right)^{1/q} \leq 1
    \]

- Ordering relationship
  \[
  \left[ \frac{1}{N} \sum_{e=1}^{N} \left\| \sigma_e^{*} \right\|^{q} \right]^{1/q} \leq \max_{e=1\ldots N} \left\| \sigma_e^{*} \right\|^{q} \leq \left[ \sum_{e=1}^{N} \left\| \sigma_e^{*} \right\|^{q} \right]^{1/q}
  \]
NUMERICAL APPLICATIONS: 3-BAR TRUSS

- Minimum compliance design

  Compliance (1,2,3) = 73.3 Nm

  Max von Mises:
  1) 229 N/m²
  2) 571 N/m²
  3) 555 N/m²

  Volume = 25%

- Stress constrained design

  Compliance
  1) 91.2 Nm
  2) 45.6 Nm
  3) 45.0 Nm

  Max Von Mises (1,2,3) = 150N/m²

  Volume = 26.4 %
NUMERICAL APPLICATIONS: 3-BAR TRUSS

- q-norm of stresses (q=4):
  
  Bound: 500 N/m²
  
  Compliance: 87.3, 59.3, 67.9 Nm
  
  Max von Mises (local) for load case 1,2, 3:
  
  230, 235, 231 N/m²
  
  Volume = 24.8%

- q-mean of stresses (q=4):
  
  Bound: 92 N/m²
  
  Compliance: 90.6, 50.3, 53.8 Nm
  
  Max von Mises (local) for load case 1,2, 3:
  
  237, 215, 207 N/m²
  
  Volume = 22.4%
Large scale optimization algorithms

- Fleury (2006) pointed out that the computation time of solution algorithm grows dramatically with the number of active constraints.

- For dual maximization algorithms the explanation is rather easy. Let’s consider the problem:

\[
\begin{align*}
\min & \quad \frac{1}{2} x^T x \\
\text{s.t.} & \quad C^T x \geq d
\end{align*}
\]

\(\text{dim } x = n\)

\(\text{dim } C = nxm\)

- Dual function

\[
\begin{align*}
\max & \quad -\frac{1}{2} \lambda^T (C^T C) \lambda + d^T \lambda \\
\text{s.t.} & \quad \lambda \geq 0
\end{align*}
\]
Large scale optimization algorithms

- Dual function maximization
  \[
  \max \quad \ell(\lambda) = -\frac{1}{2} \lambda^T (C^T C) \lambda + d^T \lambda \\
  \text{s.t.} \quad \lambda \geq 0
  \]

- Solution algorithms
  \[
  \nabla \ell(\lambda) = d - C^T x \\
  \nabla^2 \ell(\lambda) = -C^T C
  \]

Iterative Newton scheme
\[
\lambda^{(k+1)} = \lambda^{(k)} + \alpha (-C^T C)^{-1} (d - C^T \lambda^{(k)})
\]

requires solving in various ways
\[
(C^T C)^{-1} \quad \text{dim } C^T C = (m \times m)
\]
Large scale optimization algorithms

- Results based on numerical experiments by Fleury (2006) show that:
  - Computation time growths more or less linearly with the number \( n \) of design variables;
  - Computation time growths more or less like the power 3 of the number of active constraints.

- There is an urgent need for new solvers able to tackle huge problems with simultaneously a large number of design variables and a high number of active constraints.
CONCLUSIONS & PERSPECTIVES
CONCLUSIONS

- Additive manufacturing have put forward a revived interest for solving efficiently topology optimization problems with local constraints (e.g. stress constraints)

- Specific character of stress constraints
  - For several load cases
  - For unequal stress limits in tension and compression
  - Geometrical constraint
  - Several materials

- Extension of stress constraints to important problems for engineering applications:
  - Various failure criteria like unequal stress criteria
  - Fatigue
PENDING TOPICS:

- Efficient treatment of large scale optimization problems including stress constraints
  - Novel class of solution algorithms

- Accurate calculation of the stress constraints in the framework of material distribution problems:
  - Jagged / unclear boundaries
  - Stress intensity factors to take into account notches, etc.
  - Consider stress history $\sigma_i(t)$ instead of a single load case:
    - other criteria like Matake, Dang Van, Finley...
  - Consider cumulative damage Palmer Milgren

- Manufacturing constraints in order to generate designs which can be fabricated using AM
THANK YOU FOR YOUR ATTENTION
PROBLEM FORMULATION

- Homogenized failure criteria predicting failure in the microstructure from macroscopic point of view:

\[ \| \sigma^{eq}(\rho) \| = \frac{\sigma^{eq} \rho^p}{\rho} \leq \sigma_i \]

- With consistency conditions requirements: \( p=q \)

- Rank 2 layered material
- SIMP (isotropic) material
**ε-relaxation: interpretation**

- **Relaxation of stress constraints**

\[
\| \sigma^{eq}(\rho) \| \leq \sigma_l \quad \text{if} \quad \rho > 0
\]

by

\[
\begin{cases}
\| \sigma^{eq}(\rho) \| - \frac{\varepsilon}{\rho} + \varepsilon \leq 1 \\
\sigma_l \\
\varepsilon^2 \leq \rho
\end{cases}
\]

- **Solve a sequence of perturbated problems with a decreasing sequence of ε going to zero**

\[
\min_{0 \leq \rho(x) \leq 1} \quad V = \int_{\Omega} \rho(x) \ dx
\]

s.t.

\[
\| \sigma^{eq}(\rho) \| \leq \sigma_l \left(1 - \varepsilon + \frac{\varepsilon}{\rho}\right)
\]

\[
\varepsilon^2 \leq \rho
\]
NUMERICAL APPLICATIONS: 4-BAR TRUSS

Von Mises
T=C=6 N/m²

Ishai
T=6 & C=24 N/m²

Ishai
T=24 & C=6 N/m²

E=100 N/m², ν=0.3, F =1 N, L =1 m

From Swan and Kosaka (1997)
MULTI AXIAL FATIGUE CRITERIA: CROSSLAND
MULTI AXIAL FATIGUE CRITERIA: SINES

- Assuming a SIMP model, after Finite Element discretization, one can calculate the stresses at appropriate positions (e.g. the element centroid) using the tension matrix $T_e^0$

$$\sigma_{ij} = x^p E_{ij kl}^0 \varepsilon_{kl}$$

- First and second invariants can be computed by introducing the hydrostatic stress matrix $H_e^0$ and the von Mises quadratic stress matrix $M_e^0$:

$$J_{1,e}(\sigma_{ij}) = x^p H_e^0 U_e$$

$$3J_{2D,e}(\sigma_{ij}) = x^{2p} U_e^T M_e^0 U_e$$

- It is easy to recover the value of the alternate and mean stress components

$$\sigma_{a,e}^{eq} = x^p \left( c_a \sqrt{U_e^T M_e^0 U_e} \right) = x^p \bar{\sigma}_{a,e}^{eq}$$

$$\sigma_{m,e}^{eq} = x^p (c_m H_e^0 U_e) = x^p \bar{\sigma}_{m,e}^{eq}$$
MULTI AXIAL FATIGUE CRITERIA: SINES

- For topology optimization, as suggested by Duysinx & Bendsoe (1998), one should consider the micro stresses after applying the polarization factor

\[
\langle \sigma_{i,j,e} \rangle = \frac{\sigma_{i,j,e}}{x_e^q}
\]

- Sines criterion for topology optimization writes

\[
\frac{\langle \sigma_{a,e}^{eq} \rangle}{\sqrt{3} \lambda} + \kappa \frac{\langle \sigma_{m,e}^{eq} \rangle}{3 \lambda} \leq 1
\]

- The final expression Sines criterion for topology optimization reads

\[
x_e^{(p-q)} \left( \frac{\sigma_{a,e}^{eq}}{\sqrt{3}} + \frac{\kappa}{3} \sigma_{m,e}^{eq} \right) \leq 1
\]
SENSITIVITY ANALYSIS

- Sensitivity analysis of fatigue stress criteria requires the sensitivity analysis of the alternate, mean, and max components.

- Deriving the expression of the criteria, it comes

\[
\begin{align*}
\frac{\partial \langle \sigma_{a,e}^{eq} \rangle}{\partial x_k} &= \delta_{ek} (p - q) x^{p-q-1} \, e^{eq \sigma_{a,e}^{eq}} + \frac{\partial \sigma_{a,e}^{eq}}{\partial x_k} x^{p-q} \\
\frac{\partial \langle \sigma_{m,e}^{eq} \rangle}{\partial x_k} &= \delta_{ek} (p - q) x^{p-q-1} \, e^{eq \sigma_{m,e}^{eq}} + \frac{\partial \sigma_{m,e}^{eq}}{\partial x_k} x^{p-q} \\
\frac{\partial \langle \sigma_{M,e}^{eq} \rangle}{\partial x_k} &= \delta_{ek} (p - q) x^{p-q-1} \, e^{eq \sigma_{M,e}^{eq}} + \frac{\partial \sigma_{M,e}^{eq}}{\partial x_k} x^{p-q}.
\end{align*}
\]
SENSITIVITY ANALYSIS

- Selecting the adjoin methods since we have less active stress constraints that the number of design variables, one has:

\[
\frac{\partial \bar{\sigma}_{a,e}}{\partial x_k} = -\tilde{U}^T \frac{\partial K}{\partial x_k} U \quad \text{with} \quad K\tilde{U} = \left[ c_a (U^T M_e^0 U)^{-\frac{1}{2}} M_e^0 U \right]
\]

\[
\frac{\partial \bar{\sigma}_{m,e}}{\partial x_k} = -\tilde{U}^T \frac{\partial K}{\partial x_k} U, \quad \text{with} \quad K\tilde{U} = \left[ c_m H_e^0 \right]
\]

\[
\frac{\partial \bar{\sigma}_{M,e}}{\partial x_k} = -\tilde{U}^T \frac{\partial K}{\partial x_k} U, \quad \text{with} \quad K\tilde{U} = \left[ c_a H_e^0 + c_m H_e^0 \right]
\]
Sensitivity analysis

- Discretized equilibrium
  \[ KU = F \]

- Sensitivity of displacement vector
  \[ \frac{\partial U}{\partial \rho_i} = K^{-1} \left( \frac{\partial F}{\partial \rho_i} - \frac{\partial K}{\partial \rho_i} U \right) \]

- Direct approach: solve for every design variables
  \[ \sigma = TU \]

- Stress constraint
  \[ J_1^* = 3\sigma_h^* = W^T \quad \sigma = Wq \quad \text{and} \quad J_{2D}^* = \frac{1}{3} \left( \sigma_{VM}^* \right)^2 = \frac{1}{3} U^TVU \]
  \[ W = \rho^p W^0 \]
  \[ V = \rho^{2p} V^0 \]
Sensitivity analysis

- Sensitivity of unequal stress constraints: Ishai

\[
\| \sigma_{ISH}^{eq} \| = \frac{s-1}{2s} W^0 U + \frac{s+1}{2s} \sqrt{U^T V^0 U}
\]

- Derivative of criteria

\[
\frac{\partial \| \sigma_{ISH}^{eq} \|}{\partial \rho_i} = \begin{cases} \frac{s-1}{2s} W^0 + \frac{s+1}{2s} \frac{1}{\sqrt{U^T V^0 U}} V^0 \end{cases} \begin{bmatrix} q \end{bmatrix}^T \frac{\partial U}{\partial \rho_i}
\]

- Adjoin approach (for every constraint)

\[
\lambda = K^{-1} \begin{cases} \frac{s-1}{2s} W^0 + \frac{s+1}{2s} \frac{1}{\sqrt{q^T V^0 U}} V^0 \end{cases} \begin{bmatrix} U \end{bmatrix}
\]

\[
\frac{\partial \| \sigma_{ISH}^{eq} \|}{\partial \rho_i} = \lambda^T \left( \frac{\partial g}{\partial \rho_i} - \frac{\partial K}{\partial \rho_i} U \right)
\]