About the Regularity of Cantor's Bijection

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Automatic Sequences
Liège (Belgium) – May 25, 2015

Introduction

In 1877, Cantor constructed a bijection between $[0,1]$ and $[0,1]^2$, bijection defined via continued fractions.

G. Cantor, *Ein Beitrag zur Mannigfaltigkeitslehre*, Journal für die reine und angewandte Mathematik (Crelle's Journal), **Vol. 84**, 242-258, 1877.

Contents of this presentation

Notations

$$E = [0,1], \qquad D = E \cap \mathbb{Q} \qquad \text{and} \qquad I = E \setminus D.$$

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Notations

$$E=[0,1],\qquad D=E\cap\mathbb{Q}\qquad \text{and}\qquad I=E\setminus D.$$

Finite Continued Fractions

A. Ya. Khintchine, *Continued Fractions*, P. Noordhoff, 1963.

Let $\boldsymbol{a} = (a_j)_{j\in\{1,\ldots,n\}}$ a finite sequence of positive real numbers ($n \in \mathbb{N}$); the expression $[a_1, \ldots, a_n]$ is recursively defined as follows:

$$[a_1] = \frac{1}{a_1} \quad \text{and} \quad [a_1, \ldots, a_m] = \frac{1}{a_1 + [a_2, \ldots, a_m]},$$

for any $m \in \{2, \ldots, n\}$. If $\boldsymbol{a} \in \mathbb{N}^n$, we say that

$$[a_1, \ldots, a_n] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}.$$

is a (simple) **finite continued fraction**.

Proposition

For any $\boldsymbol{a} \in \mathbb{N}^n$ ($n \in \mathbb{N}$), $[a_1, \ldots, a_n]$ belongs to D. Conversely, for any $x \in D$, there exists a natural number n and a sequence $\boldsymbol{a} \in \mathbb{N}^n$ such that $x = [a_1, \ldots, a_n]$.

Convergents and Finite Continued Fractions

Let $\boldsymbol{a} \in \mathbb{N}^{\mathbb{N}}$. For all $j \in \mathbb{N}$, the quantities $p_j(\boldsymbol{a})$ and $q_j(\boldsymbol{a})$ are recursively defined as follows:

$$p_{-1}(\boldsymbol{a}) = 1,\ q_{-1}(\boldsymbol{a}) = 0,\ p_0(\boldsymbol{a}) = 0,\ q_0(\boldsymbol{a}) = 1$$

and, for $k \in \{1, \ldots, j\}$,

$$\begin{cases} p_k(\boldsymbol{a}) = a_k p_{k-1}(\boldsymbol{a}) + p_{k-2}(\boldsymbol{a}) \\ q_k(\boldsymbol{a}) = a_k q_{k-1}(\boldsymbol{a}) + q_{k-2}(\boldsymbol{a}) \end{cases}.$$

The quotient $\frac{p_j(\boldsymbol{a})}{q_j(\boldsymbol{a})}$ is called the **convergent of order** j of $\boldsymbol{a}$.

Proposition

Let $\boldsymbol{a} \in \mathbb{N}^{\mathbb{N}}$. For all $j \in \mathbb{N}$, we have

$$\frac{p_j(\boldsymbol{a})}{q_j(\boldsymbol{a})} = [a_1, \ldots, a_j].$$

Infinite Continued Fractions

With the properties of convergents, we can show that the sequence

$$\frac{p_j(\boldsymbol{a})}{q_j(\boldsymbol{a})} = [a_1, \ldots, a_j], \quad j \in \mathbb{N}$$

converges. The limit is called an **infinite continued fraction** and is denoted by

$$[a_1, a_2, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}.$$

If the real number $x \in E$ is equal to $[\boldsymbol{a}]$ where $\boldsymbol{a} \in \mathbb{N}^n$ (with $n \in \mathbb{N}$) or $\boldsymbol{a} \in \mathbb{N}^{\mathbb{N}}$, we say that $[\boldsymbol{a}]$ is a continued fraction corresponding to x.

Theorem – Representation of the real numbers (of E)

Any element of D can be expressed as a finite continued fraction. We have $x \in I$ if and only if there exists an infinite continued fraction corresponding to x; moreover, this infinite continued fraction is unique.

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Metric Theory of Continued Fractions

Let $x = [\boldsymbol{a}] \in I$; for $n \in \mathbb{N}$, we set

$$I_n(x) = \big\{ y = [\boldsymbol{b}] \in I : b_j = a_j \text{ if } j \in \{1, \ldots, n\} \big\}$$

and this set is called "interval of rank n" of I. We have $I_{n+1}(x) \subset I_n(x) \subset I$ for any $n \in \mathbb{N}$ and

$$\bigcap_{n \in \mathbb{N}} I_n(x) = \{x\}.$$

In fact, we have

$$I_n(x) = \left(\frac{p_n(\boldsymbol{a})}{q_n(\boldsymbol{a})}, \frac{p_n(\boldsymbol{a}) + p_{n-1}(\boldsymbol{a})}{q_n(\boldsymbol{a}) + q_{n-1}(\boldsymbol{a})} \right) \cap I,$$

if n is even (if n is odd, the endpoints of the interval are reversed). Every interval of rank n is partitioned into a countable infinite number of intervals of rank $n+1$. By denoting $|I_n(x)|$ the Lebesgue measure of $I_n(x)$, we have

$$|I_n(x)| = \frac{1}{q_n(\boldsymbol{a})(q_n(\boldsymbol{a}) + q_{n-1}(\boldsymbol{a}))}.$$

The properties of $I_n(x)$ will be useful to study the regularity of Cantor's bijection.

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Cantor's Bijection

G. Cantor, *Ein Beitrag zur Mannigfaltigkeitslehre*, Journal für die reine und angewandte Mathematik (Crelle's Journal), **Vol. 84**, 242-258, 1877.

If $x = [\boldsymbol{a}] \in I$, we set

$$f_1(x) = [a_1, a_3, \ldots, a_{2j+1}, \ldots] \quad \text{and} \quad f_2(x) = [a_2, a_4, \ldots, a_{2j}, \ldots].$$

The application

$$f : I \to I^2 \; ; x \mapsto (f_1(x), f_2(x))$$

is the **Cantor's Bijection** on I.

Remark

Since the cardinals of E and I are equal, f can be extended to a one-to-one mapping from E to E^2.

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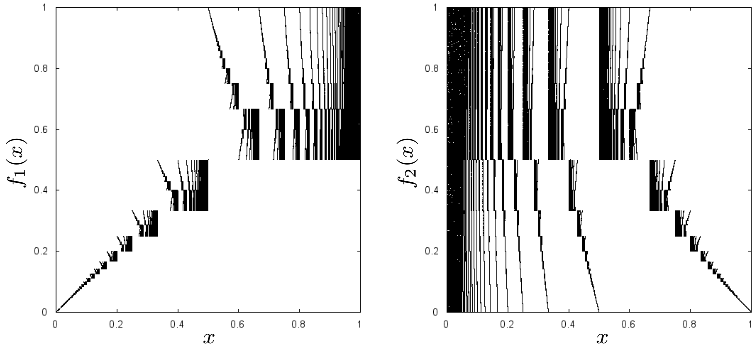
Cantor's Bijection

Representations of the functions f_1 (left panel) and f_2 (right panel)



For example, if $x \in (1/2, 1]$, then $x = [1, a_2, a_3, \ldots]$, $f_1(x) = [1, a_3, \ldots]$ and $f_1(x) \in (1/2, 1]$.

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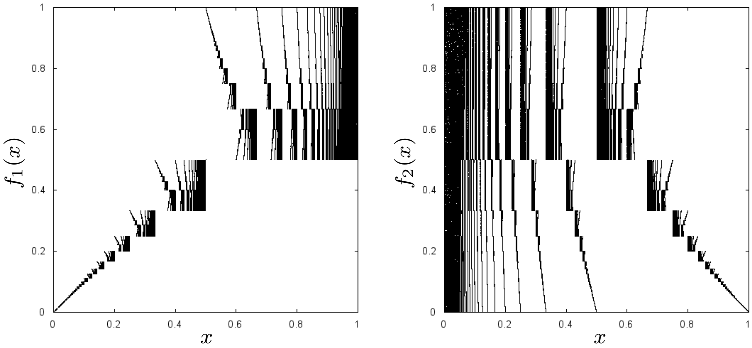

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Continuity of Cantor's Bijection

For any $n \in \mathbb{N}$ and any $x \in I$, f_1 maps the interval $I_n(x)$ to $I_m(f_1(x))$, where $m = n/2$ if n is even and $m = (n+1)/2$ if n is odd. The same argument can be applied to f_2.

Proposition

Cantor's bijection f is continuous on I.

Before giving some precisions, let us give an usual distance on $\mathbb{N}^{\mathbb{N}}$.

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A Distance on $\mathbb{N}^{\mathbb{N}}$

A usual distance on $\mathbb{N}^{\mathbb{N}}$ is given by

$$d(\boldsymbol{a},\boldsymbol{b}) = \sum_{j=1}^{\infty} 2^{-j} \frac{|a_j - b_j|}{|a_j - b_j| + 1}$$

if $\boldsymbol{a} = (a_j)_{j\in\mathbb{N}}$ and $\boldsymbol{b} = (b_j)_{j\in\mathbb{N}}$ are two elements of $\mathbb{N}^{\mathbb{N}}$. We implicitly consider that $\mathbb{N}^{\mathbb{N}}$ is equipped with this distance.

Remark

If we consider $\boldsymbol{a}$ and $\boldsymbol{b}$ as two infinite words on the alphabet $\mathbb{N}$, another equivalent distance on $\mathbb{N}^{\mathbb{N}}$ is given by

$$d'(\boldsymbol{a},\boldsymbol{b}) := \begin{cases} 0 & \text{if } \boldsymbol{a} = \boldsymbol{b} \\ 2^{-|\boldsymbol{a}\wedge\boldsymbol{b}|} & \text{if } \boldsymbol{a} \neq \boldsymbol{b} \end{cases}$$

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The sets I, D and E are endowed with the Euclidean distance.

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Continuity of Cantor's Bijection

For all $x \in I$, let us set $\varphi(x) = \boldsymbol{a}$ if $\boldsymbol{a} \in \mathbb{N}^{\mathbb{N}}$ satisfies $x = [\boldsymbol{a}]$. The application φ is a homeomorphism between I and $\mathbb{N}^{\mathbb{N}}$.

Remark

Since $(\mathbb{N}^{\mathbb{N}}, d)$ is a separable complete metric space, I is a Polish space.

Since the spaces $\mathbb{N}^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ are homeomorphic, we have the following proposition.

Proposition

Cantor's bijection f is a homeomorphism between I and I^2.

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Continuity of Cantor's Bijection

What about the continuity of f on D and then on E?

Netto's theorem

Any bijective map from E to E^2 is necessarily discontinuous.

E. Netto, *Beitrag zur Mannigfaltigkeitslehre*, Journal für die reine und angewandte Mathematik (Crelle's Journal), **Vol. 86**, 263-268, 1879.

Then, Cantor's bijection f can not be extended to a continuous bijection from E to E^2.

Proposition

Any extension of Cantor's bijection to E is discontinuous at any rational number.

Hölder Continuity and Hölder Exponent

S. Jaffard, *Wavelet Techniques in Multifractal Analysis*, In Proceedings of Symposia in Pure Mathematics, **Vol. 72**, 91-152, 2004.

Let $\alpha \in [0,1]$. A continuous and bounded real function g defined on $A \subset \mathbb{R}$ belongs to the **Hölder space** $\Lambda^\alpha(x)$ with $x \in A$ if there exists a constant $C > 0$ such that

$$|g(x) - g(y)| \leq C|x-y|^\alpha,$$

for any $y \in A$. The **Hölder exponent** $h_g(x)$ of g at x is defined as follows:

$$h_g(x) = \sup\{\alpha \in [0,1] : g \in \Lambda^\alpha(x)\}.$$

Remark

We have

$$h_g(x) = \liminf_{\substack{y \to x \\ y \in A}} \frac{\log|g(y)-g(x)|}{\log|y-x|}$$

The function g is **monofractal** if there exists $\alpha \in [0,1]$ such that $h_g(x) = \alpha$ for all $x \in A$. Otherwise, g is **multifractal**.

Hölder Continuity of Cantor's Bijection

Surrounding Theorem

Let $n \in \mathbb{N}$. If $x = [\boldsymbol{a}] \in I$ and $y \in I_n(x) \setminus I_{n+1}(x)$, then we have

$$\frac{\dfrac{1}{n}\displaystyle\sum_{j=1}^{\lceil n/2 \rceil} \log(a_{2j-1})}{\dfrac{1}{n}\displaystyle\sum_{j=1}^{n+3} \log(a_j + 1) + \dfrac{C_1(n)}{n}} \leq \frac{\log|f_1(x) - f_1(y)|}{\log|x-y|} \leq \frac{\dfrac{1}{n}\displaystyle\sum_{j=1}^{\lceil n/2 \rceil + 3} \log(a_{2j-1} + 1) + \dfrac{C_1(n)}{n}}{\dfrac{1}{n}\displaystyle\sum_{j=1}^{n} \log(a_j)}$$

with

$$C_1(n) = \frac{\log(2)}{2} + \log\left(\max\left\{\frac{a_{n+2}+2}{a_{n+2}+1}, \frac{a_{n+3}+2}{a_{n+3}+1}\right\}\right)$$

and

$$C_2(n) = \frac{\log(2)}{2} + \log\left(\max\left\{\frac{a_{2\lceil n/2 \rceil+3}+2}{a_{2\lceil n/2 \rceil+3}+1}, \frac{a_{2\lceil n/2 \rceil+5}+2}{a_{2\lceil n/2 \rceil+5}+1}\right\}\right).$$

There is a similar result for f_2.

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Hölder Continuity of Cantor's Bijection

Remark

Let $\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)}, \boldsymbol{a}^{(3)} \in \mathbb{N}^{\mathbb{N}}$ be the sequences defined by

$$a_j^{(1)} := \begin{cases} 2^j & \text{if } j \text{ is even} \\ 1 & \text{if } j \text{ is odd} \end{cases}, \quad a_j^{(2)} := 2^j \quad \text{and} \quad a_j^{(3)} := \begin{cases} 1 & \text{if } j \text{ is even} \\ 2^j & \text{if } j \text{ is odd} \end{cases}$$

for any $j \in \mathbb{N}$. Using Surrounding Theorem, we have

$$h_{f_1}([\boldsymbol{a}^{(1)}]) = 0, \quad h_{f_1}([\boldsymbol{a}^{(2)}]) = \frac{1}{2} \quad \text{and} \quad h_{f_1}([\boldsymbol{a}^{(3)}]) = 1.$$

Corollary

The functions f_1 and f_2 are multifractal. Consequently, f is multifractal.

Hölder Continuity of Cantor's Bijection

Surrounding Theorem

Let $n \in \mathbb{N}$. If $x = [\boldsymbol{a}] \in I$ and $y \in I_n(x) \setminus I_{n+1}(x)$, then we have

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There is a similar result for f_2.

Hölder Continuity of Cantor's Bijection

We say that a property P concerning sequences of $\mathbb{N}^{\mathbb{N}}$ holds almost everywhere if for almost every $x \in I$ (with respect to the Lebesgue measure), the sequence $\boldsymbol{a} \in \mathbb{N}^{\mathbb{N}}$ such that $x = [\boldsymbol{a}]$ satisfies P.

Ergodic Theorem

For any $k \in \mathbb{N} \cup \{0\}$, almost every sequence $\boldsymbol{a} \in \mathbb{N}^{\mathbb{N}}$ satisfies

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{j=1}^{n} \log(a_j + k) = \log(K_k),$$

where $K_k = \prod_{j=1}^{\infty} \left(1 + \frac{1}{j(j+2)}\right)^{\log(j+k)/\log(2)}$.

C. Ryll-Nardzewski, *On the Ergodic Theorems (II): Ergodic Theory of Continued Fractions*, Studia Mathematica **12**, 74-79, 1950.

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$$= \lim_{n \to +\infty} \frac{1}{n} \sum_{j=1}^{n} \log(a_{2j-1} + k) = \log(K_k),$$

where $K_k = \prod_{j=1}^{\infty} \left(1 + \frac{1}{j(j+2)}\right)^{\log(j+k)/\log(2)}$.

R. Nair, *On the Metrical of Continued Fractions*, In Proceedings of the Amercian Mathematicae Society **120**, 1994.

Hölder Continuity of Cantor's Bijection

By combining the surrounding theorem and the ergodic theorem, we obtain the following result.

Theorem

For almost every $x \in I$, we have

$$h_{f_1}(x), h_{f_2}(x) \in \left[\frac{\log(K_0)}{2\log(K_1)}, \frac{\log(K_1)}{2\log(K_0)} \right].$$

Then, $h_{f_1}(x)$ and $h_{f_2}(x)$ are included between 0.35 and 0.72.

S. Nicolay, L. Simons, *On the Multifractal Nature of Cantor's Bijection*, 2013, submitted.

Hölder Continuity of Cantor's Bijection

Surrounding Theorem – Some Improvements

Let $n \in \mathbb{N}$. If $x = [\boldsymbol{a}] \in I$ and $y \in I_n(x) \setminus I_{n+1}(x)$, then we have

$$\frac{2\log(q_{\lceil n/2\rceil}(\boldsymbol{a}'))}{\log(2) + 2\log(q_{n+3}(\boldsymbol{c}))} \leq \frac{\log|f_1(x) - f_1(y)|}{\log|x-y|} \leq \frac{\log(2) + 2\log(q_{\lceil n/2\rceil+3}(\boldsymbol{d}))}{2\log(q_n(\boldsymbol{a}))}$$

where $\boldsymbol{a}' = (a_{2j-1})_{j\in\mathbb{N}}$,

$$c_j = \begin{cases} a_j & \text{if } j \neq j_0 \\ a_j + 1 & \text{if } j = j_0 \end{cases} \quad \text{and} \quad d_j = \begin{cases} a_{2j-1} & \text{if } j \neq j_0' \\ a_{2j-1} + 1 & \text{if } j = j_0' \end{cases}$$

with j_0 is equal to $n+2$ or $n+3$ and j_0' to $n/2+2$ or $n/2+3$ (following the positions of y and $f_1(y)$ related to the ones of x and $f_1(x)$).

Hölder Continuity of Cantor's Bijection

Lévy's Theorem

For almost every sequence $\boldsymbol{b} \in \mathbb{N}^{\mathbb{N}}$, we have

$$\lim_{n \to +\infty} \frac{1}{n} \log(q_n(\boldsymbol{b})) = \frac{\pi^2}{12 \log(2)}.$$

P. Lévy, *Théorie de l'addition des variables aléatoires*, Gauthier-Villars, 2d édition, 1954

Moreover, we directly have

$$\lim_{n \to +\infty} \frac{1}{n+3} \log(q_{n+3}(\boldsymbol{c})) = \lim_{n \to +\infty} \frac{1}{n} \log(q_n(\boldsymbol{a}))$$

and

$$\lim_{n \to +\infty} \frac{1}{\lceil n/2 \rceil + 3} \log(q_{\lceil n/2 \rceil + 3}(\boldsymbol{d})) = \lim_{n \to +\infty} \frac{2}{n} \log(q_{\lceil n/2 \rceil}(\boldsymbol{a}'))$$

(if all these limits exist).

It only remains to show that all these limits are equal, which is not evident...

Hölder Continuity of Cantor's Bijection

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It only remains to show that all these limits are equal, which is not evident. . .

Hölder Continuity of Cantor's Bijection

Proposition

Let $x = [\boldsymbol{a}]$ be an element of I and let $\boldsymbol{a}' := (a_{2j-1})_{j \in \mathbb{N}}$. If we assume that

$$\lim_{n \to +\infty} \frac{1}{n} \log(q_n(\boldsymbol{a})) = \lim_{n \to +\infty} \frac{1}{n} \log(q_n(\boldsymbol{a}')) = \frac{\pi^2}{12 \log(2)},$$

then we have

$$h_{f_1}(x) = \frac{1}{2}.$$

There is of course a similar result for f_2.

Conjecture

For almost every $x \in [0,1]$, we have

$$h_{f_1}(x) = h_{f_2}(x) = \frac{1}{2}.$$

Hölder Continuity of Cantor's Bijection

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Hölder Continuity of Cantor's Bijection

How to show

$$\lim_{n\to+\infty}\frac{1}{n}\log(q_n(\boldsymbol{a}'))=\frac{\pi^2}{12\log(2)}\quad ?$$

An idea is to draw a proof of Lévy's theorem based on Birkhoff's ergodic theorem...

Let τ be the left shift operator on $\mathbb{N}^{\mathbb{N}}$, i.e. the application defined by

$$\tau((b_j)_{j\in\mathbb{N}}):=(b_{j+1})_{j\in\mathbb{N}}$$

We denote by τ^m the m^{th} iterate of τ for $m\in\mathbb{N}$ and by τ^0 the identity.

Lemma

For all $\boldsymbol{b}\in\mathbb{N}^{\mathbb{N}}$ and $n\in\mathbb{N}$, we have

$$\log(q_n(\boldsymbol{b}))=-\sum_{j=0}^{n-1}\log\left(\frac{p_{n-j}(\tau^j(\boldsymbol{b}))}{q_{n-j}(\tau^j(\boldsymbol{b}))}\right).$$

Hölder Continuity of Cantor's Bijection

How to show

$$\lim_{n\to+\infty} \frac{1}{n} \log(q_n(\boldsymbol{a}')) = \frac{\pi^2}{12\log(2)} \quad ?$$

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Lemma

For all $\boldsymbol{b} \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$, we have

$$\log(q_n(\boldsymbol{b})) = -\sum_{j=0}^{n-1} \log\left(\frac{p_{n-j}(\tau^j(\boldsymbol{b}))}{q_{n-j}(\tau^j(\boldsymbol{b}))}\right).$$

Hölder Continuity of Cantor's Bijection

For $\boldsymbol{b} \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$, we then have

$$\frac{1}{n}\log(q_n(\boldsymbol{b})) = -\frac{1}{n}\sum_{j=0}^{n-1}\log([\tau^j(\boldsymbol{b})]) + R_n(\boldsymbol{b})$$

where

$$R_n(\boldsymbol{b}) = \frac{1}{n}\sum_{j=0}^{n-1}\left(\log([\tau^j(\boldsymbol{b})]) - \log\left(\frac{p_{n-j}(\tau^j(\boldsymbol{b}))}{q_{n-j}(\tau^j(\boldsymbol{b}))}\right)\right).$$

Lemma

For all $b \in \mathbb{N}^{\mathbb{N}}$, we have

$$\lim_{n\to+\infty} R_n(\boldsymbol{b}) = 0$$

J. Steuding, *Ergodic Number Theory*, A Course at Vilnius University, 2013

Hölder Continuity of Cantor's Bijection

Let $x = [\boldsymbol{a}] \in I$. By definition, we have $f_1(x) = [\boldsymbol{a}']$ where $\boldsymbol{a}' := (a_{2j-1})_{j\in\mathbb{N}}$. Using the previous lemmas with $\boldsymbol{a}'$, we obtain

$$\begin{aligned}\frac{1}{n}\log(q_n(\boldsymbol{a}')) &= -\frac{1}{n}\sum_{j=0}^{n-1}\log([\tau^j(\boldsymbol{a}')]) + R_n(\boldsymbol{a}') \\ &= -\frac{1}{n}\sum_{j=0}^{n-1}\log([\tau^{2j}(\boldsymbol{a})]) + S_n(\boldsymbol{a}) + R_n(\boldsymbol{a}')\end{aligned}$$

where

$$S_n(\boldsymbol{a}) = \frac{1}{n}\sum_{j=0}^{n-1}\big(\log([\tau^{2j}(\boldsymbol{a})]) - \log([\tau^j(\boldsymbol{a}')])\big) = \frac{1}{n}\sum_{j=0}^{n-1}\log\left(\frac{[\tau^{2j}(\boldsymbol{a})]}{f_1([\tau^{2j}(\boldsymbol{a})])}\right)$$

and we know that

$$\lim_{n\to+\infty} R_n(\boldsymbol{a}') = 0.$$

Hölder Continuity of Cantor's Bijection

Theorem

For almost all $\boldsymbol{b} \in \mathbb{N}^{\mathbb{N}}$, we have

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log([\tau^{2j}(\boldsymbol{b})]) = \frac{1}{\log(2)} \int_0^1 \frac{\log(t)}{t+1}\, dt = -\frac{\pi^2}{12\log(2)}$$

R. Nair, *On the Metrical of Continued Fractions*, In Proceedings of the Amercian Mathematicae Society **120**, 1994.

We then have

$$\lim_{n\to+\infty} \frac{1}{n} \log(q_n(\boldsymbol{a}')) = \frac{\pi^2}{12\log(2)} + \lim_{n\to+\infty} S_n(\boldsymbol{a})$$

and it only remains to show that

$$\lim_{n\to+\infty} S_n(\boldsymbol{a}) = \lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log\left(\frac{[\tau^{2j}(\boldsymbol{a})]}{f_1([\tau^{2j}(\boldsymbol{a})])}\right) = 0,$$

which is not evident. It is difficult to reasonably compare $[\tau^{2j}(\boldsymbol{a})]$ and $f_1([\tau^{2j}(\boldsymbol{a})])$.

Hölder Continuity of Cantor's Bijection

Theorem

For almost all $\boldsymbol{b} \in \mathbb{N}^{\mathbb{N}}$, we have

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