Denjoy-Carleman classes and lineability

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Introduction

A $\mathcal{C}^\infty$ function f is analytic at $x_0 \in \Omega$ if its Taylor series at x_0 converges to f on an open neighbourhood of x_0. Using Cauchy's estimates, it is equivalent to have the existence of a compact neighborhood K of x_0 and of two constants $C, h > 0$ such that

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Many examples of $\mathcal{C}^\infty$ nowhere analytic functions exist. An example was given by Cellérier (1890) with the function defined for all $x \in \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{+\infty} \frac{\sin(a^n x)}{n!}$$

where a is a positive integer larger than 1.

Question.

How large is the set of nowhere analytic functions in the Fréchet space $\mathcal{C}^\infty([0,1])$?
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Lineability (Aron, Gurariy, Seoane-Sepúlveda 2005)

Let X be a topological vector space, M a subset of X, and μ a cardinal number.

(1) The set M is lineable if $M \cup \{0\}$ contains an infinite dimensional vector subspace. If the dimension of this subspace is μ, M is said to be μ-lineable.

(2) When the above linear space can be chosen to be dense in X, we say that M is (μ-)dense-lineable.

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Results. Genericity (different notions) and extension using Gevrey classes

- Morgenstern 1954
- Cater 1984
- Salzmann and Zeller 1955
- Bernal-Gonzalez 2008
- Bastin, E., Nicolay 2012
- Conejero, Jiménez-Rodríguez, Muñoz-Fernández and Seoane-Sepúlveda 2012
- Bartoszewicz, Bienias, Filipczak and Głąb 2013
- Bastin, Conejero, E. and Seoane 2014

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Roumieu classes

Let Ω be an open subset of $\mathbb{R}$ and M be a weight sequence. The space $\mathcal{E}_{\{M\}}(\Omega)$ is defined by

$$\mathcal{E}_{\{M\}}(\Omega) := \left\{f \in \mathcal{C}^\infty(\Omega) : \forall K \subseteq \Omega \text{ compact } \exists h > 0 \text{ such that } \|f\|^M_{K,h} < +\infty\right\},$$

where

$$\|f\|^M_{K,h} := \sup_{k\in\mathbb{N}_0} \sup_{x\in K} \frac{|D^k f(x)|}{h^k M_k}.$$

If $f \in \mathcal{E}_{\{M\}}(\Omega)$, we say that f is M-ultradifferentiable of Roumieu type on Ω.

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Particular case. The weight sequences $(k!)_{k\in\mathbb{N}_0}$ and $((k!)^\alpha)_{k\in\mathbb{N}_0}$ with $\alpha > 1$.

Beurling classes

Let Ω be an open subset of $\mathbb{R}$ and M be a weight sequence. The space $\mathcal{E}_{(M)}(\Omega)$ is defined by

$$\mathcal{E}_{(M)}(\Omega) := \{ f \in \mathcal{C}^\infty(\Omega) : \forall K \subseteq \Omega \text{ compact }, \forall h > 0, \ \|f\|_{K,h}^{M} < +\infty \}.$$

If $f \in \mathcal{E}_{(M)}(\Omega)$, we say that f is M-ultradifferentiable of Beurling type on Ω and we use the representation

$$\mathcal{E}_{(M)}(\Omega) = \operatorname*{proj}_{\overleftarrow{K \subseteq \Omega}} \operatorname*{proj}_{\overleftarrow{h > 0}} \mathcal{E}_{M,h}(K)$$

to endow $\mathcal{E}_{(M)}(\Omega)$ with a structure of Fréchet space.

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Questions.

- When do we have $\mathcal{E}_{\{M\}}(\Omega) \subseteq \mathcal{E}_{(N)}(\Omega)$?
- In that case, “how small” is $\mathcal{E}_{\{M\}}(\Omega)$ in $\mathcal{E}_{(N)}(\Omega)$?

General assumptions.

- We assume that any weight sequence M is logarithmically convex, i.e.

$$M_k^2 \leq M_{k-1}M_{k+1}, \quad \forall k \in \mathbb{N}\,.$$

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- We assume that any weight sequence M is such that $M_0 = 1$.
- We will often work with non-quasianalytic weight sequences M, i.e. such that

$$\sum_{k=1}^{+\infty} (M_k)^{-1/k} < +\infty.$$

  By Denjoy-Carleman theorem, it is equivalent to the fact that there exists non-zero functions with compact support in $\mathcal{E}_{\{M\}}(\mathbb{R})$.

Inclusions between Denjoy-Carleman classes

Notation. Given two weight sequences M and N, we write

$$M \lhd N \quad \Longleftrightarrow \quad \lim_{k \to +\infty} \left(\frac{M_k}{N_k} \right)^{\frac{1}{k}} = 0.$$

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Let M, N be two weight sequences and let Ω be an open subset of $\mathbb{R}$. Then

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Keys.

- If $M \lhd N$, then there exists a weight sequence L such that $M \lhd L \lhd N$.
- The function
$$\theta(x) = \sum_{k=1}^{+\infty} \frac{M_k}{2^k} \left(\frac{M_{k-1}}{M_k}\right)^k \exp\left(2i\frac{M_k}{M_{k-1}}x\right)$$
belongs to $\mathcal{E}_{\{M\}}(\mathbb{R})$. Moreover, $|D^k\theta(0)| \geq M_k \forall k \in \mathbb{N}_0$, so that $\theta \notin \mathcal{E}_{(M)}(\mathbb{R})$.

Construction

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Proposition

Assume that M and N are two weight sequences such that $M \lhd N$. If M is non-quasianalytic, there exists a function of $\mathcal{E}_{(N)}(\mathbb{R})$ which is nowhere in $\mathcal{E}_{\{M\}}$.

Proof. From Lemma 1, there is $N^\star$ such that $M \lhd N^\star \lhd N$. Applying recursively this lemma, we get a sequence $(L^{(p)})_{p\in\mathbb{N}}$ of weight sequences such that

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For every $p \in \mathbb{N}$, Lemma 2 allows us to consider a function $f_p \in \mathcal{E}_{\{L^{(p)}\}}(\mathbb{R})$ such that

$$|D^j f_p(0)| \geq L_j^{(p)}, \quad \forall j \in \mathbb{N}_0\,.$$

Since M is non-quasianalytic,

$$\exists\phi \in \mathcal{E}_{\{M\}}(\mathbb{R}), \text{ supp}\varphi \text{ compact }, \phi \equiv 1 \text{ in a nbh of } 0.$$

Let $\{x_p : p \in \mathbb{N}_0\}$ be a dense subset of $\mathbb{R}$ with $x_0 = 0$. For every $p \in \mathbb{N}$, we can find $k_p > 0$ such that the function

$$\phi_p := \phi\big(k_p(\cdot - x_p)\big)$$

has its support disjoint from $\{x_0, \dots, x_{p-1}\}$. We define g_p

$$g_p(x) := \underbrace{f_p(x - x_p)}_{\in\mathcal{E}_{\{L(p)\}}(\mathbb{R})} \; \underbrace{\phi_p(x)}_{\in\mathcal{E}_{\{M\}}(\mathbb{R})} \; \in \mathcal{E}_{(N^\star)}(\mathbb{R}).$$

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Let $\gamma_p > 0$ be such that

$$\sup_{x \in \mathbb{R}} |D^j g_p(x)| \leq \gamma_p N_j^\star, \quad \forall j \in \mathbb{N}_0$$

and define the function g by

$$g := \sum_{p=1}^{+\infty} \frac{1}{\gamma_p 2^p} g_p.$$

1. $g \in \mathcal{E}_{(N)}(\mathbb{R})$: for every $j \in \mathbb{N}_0$ and every $x \in \mathbb{R}$, we have

$$\sum_{p=1}^{+\infty} \frac{1}{\gamma_p 2^p} |D^j g_p(x)| \leq \sum_{p=1}^{+\infty} \frac{1}{2^p} N_j^\star \leq N_j^\star$$

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2. g is nowhere in $\mathcal{E}_{\{M\}}$: By contradiction, assume that there exists an open subset Ω of $\mathbb{R}$ such that $g \in \mathcal{E}_{\{M\}}(\Omega)$. Let $p_0 \in \mathbb{N}$ such that $x_{p_0} \in \Omega$. Remark that

$$\sum_{p=p_0}^{+\infty} \frac{1}{\gamma_p 2^p} g_p = \underbrace{g}_{\in \mathcal{E}_{\{M\}}(\Omega) \subseteq \mathcal{E}_{(L^{(p_0)})}(\Omega)} - \underbrace{\sum_{p=1}^{p_0-1} \frac{1}{\gamma_p 2^p} g_p}_{\in \mathcal{E}_{(L^{(p_0)})}(\Omega)} \in \mathcal{E}_{(L^{(p_0)})}(\Omega).$$

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But, since the support of g_p is disjoint of x_{p_0} for every $p > p_0$, we also have

$$\left| \sum_{p=p_0}^{+\infty} \frac{1}{\gamma_p 2^p} D^j g_p(x_{p_0}) \right| = \frac{1}{\gamma_{p_0} 2^{p_0}} \left| D^j g_{p_0}(x_{p_0}) \right| = \frac{1}{\gamma_{p_0} 2^{p_0}} \left| D^j f_{p_0}(0) \right| \geq \frac{1}{\gamma_{p_0} 2^{p_0}} L_j^{(p_0)}$$

for every $j \in \mathbb{N}_0$, hence a contradiction.

Remark. Given a sequence $(L^{(p)})_{p\in\mathbb{N}}$ such that

$$M \lhd L^{(1)} \lhd L^{(2)} \lhd \cdots \lhd L^{(p)} \lhd \cdots \lhd N^* \lhd N$$

and a dense subset $\{x_p : p \in \mathbb{N}_0\}$ of $\mathbb{R}$, we have constructed a function g such that

- $g \in \mathcal{E}_{\{N^*\}}(\mathbb{R})$
- $\notin \mathcal{E}_{(L^{(p)})}(\Omega)$ for every neighbourhood Ω of x_p.

$\longrightarrow$ Main tool for the lineability!

Lineability

Proposition

Assume that $M \lhd N$. If M is non quasianalytic, the set of functions of $\mathcal{E}_{(N)}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{M\}}$ is $\mathfrak{c}$-lineable.

Idea of the proof. For every $t \in (0,1)$, we set

$$L_k^{(t)} := (M_k)^{1-t}(N_k)^t \quad \forall k \in \mathbb{N}_0 \,.$$

Then $M \lhd L^{(t)} \lhd N$ for all $t \in (0,1)$ and $L^{(t)} \lhd L^{(s)}$ if $t < s$.

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Remark that

$$M \lhd L^{(\frac{t}{2})} \lhd L^{(\frac{2t}{3})} \lhd L^{(\frac{3t}{4})} \lhd \cdots \lhd L^{(t)} \lhd N, \quad \forall t \in (0,1).$$

and we can consider $g_t \in \mathcal{E}_{\{L^{(t)}\}}(\mathbb{R})$ which is not in $\mathcal{E}_{\left(L^{\left((1-\frac{1}{p})t\right)}\right)}(\Omega)$, for any open neighbourhood Ω of x_p and for any $p \geq 2$. Then take

$$\mathcal{D} = \mathrm{span}\{g_t : t \in (0,1)\}.$$

Dense-lineability

Remark. The set of polynomials is dense in $\mathcal{E}_{(N)}(\mathbb{R})$. Let $(t_m)_{m\in\mathbb{N}}$ be a sequence of different elements of $(0,1)$ and let $(P_{t_m})_{m\in\mathbb{N}}$ be a dense sequence of polynomials in $\mathcal{E}_{(N)}(\mathbb{R})$.

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For every $m\in\mathbb{N}$, let $k_m>0$ be such that $k_m g_{t_m}\in U_m$, where $\{U_m : m\in\mathbb{N}\}$ is a 0-neighbourhoods basis in $\mathcal{E}_{(N)}(\mathbb{R})$.

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We set

$$\mathcal{D}_d = \operatorname{span}\left\{P_t + k_t g_t : t \in (0,1)\right\}$$

where $k_t = 1$ and $P_t = 0$ if $t \neq t_m$ for every $m \in \mathbb{N}$.

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Proposition

Assume that $M \lhd N$. If M is non quasianalytic, then $\mathcal{D}_d$ is dense in $\mathcal{E}_{(N)}(\mathbb{R})$, $\dim \mathcal{D}_d = \mathfrak{c}$ and any non zero function of $\mathcal{D}_d$ is nowhere in $\mathcal{E}_{\{M\}}$. In particular, the set of functions of $\mathcal{E}_{(N)}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{M\}}$ is $\mathfrak{c}$-dense-lineable in $\mathcal{E}_{(N)}(\mathbb{R})$.

Case of countable unions

Let N be a weight sequence and let $(M^{(n)})_{n\in\mathbb{N}}$ be a sequence of weight sequences such that $M^{(n)} \lhd N$ for every $n \in \mathbb{N}$.

Question.
What about the set of functions of $\mathcal{E}_{(N)}(\mathbb{R})$ which are nowhere in $\bigcup_{n\in\mathbb{N}} \mathcal{E}_{\{M^{(n)}\}}$?

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If there is $n_0 \in \mathbb{N}$ such that the weight sequence $M^{(n_0)}$ is non quasianalytic, the set of functions of $\mathcal{E}_{(N)}(\mathbb{R})$ which are nowhere in $\bigcup_{n\in\mathbb{N}} \mathcal{E}_{\{M^{(n)}\}}$ is $\mathfrak{c}$-dense-lineable in $\mathcal{E}_{(N)}(\mathbb{R})$.

Idea. Construct a weight sequence P such that

$$\bigcup_{n\in\mathbb{N}} \mathcal{E}_{\{M^{(n)}\}} \subseteq \mathcal{E}_{\{P\}} \subsetneq \mathcal{E}_{(N)}.$$

Gevrey classes

They correspond to Roumieu classes given by the weight sequence

$$M_k := (k!)^\alpha, \quad k \in \mathbb{N}_0 \, .$$

Corollary

Let $\alpha > 1$. The set of functions of $\mathcal{E}_{((k!)^\alpha)}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{(k!)^\beta\}}$ for every $\beta \in (1, \alpha)$, is $\mathfrak{c}$-dense-lineable in $\mathcal{E}_{((k!)^\alpha)}(\mathbb{R})$.

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Proof. It suffices to apply the previous result the weight sequences $M^{(n)}$ ($n \in \mathbb{N}$) given by

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where $(\beta_n)_{n\in\mathbb{N}}$ is an increasing sequence of $(1, \alpha)$ that converges to α.

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Proposition (Schmets, Valdivia 1991)

Let $\alpha > 1$. The set of functions of $\mathcal{E}_{((k!)^\alpha)}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{(k!)^\beta\}}$ for every $\beta \in (1, \alpha)$ is residual in $\mathcal{E}_{((k!)^\alpha)}(\mathbb{R})$.

More with weight functions

Definition

A function $\omega : [0, +\infty) \to [0, +\infty)$ is called a weight function if it is continuous, increasing and satisfies $\omega(0) = 0$ as well as the following conditions

(α) There exists $L \geq 1$ such that $\omega(2t) \leq L\omega(t) + L,\ t \geq 0$,

(β) $\displaystyle\int_1^{+\infty} \frac{\omega(t)}{t^2} dt < +\infty$,

(γ) $\log(t) = o(\omega(t))$ as t tends to infinity,

(δ) $\varphi_\omega : t \mapsto \omega(e^t)$ is convex on $[0, +\infty)$.

The Young conjugate of φ_ω is defined by

$$\varphi_\omega^*(x) := \sup\{xy - \varphi_\omega(y) : y > 0\}, \quad x \geq 0.$$

Roumieu classes

If ω is a weight function and if Ω is an open subset of $\mathbb{R}^n$, we define the space $\mathcal{E}_{\{\omega\}}(\Omega)$ of ω-ultradifferentiable functions of Roumieu type on Ω by

$$\mathcal{E}_{\{\omega\}}(\Omega) := \{f \in \mathcal{E}(\Omega) : \forall K \subset \Omega \text{ compact } \exists m \in \mathbb{N} \text{ such that } \|f\|_{K,m} < +\infty\},$$

where $\|f\|_{K,m} := \sup_{\alpha \in \mathbb{N}_0^n} \sup_{x \in K} |D^\alpha f(x)| \exp\left(-\frac{1}{m}\varphi_\omega^*(m|\alpha|)\right) < +\infty$.

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Beurling classes

If ω is a weight function and if Ω is an open subset of $\mathbb{R}^n$, the space $\mathcal{E}_{(\omega)}(\Omega)$ of ω-ultradifferentiable functions of Beurling type on Ω is defined by

$$\mathcal{E}_{(\omega)}(\Omega) := \{f \in \mathcal{E}(\Omega) : \forall K \subset \Omega \text{ compact }, \forall m \in \mathbb{N}, \ p_{K,m}(f) < +\infty\},$$

where for every compact subset K of $\mathbb{R}^n$ and every $m \in \mathbb{N}$

$$p_{K,m}(f) := \sup_{\alpha \in \mathbb{N}_0^n} \sup_{x \in K} |D^\alpha f(x)| \exp\left(-m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right).$$

We endow the space $\mathcal{E}_{(\omega)}(\Omega)$ with its natural Fréchet space topology.

Given two weight functions ω and σ, we write

$$\omega \lhd \sigma \iff \sigma(t) = o(\omega(t)) \text{ as } t \to +\infty.$$

Proposition

Let ω and σ be two weight functions such that $\omega \lhd \sigma$. If Ω is a convex open subset of $\mathbb{R}^n$, then $\mathcal{E}_{\{\omega\}}(\Omega)$ is strictly included in $\mathcal{E}_{(\sigma)}(\Omega)$.

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Proposition

Let ω and σ be two weight functions such that $\omega \lhd \sigma$. The set of functions of $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$ which are nowhere in $\mathcal{E}_{\{\omega\}}$ is dense-lineable in $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$.

Idea.

- Existence: Baire category theorem.
- Lineability: Construct a sequence $(\omega^{(p)})_{p\in\mathbb{N}}$ of weight functions such that

$$\omega \lhd \omega^{(1)} \lhd \cdots \lhd \omega^{(p)} \lhd \omega^{(p+1)} \lhd \cdots \lhd \sigma.$$

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Happy birthday!

I hope that you are enjoying the party!