
MATH0488 – Stochastic Processes

Stochastically perturbed bifurcation

Part 3 of 3: Theoretical study of buckling of randomly imperfect beam

Maarten Arnst, Marco Lucio Cerquaglia, and Kavita Goyal

April 28, 2015

- Two key ideas.
- Bifurcation analysis.
- Perfect beam.
- Imperfect beam
- Assignment.
- References.

Law of large numbers (LLN), central limit theorem (CLT), and ergodicity

- **Law of large numbers (LLN):** Let $\{X_\ell\}_{\ell=1}^{+\infty}$ be a sequence of independent and identically distributed (i.i.d.) copies of a random variable X with values in \mathbb{R} . If $\int |X|dP < +\infty$, then the sequence $\{S_\nu = \frac{1}{\nu} \sum_{\ell=1}^{\nu} X_\ell\}_{\nu=1}^{+\infty}$ converges almost surely to $\bar{x} = \int XdP$, that is,

$$\lim_{\nu \rightarrow +\infty} \frac{1}{\nu} \sum_{\ell=1}^{\nu} X_\ell \stackrel{\text{a.s.}}{=} \bar{x}.$$

- **Central limit theorem (CLT):** Let $\{X_\ell\}_{\ell=1}^{+\infty}$ be a sequence of i.i.d. copies of a random variable X with values in \mathbb{R} . If $\int |X|^2dP < +\infty$, then $\{\sqrt{\nu}(S_\nu - \bar{x}) = \sqrt{\nu} \left(\frac{1}{\nu} \sum_{\ell=1}^{\nu} X_\ell - \bar{x} \right)\}_{\nu=1}^{+\infty}$ converges in distribution to a Gaussian r.v. with mean 0 and variance σ_X^2 , that is,

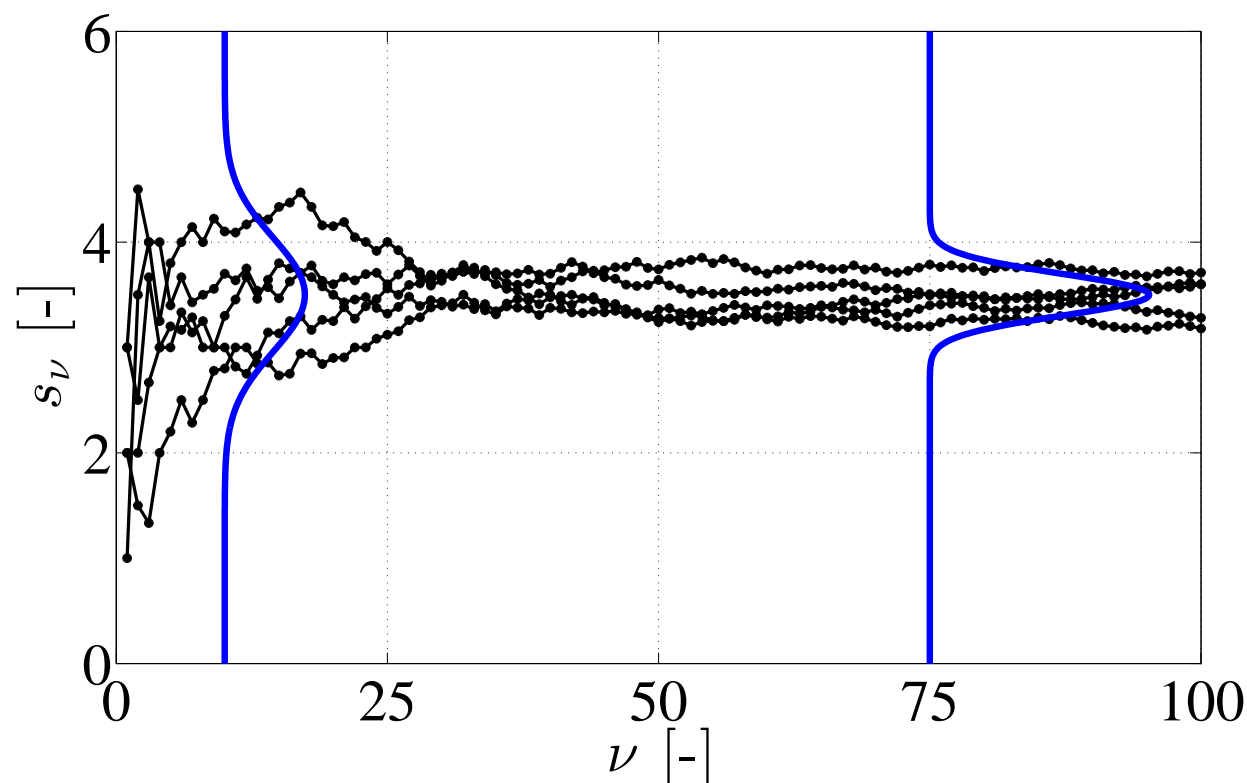
$$\lim_{\nu \rightarrow +\infty} \sqrt{\nu} \left(\frac{1}{\nu} \sum_{\ell=1}^{\nu} X_\ell - \bar{x} \right) \stackrel{d}{=} N(0, \sigma_X^2).$$

Two key ideas

Law of large numbers (LLN), central limit theorem (CLT), and ergodicity (continued)

■ Example “rolling a die:”

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
	4	3	1	1	5	4	4	1	4	6	4	5	3	1
$\frac{1}{\nu} \sum_{l=1}^{\nu} x_l$	4.0	3.5	2.6	2.2	2.8	3.0	3.1	2.8	3.0	3.3	3.3	3.5	3.4	3.2



Law of large numbers (LLN), central limit theorem (CLT), and ergodicity (continued)

- A stationary, second-order stochastic process $\{Z(t), t \in \mathbb{R}\}$ indexed by \mathbb{R} with values in \mathbb{R} is **mean-ergodic** if the mean $\bar{z} = \int Z(t)dP$ can be computed from the average of a sample path:

$$\lim_{\tau \rightarrow +\infty} \frac{1}{2\tau} \int_{-\tau}^{+\tau} Z(t)dt \stackrel{\text{m.s.}}{=} \bar{z}.$$

To establish the mean-ergodicity of a given stationary, second-order s.p. $\{Z(t), t \in \mathbb{R}\}$, one has to prove the mean-square convergence of $\frac{1}{2\tau} \int_{-\tau}^{+\tau} Z(t)dt$ to \bar{z} as $\tau \rightarrow +\infty$.

- The law of large numbers for independent random variables is related to the property of ergodicity for stochastic processes. The central limit theorem for independent random variables can also be extended to stochastic processes.
- In summary, under certain conditions, the behavior of the arithmetic mean of a sufficiently large number of random variables and the behavior of the sample-path average of a stationary second-order stochastic process can be described in terms of simple probabilistic laws.

Linear and nonlinear elimination

- Example linear elimination:

$$\begin{cases} x + 2y = 3 \\ 4x + 5y = 6 \end{cases} \implies \begin{cases} x + 2y = 3 \\ y = \frac{6}{5} - \frac{4}{5}x \end{cases} \implies x + 2 \left(\frac{6}{5} - \frac{4}{5}x \right) = 3.$$

- Example nonlinear elimination:

$$\begin{cases} g(x, y) = 0 \\ h(x, y) = 0 \end{cases} \implies \begin{cases} g(x, y) = 0 \\ y = \psi(x) \text{ s.t. } h(x, \psi(x)) = 0 \end{cases} \implies g(x, \psi(x)) = 0.$$

- In summary, linear and nonlinear elimination allow the dimensionality of a problem to be reduced.

Bifurcation analysis

Problem setting

- Let us consider again the boundary-value problem

$$\begin{cases} \frac{d^2(\theta - \sigma\theta_\epsilon)}{ds^2} + \lambda \sin(\theta) = 0 & \text{with } \lambda = \frac{p}{yj}, \\ \theta(0) = \sigma\theta_\epsilon(0) \quad \text{and} \quad \frac{d\theta}{ds}(\ell) = \sigma \frac{d\theta_\epsilon}{ds}(\ell). \end{cases}$$

- Introducing the auxiliary function $w = \theta - \sigma\theta_\epsilon$, we obtain the boundary-value problem

$$\begin{cases} \frac{d^2w}{ds^2} + \lambda \sin(w + \sigma\theta_\epsilon) = 0 & \text{with } \lambda = \frac{p}{yj}, \\ w(0) = \frac{dw}{ds}(\ell) = 0. \end{cases}$$

- The finite-difference approximation of this boundary-value problem leads to

$$[K]\mathbf{w} + \lambda \mathbf{f}(\mathbf{w} + \sigma\boldsymbol{\theta}_\epsilon) = \mathbf{0}.$$

where $\mathbf{f}(\mathbf{w} + \sigma\boldsymbol{\theta}_\epsilon)$ is the vector with components $\sin(w_j + \sigma\theta_\epsilon(s_j))$, $j = 1, \dots, \mu - 1$.

- If $\sigma\boldsymbol{\theta}_\epsilon = \mathbf{0}$, then the first bifurcation occurs at $(\lambda = \lambda_0, \mathbf{w} = \mathbf{0})$, where λ_0 is the smallest real value for which the Jacobian matrix has a vanishing eigenvalue, that is, for which there exists $\boldsymbol{\phi}_0 \neq \mathbf{0}$ for which $[K]\boldsymbol{\phi}_0 + \lambda_0[\mathbf{D}_\theta \mathbf{f}(\mathbf{0})]\boldsymbol{\phi}_0 = [K]\boldsymbol{\phi}_0 + \lambda_0\boldsymbol{\phi}_0 = \mathbf{0}$.

Lyapunov-Schmidt reduction

- The Lyapunov-Schmidt reduction involves deducing from the “full” problem a “reduced” problem by **nonlinear elimination** of the component of the solution orthogonal to the eigenvector corresponding to the vanishing eigenvalue of the Jacobian at the bifurcation.
- We begin with writing the solution as follows:

$$\mathbf{w} = (\phi_0 \cdot \mathbf{w})\phi_0 + \sum_{j=1}^{\mu-2} (\phi_j \cdot \mathbf{w})\phi_j,$$

and with writing the algebraic problem as follows:

$$\begin{cases} \phi_0 \cdot ([K]\mathbf{w} + \lambda\mathbf{f}(\mathbf{w} + \sigma\boldsymbol{\theta}_\epsilon)) = 0, \\ \phi_k \cdot ([K]\mathbf{w} + \lambda\mathbf{f}(\mathbf{w} + \sigma\boldsymbol{\theta}_\epsilon)) = 0, \quad k = 1, \dots, \mu - 2. \end{cases}$$

- Then, introducing this representation of the solution into this representation of the algebraic problem and introducing the variables ζ and ξ such that $\zeta = \phi_0 \cdot \mathbf{w}$ and $\lambda = \lambda_0(1 + \xi)$, we obtain

$$\begin{cases} \phi_0 \cdot \left([K] \left(\zeta\phi_0 + \sum_{j=1}^{\mu-2} (\phi_j \cdot \mathbf{w})\phi_j \right) + \lambda_0(1 + \xi)\mathbf{f} \left(\zeta\phi_0 + \sum_{j=1}^{\mu-2} (\phi_j \cdot \mathbf{w})\phi_j + \sigma\boldsymbol{\theta}_\epsilon \right) \right) = 0, \\ \phi_k \cdot \left([K] \left(\zeta\phi_0 + \sum_{j=1}^{\mu-2} (\phi_j \cdot \mathbf{w})\phi_j \right) + \lambda_0(1 + \xi)\mathbf{f} \left(\zeta\phi_0 + \sum_{j=1}^{\mu-2} (\phi_j \cdot \mathbf{w})\phi_j + \sigma\boldsymbol{\theta}_\epsilon \right) \right) = 0, \quad k = 1, \dots, \mu - 2. \end{cases}$$

Lyapunov-Schmidt reduction (continued)

- Subsequently, by the implicit function theorem, for sufficiently small (ζ, ξ, σ) , the latter equations uniquely determine the component of the solution orthogonal to the eigenvector corresponding to the vanishing eigenvalue of the Jacobian at the bifurcation, which we denote by $\psi(\zeta, \xi, \sigma)$, hence,

$$q_k(\zeta, \xi, \sigma) = \phi_k \cdot \left([K]\psi(\zeta, \xi, \sigma) + \lambda_0(1 + \xi)\mathbf{f}\left(\zeta\phi_0 + \psi(\zeta, \xi, \sigma) + \sigma\boldsymbol{\theta}_\epsilon\right) \right) = 0, \quad k = 1, \dots, \mu - 2.$$

- Finally, inserting $\psi(\zeta, \xi, \sigma)$ into the former equation, we obtain the “reduced” problem

$$p(\zeta, \xi, \sigma) = -\zeta + (1 + \xi)\phi_0 \cdot \mathbf{f}\left(\zeta\phi_0 + \psi(\zeta, \xi, \sigma) + \sigma\boldsymbol{\theta}_\epsilon\right) = 0.$$

- In summary, the Lyapunov-Schmidt reduction involves the reduction of the algebraic problem through nonlinear elimination to a “reduced” problem with only a few active independent variables:
 - ◆ ζ : component of solution along critical eigenvector for perfect beam,
 - ◆ ξ : excess of axial force over buckling load for perfect beam,
 - ◆ σ : magnitude of imperfection.

Power-series expansion

- It is a fundamental strategy of bifurcation theory to extract information about bifurcation behavior by examining a power-series expansion of the bifurcation equation:

$$p(\zeta, \xi, \sigma) = \sum_{\alpha, \beta, \gamma} \frac{\zeta^\alpha \xi^\beta \sigma^\gamma}{\alpha! \beta! \gamma!} \frac{\partial^{\alpha+\beta+\gamma} p}{\partial \zeta^\alpha \partial \xi^\beta \partial \sigma^\gamma} (0, 0, 0).$$

- Many coefficients vanish. In fact, among the leading terms, the only nonvanishing ones are $p_{001} = \partial p / \partial \sigma(\mathbf{0})$, $p_{110} = \partial^2 p / \partial \zeta \partial \xi(\mathbf{0})$, $p_{011} = \partial^2 p / \partial \xi \partial \sigma(\mathbf{0})$, $p_{300} = \partial^3 p / \partial \zeta^3(\mathbf{0})$, $p_{201} = \partial^3 p / \partial \zeta^2 \partial \sigma(\mathbf{0})$, $p_{102} = \partial^3 p / \partial \zeta \partial \sigma^2(\mathbf{0})$, and $p_{003} = \partial^3 p / \partial \sigma^3(\mathbf{0})$, hence,

$$\begin{aligned} p(\zeta, \xi, \sigma) &= p_{001} \sigma + p_{110} \zeta \xi + p_{011} \xi \sigma \\ &\quad + \frac{1}{6} p_{300} \zeta^3 + \frac{1}{2} p_{201} \zeta^2 \sigma + \frac{1}{2} p_{102} \zeta \sigma^2 + \frac{1}{6} p_{003} \sigma^3 \\ &\quad + \text{higher-order terms.} \end{aligned}$$

Proof: Power-series expansion:

- Differentiating the eqns. with respect to ζ , ξ , and σ and then applying the chain rule, we can obtain a system of eqns. from which the coefficients in the power-series expansion can be determined.
- Power-series expansion of the solution component orthogonal to the critical eigenvector:

$$\psi(\zeta, \xi, \sigma) = \sum_{\alpha, \beta, \gamma} \frac{\zeta^\alpha \xi^\beta \sigma^\gamma}{\alpha! \beta! \gamma!} \frac{\partial^{\alpha+\beta+\gamma} \psi}{\partial \zeta^\alpha \partial \xi^\beta \partial \sigma^\gamma} (0, 0, 0).$$

- $\frac{\partial q_k}{\partial \zeta} (0, 0, 0) = 0, \quad k = 1, \dots, \mu - 2,$

$$\implies \phi_k \cdot \left([K] \frac{\partial \psi}{\partial \zeta} (\zeta, \xi, \sigma) + \lambda_0 (1 + \xi) \left(\sum_{j=1}^{\mu-1} \cos(\zeta \phi_{0,j} + \psi_j(\zeta, \xi, \sigma) + \sigma \theta_\epsilon(s_j)) \right. \right.$$

$$\left. \left. (\phi_{0,j} + \frac{\partial \psi_j}{\partial \zeta} (\zeta, \xi, \sigma)) e_j \right) \right) \Big|_{(\zeta, \xi, \sigma) = (0, 0, 0)} = 0, \quad k = 1, \dots, \mu - 2,$$

$$\implies \phi_k \cdot \left([K] \frac{\partial \psi}{\partial \zeta} (0, 0, 0) + \lambda_0 \frac{\partial \psi}{\partial \zeta} (0, 0, 0) \right) = 0, \quad k = 1, \dots, \mu - 2,$$

$$\implies \frac{\partial \psi}{\partial \zeta} (0, 0, 0) = \mathbf{0}.$$

- ...

- Among the leading terms, the only nonvanishing coefficients are $\partial \psi / \partial \sigma(\mathbf{0})$, $\partial^2 \psi / \partial \xi \partial \sigma(\mathbf{0})$, $\partial^3 \psi / \partial \zeta^3(\mathbf{0})$, $\partial^3 \psi / \partial \zeta^2 \partial \sigma(\mathbf{0})$, $\partial^3 \psi / \partial \zeta \partial \sigma^2(\mathbf{0})$, $\partial^3 \psi / \partial \xi^2 \partial \sigma(\mathbf{0})$, and $\partial^3 \psi / \partial \sigma^3 \partial \sigma(\mathbf{0})$.

Proof: Power-series expansion (continued):

$$\blacksquare \quad \frac{\partial p}{\partial \sigma}(0, 0, 0) = (1 + \xi)\phi_0 \cdot \left(\sum_{j=1}^{\mu-1} \cos(\zeta\phi_{0,j} + \psi_j(\zeta, \xi, \sigma) + \sigma\theta_\epsilon(s_j)) \left(\frac{\partial \psi_j}{\partial \sigma}(\zeta, \xi, \sigma) + \theta_\epsilon(s_j) \right) \mathbf{e}_j \right) \Big|_{(\zeta, \xi, \sigma) = (0, 0, 0)}$$

$$\implies \frac{\partial p}{\partial \sigma}(0, 0, 0) = \phi_0 \cdot \boldsymbol{\theta}_\epsilon.$$

$$\blacksquare \quad \frac{\partial p}{\partial \zeta}(0, 0, 0) = -1 + (1 + \xi)\phi_0 \cdot \left(\sum_{j=1}^{\mu-1} \cos(\zeta\phi_{0,j} + \psi_j(\zeta, \xi, \sigma) + \sigma\theta_\epsilon(s_j)) \left(\phi_{0,j} + \frac{\partial \psi_j}{\partial \zeta}(\zeta, \xi, \sigma) \right) \mathbf{e}_j \right) \Big|_{(\zeta, \xi, \sigma) = (0, 0, 0)},$$

$$\implies \frac{\partial p}{\partial \zeta}(0, 0, 0) = 0.$$

$$\blacksquare \quad \begin{aligned} \frac{\partial^2 p}{\partial \zeta \partial \xi}(0, 0, 0) &= \phi_0 \cdot \left(\sum_{j=1}^{\mu-1} \cos(\zeta\phi_{0,j} + \psi_j(\zeta, \xi, \sigma) + \sigma\theta_\epsilon(s_j)) \left(\phi_{0,j} + \frac{\partial \psi_j}{\partial \zeta}(\zeta, \xi, \sigma) \right) \mathbf{e}_j \right) \\ &+ (1 + \xi)\phi_0 \cdot \left(\sum_{j=1}^{\mu-1} -\sin(\zeta\phi_{0,j} + \psi_j(\zeta, \xi, \sigma) + \sigma\theta_\epsilon(s_j)) \frac{\partial \psi_j}{\partial \xi}(\zeta, \xi, \sigma) \left(\phi_{0,j} + \frac{\partial \psi_j}{\partial \zeta}(\zeta, \xi, \sigma) \right) \mathbf{e}_j \right) \\ &+ (1 + \xi)\phi_0 \cdot \left(\sum_{j=1}^{\mu-1} \cos(\zeta\phi_{0,j} + \psi_j(\zeta, \xi, \sigma) + \sigma\theta_\epsilon(s_j)) \frac{\partial^2 \psi_j}{\partial \zeta \partial \xi}(\zeta, \xi, \sigma) \mathbf{e}_j \right) \Big|_{(\zeta, \xi, \sigma) = (0, 0, 0)}, \end{aligned}$$

$$\implies \frac{\partial^2 p}{\partial \zeta \partial \xi}(0, 0, 0) = 1.$$

■ ...

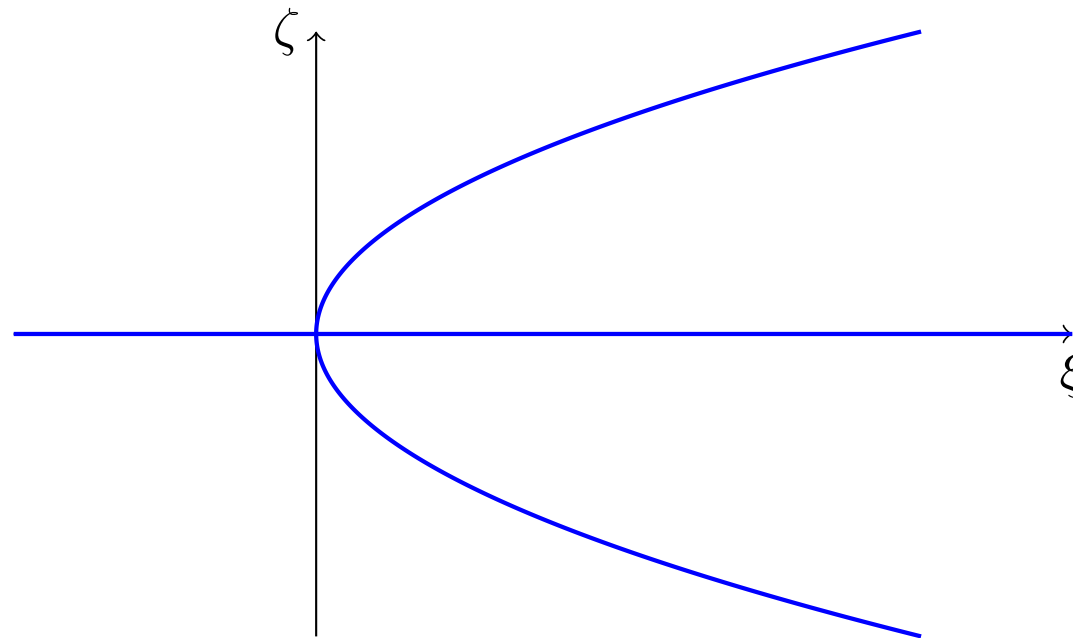
Perfect beam

- If $\sigma = 0$, then the bifurcation equation reduces to

$$p_{110}\zeta\xi + \frac{1}{6}p_{300}\zeta^3 + \text{higher-order terms} = 0.$$

- The solution reads as

$$\begin{cases} \zeta = 0 & \text{trivial path,} \\ \xi = -\frac{1}{6} \frac{p_{300}}{p_{110}} \zeta^2 + \text{higher-order terms} & \text{bifurcated path.} \end{cases}$$



Imperfect beam

Law of large numbers, central limit theorem, and ergodicity

- Let us recall that if $\sigma \neq 0$, then the bifurcation equation reads as

$$\begin{aligned} & p_{001}\sigma + p_{110}\zeta\xi + p_{011}\xi\sigma \\ & + \frac{1}{6}p_{300}\zeta^3 + \frac{1}{2}p_{201}\zeta^2\sigma + \frac{1}{2}p_{102}\zeta\sigma^2 + \frac{1}{6}p_{003}\sigma^3 \\ & + \text{higher-order terms} = 0. \end{aligned}$$

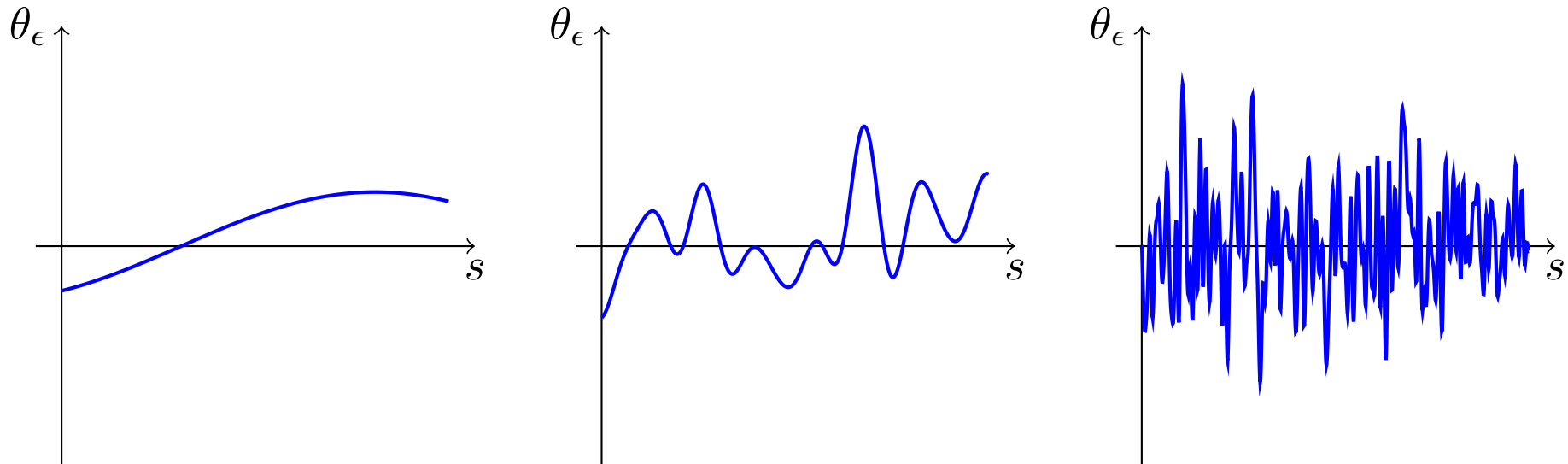
Here, the coefficients p_{001} , p_{011} , p_{201} , p_{102} , and p_{003} are random because they depend on θ_ϵ .

- We wish to use the property that under certain conditions, the behavior of the arithmetic mean of a sufficiently large number of random variables and the behavior of the sample-path average of a stationary second-order stochastic process can be described in terms of simple probabilistic laws, that is, we wish to use the **law of large numbers**, the **central limit theorem**, and **ergodicity**.
- To place ourselves in a context wherein a “sufficiently large number of random variables” contribute, we will consider the **asymptotic behavior of the bifurcation equation when $\epsilon \rightarrow 0$** .

Let us recall that ϵ determines the spatial correlation of the random imperfection.

Law of large numbers, central limit theorem, and ergodicity (continued)

- We found that $p_{001} = \phi_0 \cdot \theta_\epsilon$. Here, let us recall that ϕ_0 and θ_ϵ are vectors that collect the values taken by the critical eigenvector and the normalized imperfection, respectively, at the grid points of the finite-difference method with grid spacing $h = \ell/\mu$.
- As ϵ tends to 0, we must let the grid spacing h tend to 0 as well:

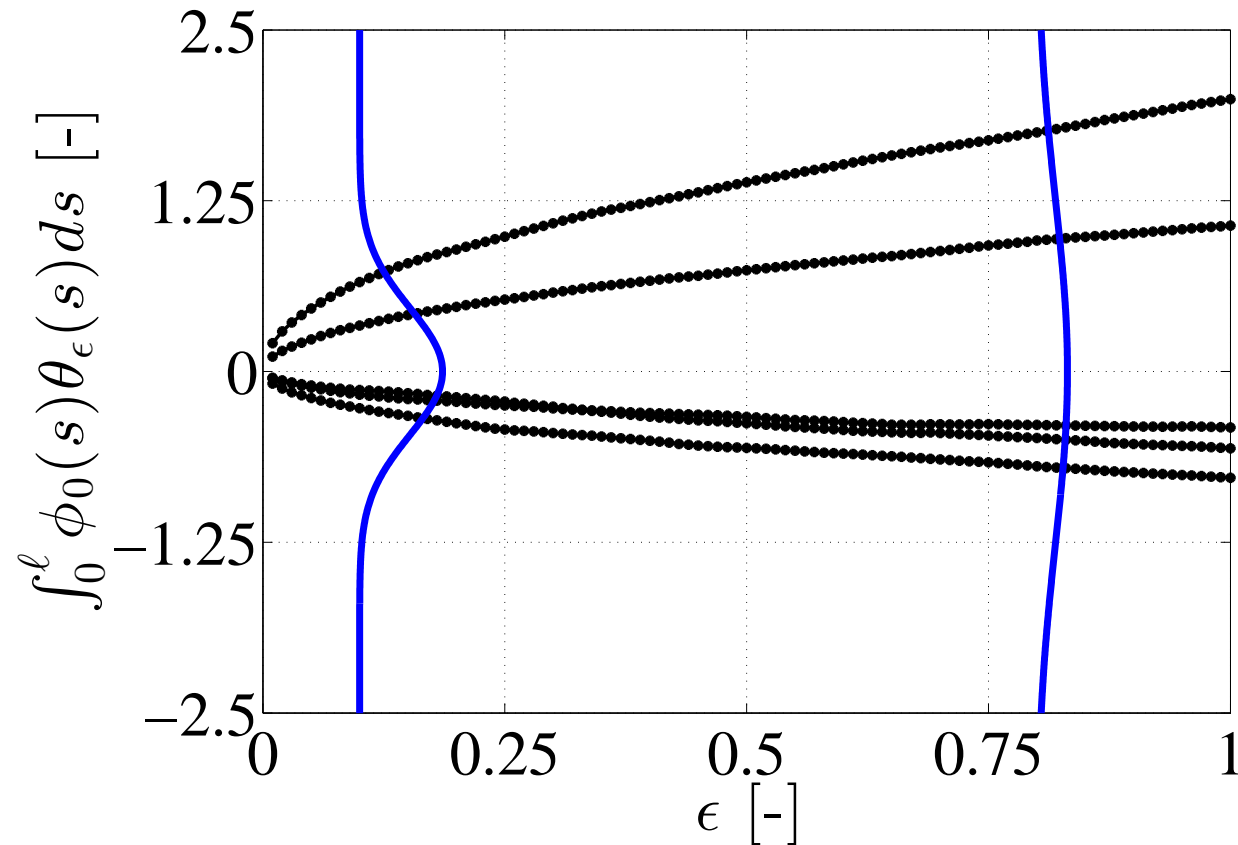


- As the grid spacing h tends to 0, the inner product $\phi_0 \cdot \theta_\epsilon$ tends to an integral:

$$\lim_{h \rightarrow 0} h p_{001} = \lim_{h \rightarrow 0} h \sum_{j=1}^{\mu-1} \phi_{0,j} \theta_\epsilon(s_j) = \int_0^\ell \phi_0(s) \theta_\epsilon(s) ds.$$

Law of large numbers, central limit theorem, and ergodicity (continued)

- As ϵ tends to zero, the integral $\int_0^\ell \phi_0(s)\theta_\epsilon(s)ds$ tends in distribution to a zero-mean Gaussian r.v.:



$$\lim_{\epsilon \rightarrow 0} \frac{1}{\sqrt{\epsilon}} \int_0^\ell \phi_0(s)\theta_\epsilon(s)ds \stackrel{d}{=} \eta = N \left(0, \int_{-\infty}^{+\infty} r_{\theta_1}(u)du \right).$$

Proof: limit of $\eta_\epsilon = \int_0^\ell \phi_0(s)\theta_\epsilon(s)ds$:

- The linear transformation of a Gaussian stochastic process is Gaussian.
- Because we assumed the random imperfection to be zero mean, we obtain

$$\int \eta_\epsilon dP = \frac{1}{\sqrt{\epsilon}} \int_0^\ell \int \theta_\epsilon(s) dP \phi_0(s) ds = 0.$$

- The variance of η_ϵ reads as

$$\begin{aligned} \int \eta_\epsilon^2 dP &= \frac{1}{\epsilon} \int_0^\ell \int_0^\ell \phi_0(s)\phi_0(t)r_{\theta_\epsilon}(s-t) ds dt \\ &= \int_0^\ell \int_{-t/\epsilon}^{(\ell-t)/\epsilon} \phi_0(t+\epsilon u)\phi_0(t)r_{\theta_1}(u) du dt, \end{aligned}$$

in which we carried out the change of variables $u = (s-t)/\epsilon$. As $\epsilon \rightarrow 0$, we thus obtain

$$\lim_{\epsilon \rightarrow 0} \int \eta_\epsilon^2 dP = \int_0^\ell \phi_0^2(t) dt \int_{-\infty}^{+\infty} r_{\theta_1}(u) du = \int_{-\infty}^{+\infty} r_{\theta_1}(u) du.$$

- In conclusion, provided that the autocorrelation function is integrable, the random variable η_ϵ converges as $\epsilon \rightarrow 0$ to a Gaussian random variable with mean 0 and variance $\int_{-\infty}^{+\infty} r_{\theta_1}(u) du$.

Law of large numbers, central limit theorem, and ergodicity (continued)

- Extending this analysis to the other coefficients ultimately leads to

$$\lim_{\epsilon \rightarrow 0} \lim_{h \rightarrow 0} \frac{1}{\sqrt{\epsilon}} h p_{001} = \lim_{\epsilon \rightarrow 0} \frac{1}{\sqrt{\epsilon}} \int_0^\ell \theta_\epsilon(s) \phi_0(s) ds \stackrel{d}{=} \eta,$$

$$\lim_{\epsilon \rightarrow 0} \lim_{h \rightarrow 0} h p_{110} = \lim_{\epsilon \rightarrow 0} \int_0^\ell \phi_0^2(s) ds = 1,$$

$$\lim_{\epsilon \rightarrow 0} \lim_{h \rightarrow 0} \frac{1}{\sqrt{\epsilon}} h p_{011} = \lim_{\epsilon \rightarrow 0} \frac{1}{\sqrt{\epsilon}} \int_0^\ell \theta_\epsilon(s) \phi_0(s) ds \stackrel{d}{=} \eta,$$

$$\lim_{\epsilon \rightarrow 0} \lim_{h \rightarrow 0} h p_{300} = - \lim_{\epsilon \rightarrow 0} \int_0^\ell \phi_0^4(s) ds = -\bar{a},$$

$$\lim_{\epsilon \rightarrow 0} \lim_{h \rightarrow 0} \frac{1}{\sqrt{\epsilon}} h p_{201} = - \lim_{\epsilon \rightarrow 0} \frac{1}{\sqrt{\epsilon}} \int_0^\ell \phi_0^3(s) \left(\frac{\partial \psi}{\partial \sigma}(s) + \theta_\epsilon(s) \right) ds \stackrel{d}{=} -\iota,$$

$$\lim_{\epsilon \rightarrow 0} \lim_{h \rightarrow 0} h p_{102} = - \lim_{\epsilon \rightarrow 0} \int_0^\ell \phi_0^2(s) \left(\frac{\partial \psi}{\partial \sigma}(s) + \theta_\epsilon(s) \right)^2 ds = -\bar{b} = -1,$$

$$\lim_{\epsilon \rightarrow 0} \lim_{h \rightarrow 0} h p_{003} = - \lim_{\epsilon \rightarrow 0} \int_0^\ell \phi_0(s) \left(\frac{\partial \psi}{\partial \sigma}(s) + \theta_\epsilon(s) \right)^3 ds = -\bar{c} = 0.$$

Scaling

- We let ζ , ξ , and σ depend on ϵ by **scaling** ζ , ξ , and σ as follows:

$$\zeta = \epsilon^\rho \bar{\zeta}, \quad \xi = \epsilon^{2\rho} \bar{\xi}, \quad \sigma = \epsilon^\kappa \bar{\sigma}.$$

- Inserting these scaling relations into the bifurcation equation, we obtain

$$\begin{aligned} & p_{001} \epsilon^\kappa \bar{\sigma} + p_{110} \epsilon^\rho \bar{\zeta} \epsilon^{2\rho} \bar{\xi} + p_{011} \epsilon^{2\rho} \bar{\xi} \epsilon^\kappa \bar{\sigma} \\ & + \frac{1}{6} p_{300} \epsilon^{3\rho} \bar{\zeta}^3 + \frac{1}{2} p_{201} \epsilon^{2\rho} \bar{\zeta}^2 \epsilon^\kappa \bar{\sigma} + \frac{1}{2} p_{102} \epsilon^\rho \bar{\zeta} \epsilon^{2\kappa} \bar{\sigma}^2 + \frac{1}{6} p_{003} \epsilon^{3\kappa} \bar{\sigma}^3 \\ & + \text{higher-order terms} = 0. \end{aligned}$$

- Multiplying with $\epsilon^{-3\rho}$, we obtain

$$\begin{aligned} & \epsilon^{\kappa-3\rho+1/2} \bar{\sigma} \underbrace{\frac{1}{\sqrt{\epsilon}} p_{001}}_{\rightarrow \eta} + \bar{\zeta} \bar{\xi} + \epsilon^{\kappa-\rho+1/2} \bar{\xi} \bar{\sigma} \underbrace{\frac{1}{\sqrt{\epsilon}} p_{011}}_{\rightarrow \eta} \\ & + \frac{1}{6} \bar{\zeta}^3 (-\bar{a}) + \frac{1}{2} \epsilon^{\kappa-\rho+1/2} \bar{\zeta}^2 \bar{\sigma} \underbrace{\frac{1}{\sqrt{\epsilon}} p_{201}}_{\rightarrow -\iota} + \frac{1}{2} \epsilon^{2\kappa-2\rho} \bar{\zeta} \bar{\sigma} \underbrace{p_{102}}_{\rightarrow -1} + \frac{1}{6} \epsilon^{3\kappa-3\rho} \bar{\sigma}^3 \underbrace{p_{003}}_{\rightarrow 0} \\ & + \text{higher-order terms} = 0. \end{aligned}$$

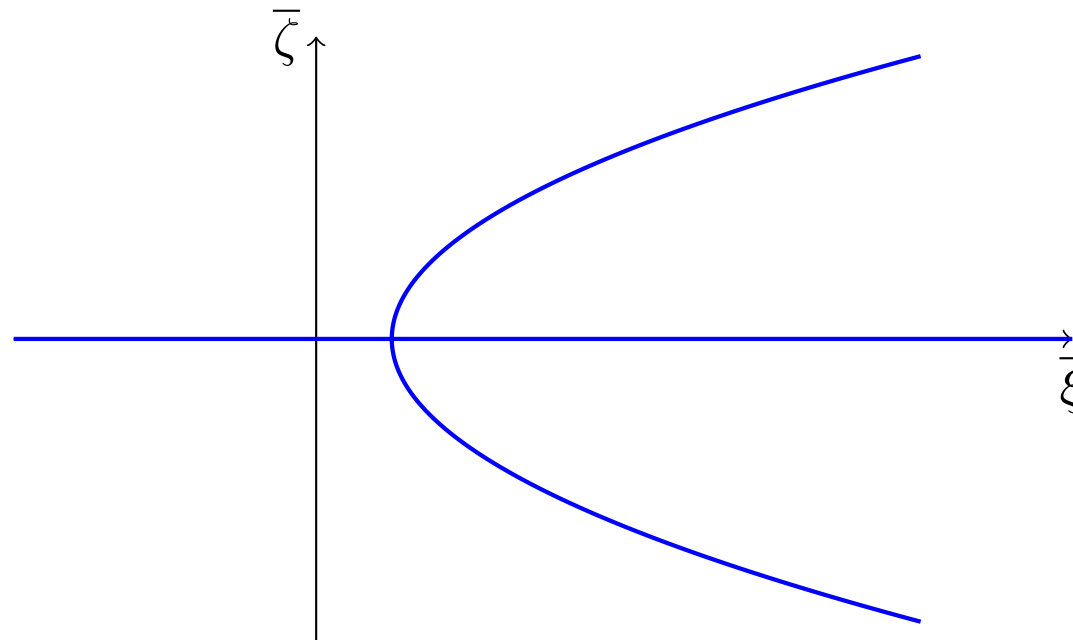
Case of large-magnitude random imperfection

- If $0 < \kappa < \frac{1}{4}$ (large-magnitude) and $\rho = \kappa$, the bifurcation equation leads as $\epsilon \rightarrow 0$ to

$$\bar{\zeta} \bar{\xi} - \frac{1}{6} \bar{a} \bar{\zeta}^3 - \bar{\zeta} \bar{\sigma}^2 = 0.$$

- The solution to this equation reads as

$$\begin{cases} \bar{\zeta} = 0 & \text{trivial path,} \\ \bar{\xi} = \frac{1}{6} \bar{a} \bar{\zeta}^2 + \bar{\sigma}^2 & \text{bifurcated path.} \end{cases}$$



- For large-magnitude random imperfection, as $\epsilon \rightarrow 0$, the bifurcation equation becomes deterministic, and it determines a critical buckling load that is always strictly greater than the critical buckling load for the perfect beam.

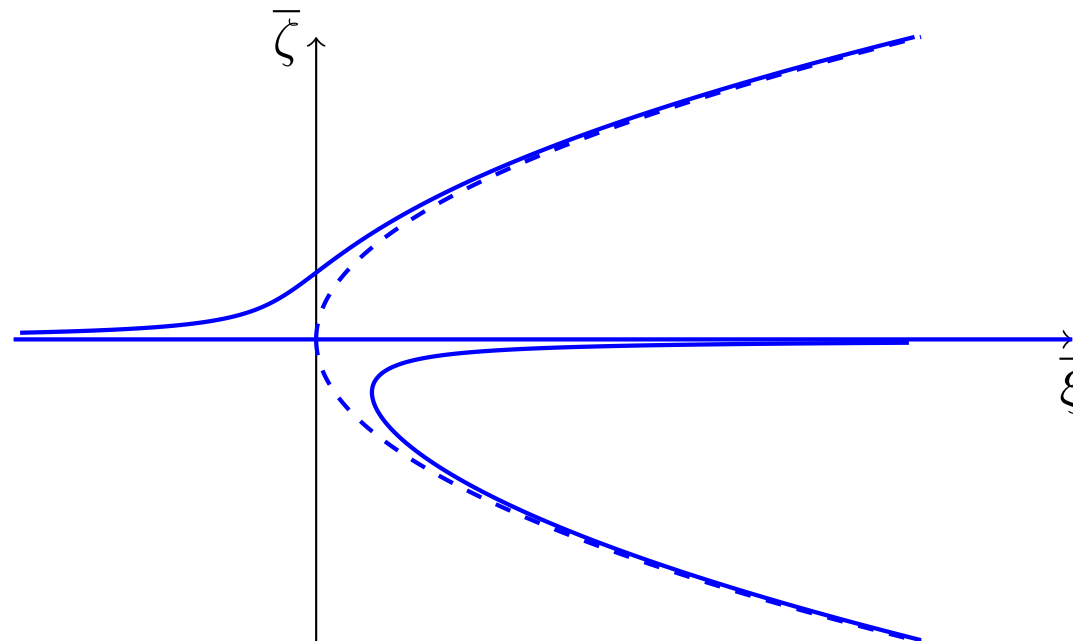
Case of small-magnitude random imperfection

- If $\kappa > \frac{1}{4}$ (small-magnitude) and $\rho = \frac{1}{3}\kappa + \frac{1}{6}$, the bifurcation equation leads as $\epsilon \rightarrow 0$ to

$$\bar{\sigma}\eta + \bar{\zeta}\bar{\xi} - \frac{1}{6}\bar{a}\bar{\zeta}^3 = 0.$$

- The solution to this equation reads as

$$\bar{\xi} = \frac{1}{6}\bar{a}\bar{\zeta}^2 - \bar{\sigma}\eta/\bar{\zeta}.$$



- For small-magnitude random imperfection, as $\epsilon \rightarrow 0$, the bifurcation equation remains random. The random bifurcation diagram is randomly perturbed about the classical pitchfork. The random critical buckling load is always greater than the critical buckling load for the perfect beam.

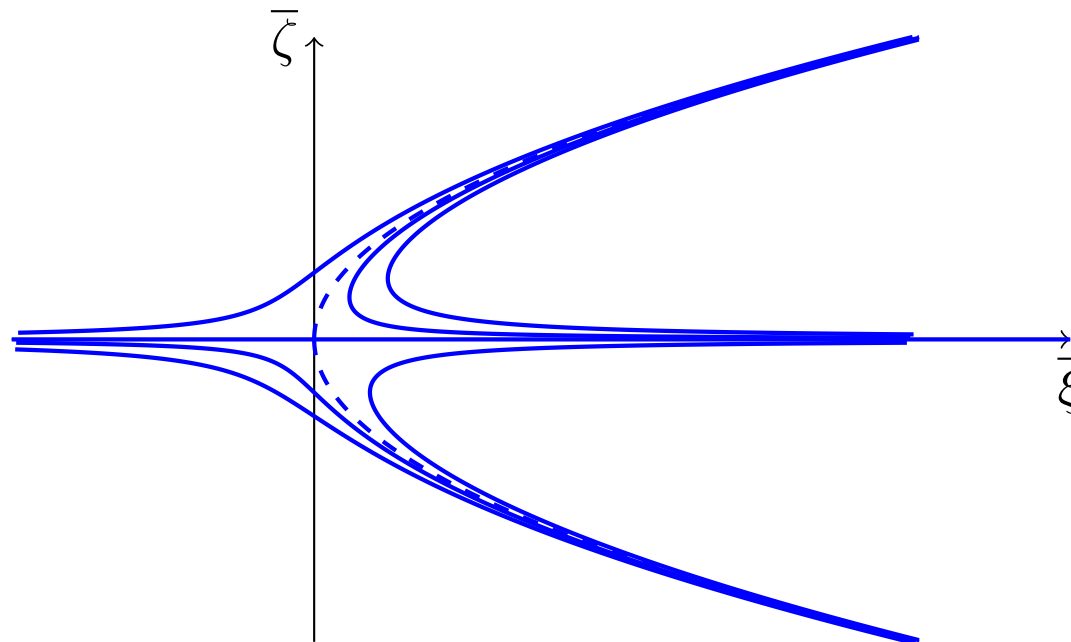
Case of small-magnitude random imperfection

- If $\kappa > \frac{1}{4}$ (small-magnitude) and $\rho = \frac{1}{3}\kappa + \frac{1}{6}$, the bifurcation equation leads as $\epsilon \rightarrow 0$ to

$$\bar{\sigma}\eta + \bar{\zeta}\bar{\xi} - \frac{1}{6}\bar{a}\bar{\zeta}^3 = 0.$$

- The solution to this equation reads as

$$\bar{\xi} = \frac{1}{6}\bar{a}\bar{\zeta}^2 - \bar{\sigma}\eta/\bar{\zeta}.$$



- For small-magnitude random imperfection, as $\epsilon \rightarrow 0$, the bifurcation equation remains random. The random bifurcation diagram is randomly perturbed about the classical pitchfork. The random critical buckling load is always greater than the critical buckling load for the perfect beam.

Assignment

- Please establish connections between your numerical study for the assignment in the previous lecture and the present theoretical study. Can you find back in your numerical results behavior suggested by the present theoretical study?
- General recommendations for the report for the combination of the three assignments:
 - ◆ The report must be neat, well organized, and professionally written. You may write in either French or English. Indicate the first name and the last name of each group member and the total number of pages on the first page of your report. Pay attention to spelling and grammar: use a spell checker! Pay attention to the clarity of your figures: all figures must be computer plots with adequate ranges, ticks, and scales; axes must be labeled; and proper units must be included in labels and legends where appropriate. Include a list of references.
 - ◆ The report must be sent in PDF format by email to M. Arnst before/on Wednesday May 6, 2015. Late reports will not be accepted unless the lateness is appropriately justified or prior arrangements were made. Please attach to your email a file with your Matlab code.
- General recommendations for the presentation for the combination of the three assignments:
 - ◆ The presentation should emphasize the understanding that you gained.
 - ◆ The presentation must be neat and well organized. You may present in either French or English. Pay attention to the clarity of your figures: for the sake of readability, axis and tick labels may have to be larger in your presentation than in your report.
 - ◆ Length of 10 slides. The first slide, that is, the slide after the title slide, must provide a general overview. Please include one slide that summarizes the theoretical analysis of Lecture 3.

References consulted to prepare this lecture

- W. Day, A. Karkowski, and G. Papanicolaou. Buckling of randomly imperfect beams. *Acta Applicandae Mathematicae*, 17:269–186, 1989.
- S. Krantz and H. Parks. *The implicit function theorem*. Springer, 2013.
- A. Papoulis. Probability, random variables, and stochastic processes. McGraw-Hill, 1991.
- R. Serfling. Approximation theorems of mathematical statistics. John Wiley & Sons, 1980.