MATH0488 – Stochastic Processes

Stochastically perturbed bifurcation

Part 2 of 3: Numerical study of buckling of randomly imperfect beam

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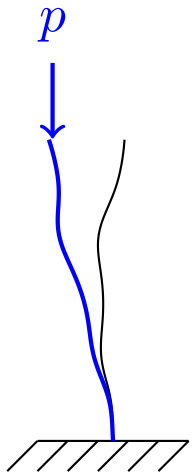
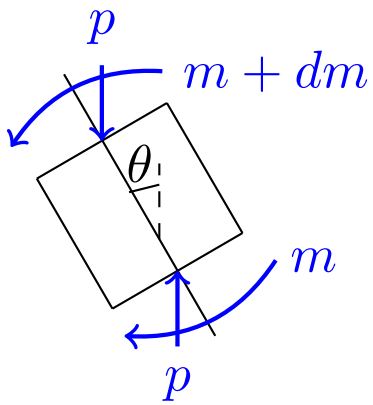
Outline

- Problem formulation.
- Implicit function theorem.
- Stochastic model.
- Sampling method.
- Assignment.
- References.

Problem formulation

- We consider the following problem setting:





  - We denote by $\theta(s)$ and $\sigma\theta_\epsilon(s)$ the angles that the tangent vectors to the deformed imperfect beam and the undeformed imperfect beam, respectively, make with the vertical axis.
  - Equilibrium: $m - (m + dm) - p\sin(\theta)ds = 0 \implies -\frac{dm}{ds} - p\sin(\theta) = 0.$
  - Constitutive equation: $\frac{d(\theta - \sigma\theta_\epsilon)}{ds} = \frac{m}{yj}$.
  - At the one end, the rod is fixed; at the other end, it is subjected to a constant vertical force p.

- We obtain the following boundary-value problem:

$$\begin{cases} \dfrac{d^2(\theta - \sigma\theta_\epsilon)}{ds^2} + \lambda\sin(\theta) = 0 \quad \text{with} \quad \lambda = \dfrac{p}{yj}, \\ \theta(0) = \sigma\theta_\epsilon(0) \quad \text{and} \quad \dfrac{d\theta}{ds}(\ell) = \sigma\dfrac{d\theta_\epsilon}{ds}(\ell). \end{cases}$$

Problem formulation

- For the perfect beam, that is, $\sigma\theta_\epsilon = 0$, we found in the previous lecture that the trivial solution $\theta = 0$ solves the problem for any λ in $\mathbb{R}$, and the linearization of the problem close to this trivial solution leads to an eigenproblem, whose solution immediately provides the bifurcations.

- For the imperfect beam, the function $\theta = 0$ is generally no longer a solution to the problem, and it is no longer immediate to determine bifurcations. In the Section “Implicit function theorem,” we will explain how to “detect bifurcations” in an algorithm that follows a “branch” of solutions.

- We will consider the imperfection, that is, $\sigma\theta_\epsilon$, to be random. In the Section “Stochastic model,” we will model this imperfection as a stochastic process. And in the Section “Sampling method,” we will explain how to numerically generate samples of this stochastic process, thus enabling a sampling-based numerical study of the impact of the random imperfection on the bifurcations.

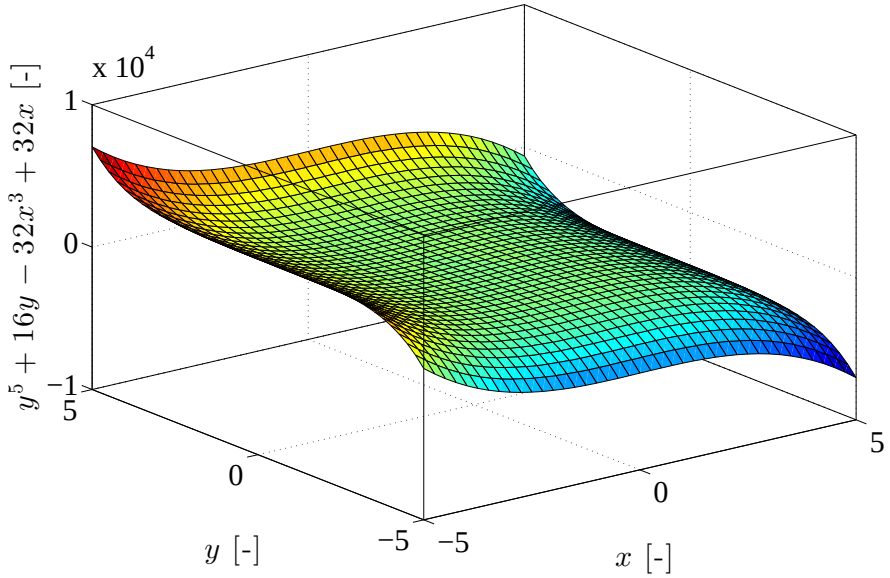
- In the next lecture, we will carry out a theoretical analysis of the impact of the random imperfection on the bifurcations. The theoretical insight to be gained in the next lecture will help us better understand the numerical results already obtained in this lecture.

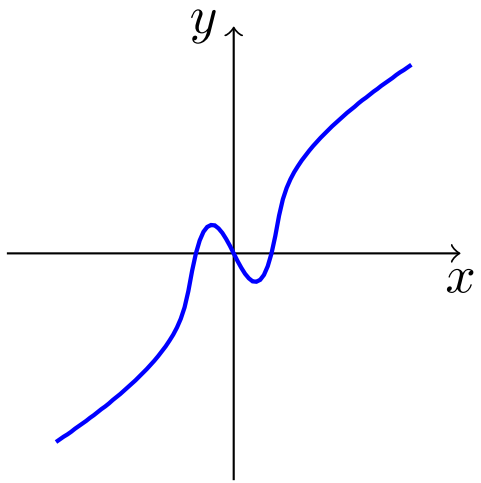
Implicit function theorem

Implicit function theorem

- Often, a function is given to us **explicitly**, that is, in the form of a formula, for example,
$$x \mapsto y = \psi(x) = x^3 + 5x^2 - x - 3.$$

- This notion of "function given in the form of an explicit formula" is too limited for many purposes. For example, the subset of points (x, y) in the plane $\mathbb{R}^2$ that satisfy
$$g(x, y) = y^5 + 16y - 32x^3 + 32x = 0$$



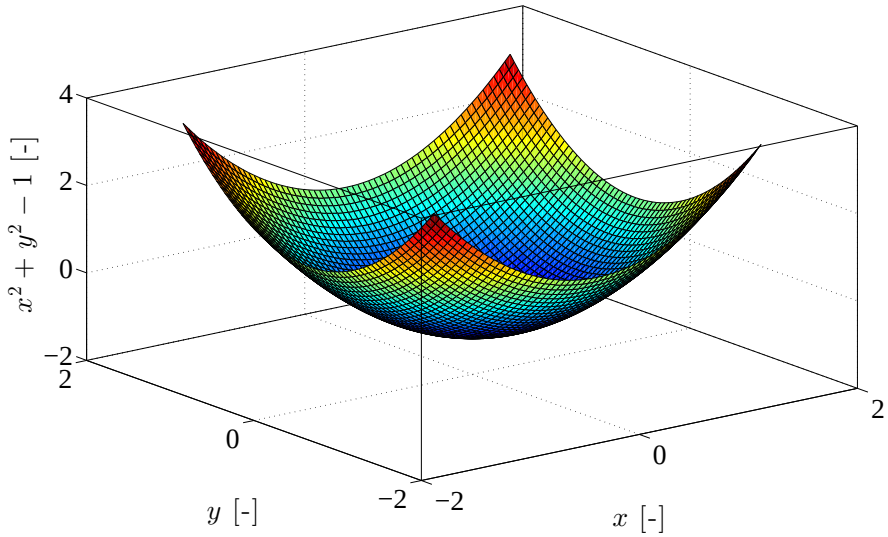


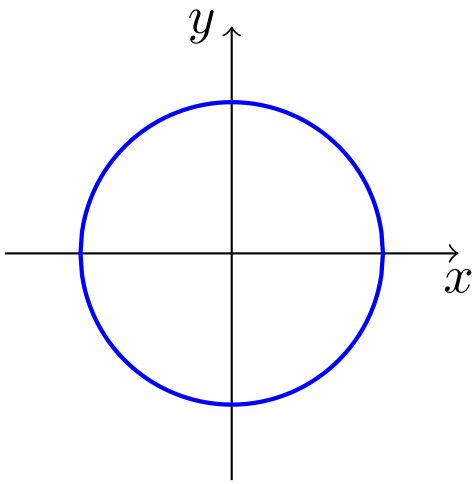
This figure suggests that there is a unique function ψ that maps any x onto $y = \psi(x)$ such that $g(x, \psi(x)) = 0$; however, no explicit formula for ψ exists. We say that ψ is defined **implicitly**.

Implicit function theorem

- A given equation does not always uniquely define a subset of its variables as an implicit function of its remaining variables. For example, the subset of points (x, y) in the plane $\mathbb{R}^2$ that satisfy

$$g(x, y) = x^2 + y^2 - 1 = 0$$





The problem is that for a given $-1 < x < 1$, the eqn. is solved by multiple y, i.e., $y = \pm\sqrt{1 - x^2}$.

- For any solution (x_0, y_0) with $y_0 > 0$, there exists an open interval $]x_0 - \delta, x_0 + \delta[$ within which we can express y explicitly as $y = \sqrt{1 - x^2}$; and for any solution (x_0, y_0) with $y_0 < 0$, there exists an open interval $]x_0 - \delta, x_0 + \delta[$ within which we can express y explicitly as $y = -\sqrt{1 - x^2}$.

- Around $x_0 = \pm 1$, we cannot uniquely express y as a function of x. Note that $\frac{\partial g}{\partial y}(x_0, y_0) = 0$.

Implicit function theorem

- The implicit function theorem can help us determine when a given equation uniquely defines a subset of its variables implicitly as a function of its remaining variables.

- Let g be a continuously differentiable function from $\mathbb{R}^2$ into $\mathbb{R}$ with $g(x_0, y_0) = 0$. If the linearization of the problem $g(x, y) = 0$ with respect to y at (x_0, y_0) is nondegenerate, that is, the derivative of $g(x_0, \cdot)$ with respect to y at y_0 is nonvanishing,

$$g(x_0, y_0) = 0,$$
$$\frac{\partial g}{\partial y}(x_0, y_0) \neq 0,$$

  then at least locally the problem uniquely defines y implicitly as a function of x, that is, there exists an open interval $]x_0 - \delta, x_0 - \delta[$ around x_0, an open interval $]y_0 - \epsilon, y_0 - \epsilon[$ around y_0, and a unique function ψ from $]x_0 - \delta, x_0 - \delta[$ into $]y_0 - \epsilon, y_0 - \epsilon[$ that satisfy

$$g\big(x, \psi(x)\big) = 0 \quad \text{for all } x \text{ in }]x_0 - \delta, x_0 - \delta[,$$

  and this function ψ is continuously differentiable.

Implicit function theorem

- The implicit function theorem can be generalized to a higher dimensional context as follows.

- Let $\boldsymbol{g}$ be a continuously differentiable function from $\mathbb{R} \times \mathbb{R}^m$ into $\mathbb{R}^m$ with $\boldsymbol{g}(x_0, \boldsymbol{y}_0) = \boldsymbol{0}$. If the linearization of the problem $\boldsymbol{g}(x, \boldsymbol{y}) = \boldsymbol{0}$ with respect to $\boldsymbol{y}$ at $(x_0, \boldsymbol{y}_0)$ is nondegenerate, that is, the directional derivative of $\boldsymbol{g}(x_0, \cdot)$ at $\boldsymbol{y}_0$ is nonvanishing in any direction $\boldsymbol{e}$,

$$\boldsymbol{g}(x_0, \boldsymbol{y}_0) = \boldsymbol{0},$$
$$d_{\boldsymbol{e}}\boldsymbol{g}(x_0, \boldsymbol{y}_0) = \lim_{t \to 0} \frac{\boldsymbol{g}(x_0, \boldsymbol{y}_0 + t\boldsymbol{e}) - \boldsymbol{g}(x_0, \boldsymbol{y}_0)}{t} \neq \boldsymbol{0} \quad \text{for any} \quad \boldsymbol{e} \neq \boldsymbol{0},$$

  then at least locally the problem uniquely defines $\boldsymbol{y}$ implicitly as a function of x, that is, there exists an open interval $]x_0 - \delta, x_0 - \delta[$ around x_0, an open ball $\{\boldsymbol{y} \in \mathbb{R}^m : \|\boldsymbol{y} - \boldsymbol{y}_0\| < \epsilon\}$ around $\boldsymbol{y}_0$, and a unique function $\boldsymbol{\psi}$ from $]x_0 - \delta, x_0 - \delta[$ into $\{\boldsymbol{y} \in \mathbb{R}^m : \|\boldsymbol{y} - \boldsymbol{y}_0\| < \epsilon\}$ satisfying

$$\boldsymbol{g}\big(x, \boldsymbol{\psi}(x)\big) = \boldsymbol{0} \quad \text{for all } x \text{ in }]x_0 - \delta, x_0 - \delta[,$$

  and this function $\boldsymbol{\psi}$ is continuously differentiable.

Implicit function theorem

- To elaborate this generalization in more detail, let us recall a few notions from differential calculus.
- Let g be a function from $\mathbb{R}$ into $\mathbb{R}$ that maps any x onto $y = g(x)$.
  Then, the **derivative** of g at x_0 is given by

$$\frac{dg}{dx}(x_0) = \lim_{t \to 0} \frac{g(x_0 + t) - g(x_0)}{t}.$$

- Let g be a function from $\mathbb{R}^m$ into $\mathbb{R}$ that maps any $\boldsymbol{x} = (x_1, \ldots, x_m)$ onto $y = g(\boldsymbol{x})$.
  Then, the **directional derivative** of g in direction $\boldsymbol{e} = (e_1, \ldots, e_m)$ at $\boldsymbol{x_0}$ is given by

$$d_{\boldsymbol{e}} g(\boldsymbol{x}_0) = \lim_{t \to 0} \frac{g(\boldsymbol{x}_0 + t\boldsymbol{e}) - g(\boldsymbol{x}_0)}{t}.$$

  This directional derivative can be represented as follows:

$$d_{\boldsymbol{e}} g(\boldsymbol{x}_0) = \boldsymbol{\nabla}_{\boldsymbol{x}} \boldsymbol{g}(\boldsymbol{x}_0) \cdot \boldsymbol{e} = \begin{bmatrix} \frac{\partial g}{\partial x_1}(\boldsymbol{x}_0) \\ \vdots \\ \frac{\partial g}{\partial x_m}(\boldsymbol{x}_0) \end{bmatrix} \cdot \begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix},$$

  where the **gradient vector** $\boldsymbol{\nabla}_{\boldsymbol{x}} \boldsymbol{g}(\boldsymbol{x}_0)$ collects the **partial derivatives** of g at $\boldsymbol{x}_0$ given by

$$\frac{\partial g}{\partial x_i}(\boldsymbol{x}_0) = \lim_{t \to 0} \frac{g(x_{01}, \ldots, x_{0i} + t, \ldots, x_{0m}) - g(x_{01}, \ldots, x_{0i}, \ldots, x_{0m})}{t}, \quad 1 \leq i \leq m.$$

Implicit function theorem

- Let $\boldsymbol{g}$ be a function from $\mathbb{R}^m$ into $\mathbb{R}^n$ that maps any $\boldsymbol{x} = (x_1, \ldots, x_m)$ onto $\boldsymbol{y} = (y_1, \ldots, y_n)$ such that $\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x})$. Then, the **directional derivative** of $\boldsymbol{g}$ in direction $\boldsymbol{e}$ at $\boldsymbol{x_0}$ is given by

$$d_{\boldsymbol{e}}\boldsymbol{g}(\boldsymbol{x}_0) = \lim_{t \to 0} \frac{\boldsymbol{g}(\boldsymbol{x}_0 + t\boldsymbol{e}) - \boldsymbol{g}(\boldsymbol{x}_0)}{t}.$$

This directional derivative can be represented as follows:

$$d_{\boldsymbol{e}}\boldsymbol{g}(\boldsymbol{x}_0) = [\mathsf{D}_{\boldsymbol{x}}\boldsymbol{g}(\boldsymbol{x}_0)]\boldsymbol{e} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1}(\boldsymbol{x}_0) & \cdots & \frac{\partial g_1}{\partial x_m}(\boldsymbol{x}_0) \\ \vdots & & \vdots \\ \frac{\partial g_n}{\partial x_1}(\boldsymbol{x}_0) & \cdots & \frac{\partial g_n}{\partial x_m}(\boldsymbol{x}_0) \end{bmatrix} \begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix},$$

where the **Jacobian matrix** $[\mathsf{D}_{\boldsymbol{x}}\boldsymbol{g}(\boldsymbol{x}_0)]$ collects the **partial derivatives** of $\boldsymbol{g}$ at $\boldsymbol{x}_0$ given by

$$\frac{\partial g_j}{\partial x_i}(\boldsymbol{x}_0) = \lim_{t \to 0} \frac{g_j(x_{01}, \ldots, x_{0i} + t, \ldots, x_{0m}) - g_j(x_{01}, \ldots, x_{0i}, \ldots, x_{0m})}{t}, \quad 1 \leq i \leq m, \quad 1 \leq j \leq n.$$

Implicit function theorem

- Thus, let $\boldsymbol{g}$ be a continuously differentiable function from $\mathbb{R} \times \mathbb{R}^m$ into $\mathbb{R}^m$ with $\boldsymbol{g}(x_0, \boldsymbol{y}_0) = \boldsymbol{0}$. If the linearization of the problem $\boldsymbol{g}(x, \boldsymbol{y}) = \boldsymbol{0}$ with respect to $\boldsymbol{y}$ at $(x_0, \boldsymbol{y}_0)$ is nondegenerate in that the directional derivative of $\boldsymbol{g}(x_0, \cdot)$ at $\boldsymbol{y}_0$ is nonvanishing in any direction $\boldsymbol{e}$,

$$\boldsymbol{g}(x_0, \boldsymbol{y}_0) = \boldsymbol{0},$$
$$d_{\boldsymbol{e}}\boldsymbol{g}(x_0, \boldsymbol{y}_0) = \lim_{t \to 0} \frac{\boldsymbol{g}(x_0, \boldsymbol{y}_0 + t\boldsymbol{e}) - \boldsymbol{g}(x_0, \boldsymbol{y}_0)}{t} \neq \boldsymbol{0} \quad \text{for any} \quad \boldsymbol{e} \neq \boldsymbol{0},$$

or, equivalently, if the Jacobian matrix $[\mathrm{D}_{\boldsymbol{y}}\boldsymbol{g}(x_0, \boldsymbol{y}_0)]$ does not have a vanishing eigenvalue,

$$\begin{bmatrix} \frac{\partial g_1}{\partial y_1}(x_0, \boldsymbol{y}_0) & \cdots & \frac{\partial g_1}{\partial y_m}(x_0, \boldsymbol{y}_0) \\ \vdots & & \vdots \\ \frac{\partial g_n}{\partial y_1}(x_0, \boldsymbol{y}_0) & \cdots & \frac{\partial g_n}{\partial y_m}(x_0, \boldsymbol{y}_0) \end{bmatrix} \begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix} \neq \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{for any} \quad \begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix} \neq \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix},$$

then at least locally the problem uniquely defines $\boldsymbol{y}$ implicitly as a function of x, that is, there exists an open interval $]x_0 - \delta, x_0 - \delta[$ around x_0, an open ball $\{\boldsymbol{y} \in \mathbb{R}^m : \|\boldsymbol{y} - \boldsymbol{y}_0\| < \epsilon\}$ around $\boldsymbol{y}_0$, and a unique function $\boldsymbol{\psi}$ from $]x_0 - \delta, x_0 - \delta[$ into $\{\boldsymbol{y} \in \mathbb{R}^m : \|\boldsymbol{y} - \boldsymbol{y}_0\| < \epsilon\}$ satisfying

$$\boldsymbol{g}\big(x, \boldsymbol{\psi}(x)\big) = \boldsymbol{0} \quad \text{for all } x \text{ in }]x_0 - \delta, x_0 - \delta[,$$

and this function $\boldsymbol{\psi}$ is continuously differentiable.

Buckling of perfect beam

- Let us consider again our boundary-value problem for the perfect beam. In the previous lecture, the finite-difference approximation of this boundary-value problem led to the algebraic problem

$$[K]\boldsymbol{\theta} + \lambda \boldsymbol{f}(\boldsymbol{\theta}) = \mathbf{0}.$$

- We look at this problem as an equation $\boldsymbol{g}(\lambda, \boldsymbol{\theta}) = \mathbf{0}$ in two "variables" λ and $\boldsymbol{\theta}$:

$$\boldsymbol{g}(\lambda, \boldsymbol{\theta}) = \mathbf{0} \quad \text{with} \quad \boldsymbol{g}(\lambda, \boldsymbol{\theta}) \equiv [K]\boldsymbol{\theta} + \lambda \boldsymbol{f}(\boldsymbol{\theta}).$$

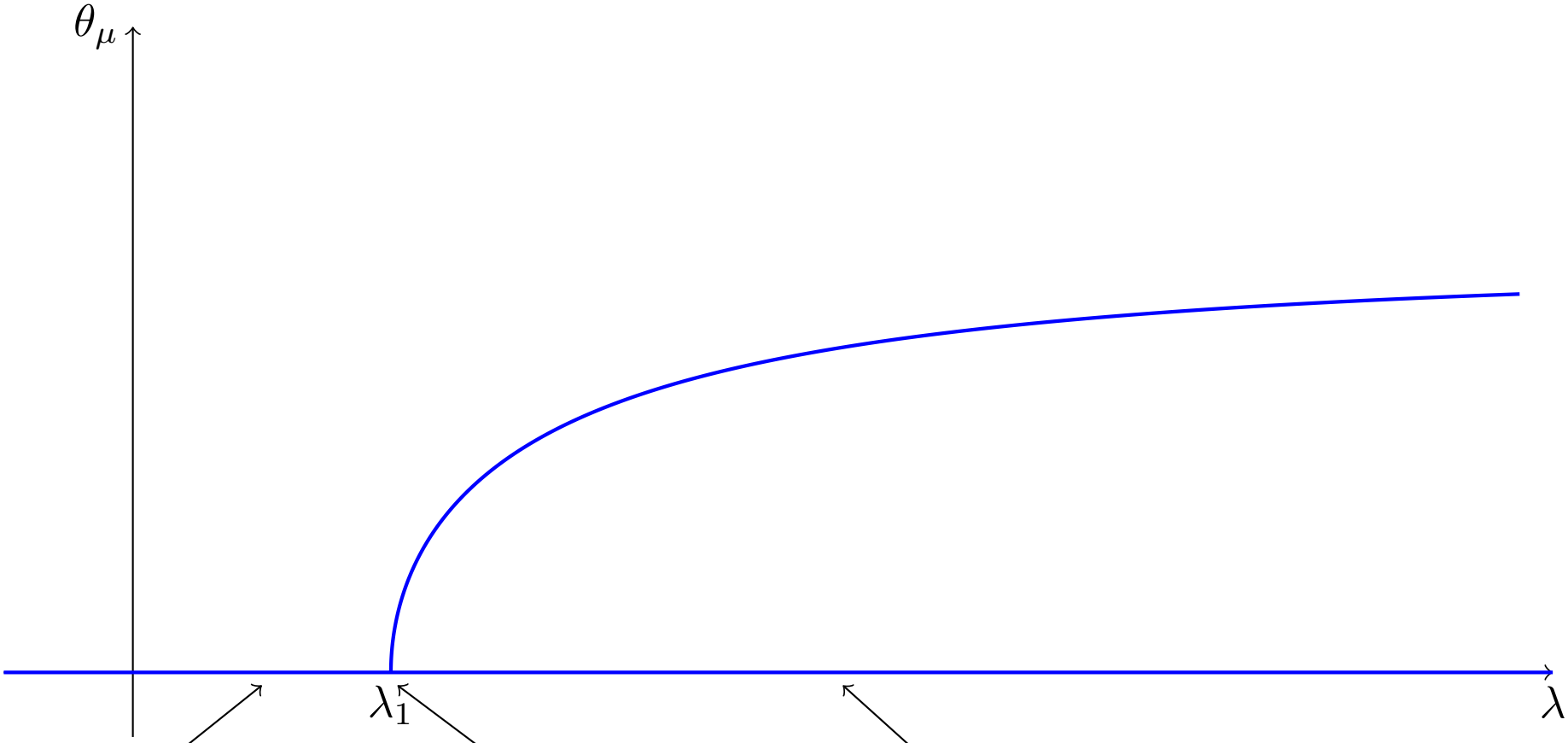
- This problem admits for any λ in $\mathbb{R}$ the trivial solution $(\lambda, \boldsymbol{\theta}) = (\lambda, \mathbf{0})$, that is, $\boldsymbol{g}(\lambda, \mathbf{0}) = \mathbf{0}$.

- The implicit function theorem asserts that nontrivial solutions can exist close to the trivial solution $(\lambda, \mathbf{0})$ only if the linearization of the problem $\boldsymbol{g}(\lambda, \boldsymbol{\theta}) = \mathbf{0}$ with respect to $\boldsymbol{\theta}$ at $(\lambda, \mathbf{0})$ is degenerate, or, equivalently, if the Jacobian matrix has a vanishing eigenvalue, that is,

$$\text{there exists} \quad \boldsymbol{\phi} \neq \mathbf{0} \quad \text{for which} \quad [\mathrm{D}_{\boldsymbol{\theta}} \boldsymbol{g}(\lambda, \mathbf{0})]\boldsymbol{\phi} = [K]\boldsymbol{\phi} + \lambda [\mathrm{D}_{\boldsymbol{\theta}} \boldsymbol{f}(\mathbf{0})]\boldsymbol{\phi} = [K]\boldsymbol{\phi} + \lambda \boldsymbol{\phi} = \mathbf{0},$$

thus confirming what we already found in the previous lecture, namely, that bifurcations occur at the eigenvalues λ_k of the eigenproblem $-[K]\boldsymbol{\phi}_k = \lambda_k \boldsymbol{\phi}_k$.

Buckling of perfect beam (continued)



$[\mathrm{D}_{\boldsymbol{\theta}}\boldsymbol{g}(\lambda, \mathbf{0})]$ has strictly positive eigenvalues

$[\mathrm{D}_{\boldsymbol{\theta}}\boldsymbol{g}(\lambda, \mathbf{0})]$ has zero eigenvalue

$[\mathrm{D}_{\boldsymbol{\theta}}\boldsymbol{g}(\lambda, \mathbf{0})]$ has strictly positive and negative eigenvalues

Buckling of imperfect beam

- Let us now consider again the boundary-value problem for the imperfect beam.

$$\begin{cases} \dfrac{d^2(\theta - \sigma\theta_\epsilon)}{ds^2} + \lambda \sin(\theta) = 0 \quad \text{with} \quad \lambda = \dfrac{p}{yj}, \\ \theta(0) = \sigma\theta_\epsilon(0) \quad \text{and} \quad \dfrac{d\theta}{ds}(\ell) = \sigma \dfrac{d\theta_\epsilon}{ds}(\ell). \end{cases}$$

- Introducing the auxiliary function $w = \theta - \sigma\theta_\epsilon$, we obtain the boundary-value problem

$$\begin{cases} \dfrac{d^2w}{ds^2} + \lambda \sin(w + \sigma\theta_\epsilon) = 0 \quad \text{with} \quad \lambda = \dfrac{p}{yj}, \\ w(0) = \dfrac{dw}{ds}(\ell) = 0. \end{cases}$$

- As in the previous lecture, the finite-difference approximation of this boundary-value problem leads to the algebraic problem of the form

$$[K]\boldsymbol{w} + \lambda \boldsymbol{f}(\boldsymbol{w} + \sigma\boldsymbol{\theta}_\epsilon) = \mathbf{0}.$$

where $\boldsymbol{f}(\boldsymbol{w} + \sigma\boldsymbol{\theta}_\epsilon)$ is the vector with components $\sin(w_j + \sigma\theta_\epsilon(s_j)),\ j = 1, \ldots, \mu - 1$.

Buckling of imperfect beam (continued)

- In a computation under "load control," one repeats a Newton-Raphson procedure for a sequence of increasing values of λ. To "follow" a particular "branch" of solutions, one can systematically choose as initial approximation to the solution for a subsequent value of λ the final approximation to the solution obtained for the previous value of λ.

- A simple method for "detecting" bifurcations involves systematically inspecting the eigenvalues of the Jacobian matrix. At a bifurcation, the Jacobian matrix must have at least one vanishing eigenvalue. An increase in λ that changes the sign of the smallest magnitude eigenvalue of the Jacobian matrix is an increase in λ that passes the first bifurcation.

- For a value of λ slightly larger than that at a bifurcation, one can make the Newton-Raphson converge to a solution on the new "branch" of solutions by using as initial approximation the sum of the final approximation to the solution obtained for the previous value of λ and a multiple of the eigenvector of the Jacobian matrix corresponding to the eigenvalue that changed its sign, where the premultiplication constant must be appropriately chosen.

Stochastic model

Review: Notations and conventions

- Let $f : \mathbb{R} \to \mathbb{R}$ be an **integrable function**, that is, $\int_{\mathbb{R}} |f(t)|dt < +\infty$. Then, the **Fourier transform** $\hat{f}$ of f is the bounded, continuous function $\hat{f}$ from $\mathbb{R}$ into $\mathbb{C}$ such that

$$\hat{f}(\omega) = \mathcal{F}f(\omega) = \int_{\mathbb{R}} \exp(-i\omega t) f(t) dt.$$

  The Fourier transform of an integrable function is not necessarily integrable itself.

- Let $f : \mathbb{R} \to \mathbb{R}$ be a **square-integrable function** , that is, $\int_{\mathbb{R}} |f(t)|^2 dt < +\infty$. Then, the **Fourier transform** $\hat{f}$ of f is the square-integrable function $\hat{f}$ from $\mathbb{R}$ into $\mathbb{C}$ such that

$$\begin{cases} \hat{f}(\omega) = \mathcal{F}f(\omega) = \displaystyle\int_{\mathbb{R}} \exp(-i\omega t) f(t) dt, \\ f(t) = \mathcal{F}^{-1}\hat{f}(t) = \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}} \exp(i\omega t) \hat{f}(\omega) d\omega. \end{cases}$$

- These definitions indicate that one cannot take the Fourier transform of any function: these definitions provide the Fourier transform only for integrable and square-integrable functions.

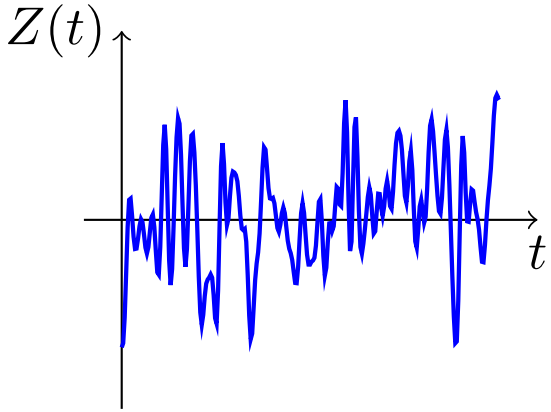
- We include the minus sign in the forward transform and the factor $\frac{1}{2\pi}$ in the inverse transform.

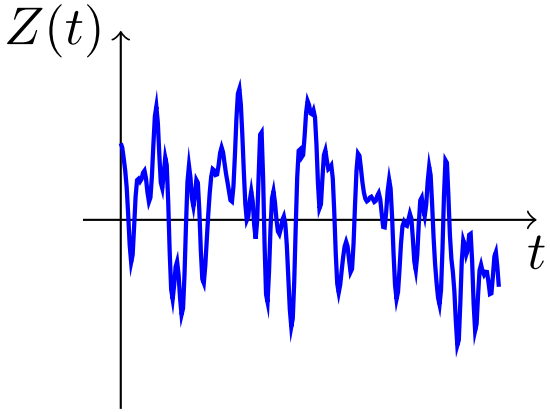
Review: Outlook

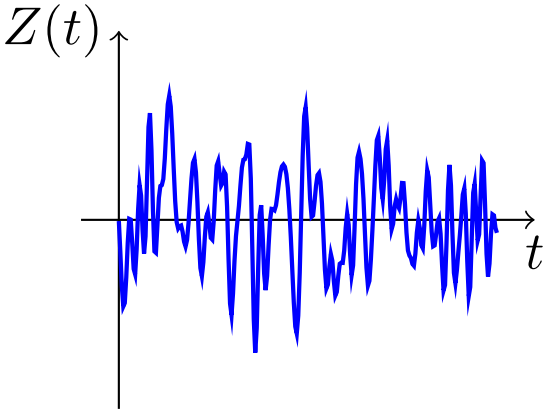
- **Random variables** (samples are scalars, vectors, matrices,. . .):



- **Stochastic processes** (samples are functions of one variable):







Review: Random variables

- The **probability distribution** P_Z of a random variable Z with values in $\mathbb{R}$ is the function that associates to any meaningful subset $\mathcal{B}$ of $\mathbb{R}$ the probability that the value taken by Z is in $\mathcal{B}$, that is,
$$P_Z(\mathcal{B}) = P(Z \in \mathcal{B}).$$

- The **probability density function** ρ_Z of a probability distribution P_Z with respect to dz, if it exists, is the function from $\mathbb{R}$ into $\mathbb{R}^+$ such that for any meaningful subset $\mathcal{B}$ of $\mathbb{R}$, we have
$$P_Z(\mathcal{B}) = \int_{\mathcal{B}} \rho_Z(z) dz.$$
The probability density function is normalized in that $P_Z(\mathbb{R}) = \int_{\mathbb{R}} \rho_Z(z) dz = 1$.

- A random variable Z with values in $\mathbb{R}$ is a **Gaussian** random variable with mean $\overline{z}$ and variance σ_Z^2 if it admits the probability density function
$$\rho_Z(z) = \frac{1}{\sqrt{2\pi}\sigma_Z} \exp\left(-\frac{(z-\overline{z})^2}{2\sigma_Z^2}\right).$$

Review: Random variables

- A random variable Z with values in $\mathbb{R}$ is of the **second order** if
$$E(Z^2) = \int_{\mathbb{R}} z^2 \rho_Z(z) dz < +\infty.$$

- The **mean** $\overline{z}$ of a second-order random variable Z with values in $\mathbb{R}$ is defined by
$$\overline{z} = E(Z) = \int_{\mathbb{R}} z \rho_Z(z) dz.$$

- The **variance** σ_Z^2 of a second-order random variable Z with values in $\mathbb{R}$ is defined by
$$\sigma_Z^2 = E\big((Z - \overline{z})^2\big) = \int_{\mathbb{R}} (z - \overline{z})^2 \rho_Z(z) dz.$$

- Please note that $\sigma_Z^2 = E\big((Z - \overline{z})^2\big) = E(Z^2) - \overline{z}^2$.

Review: Stochastic processes

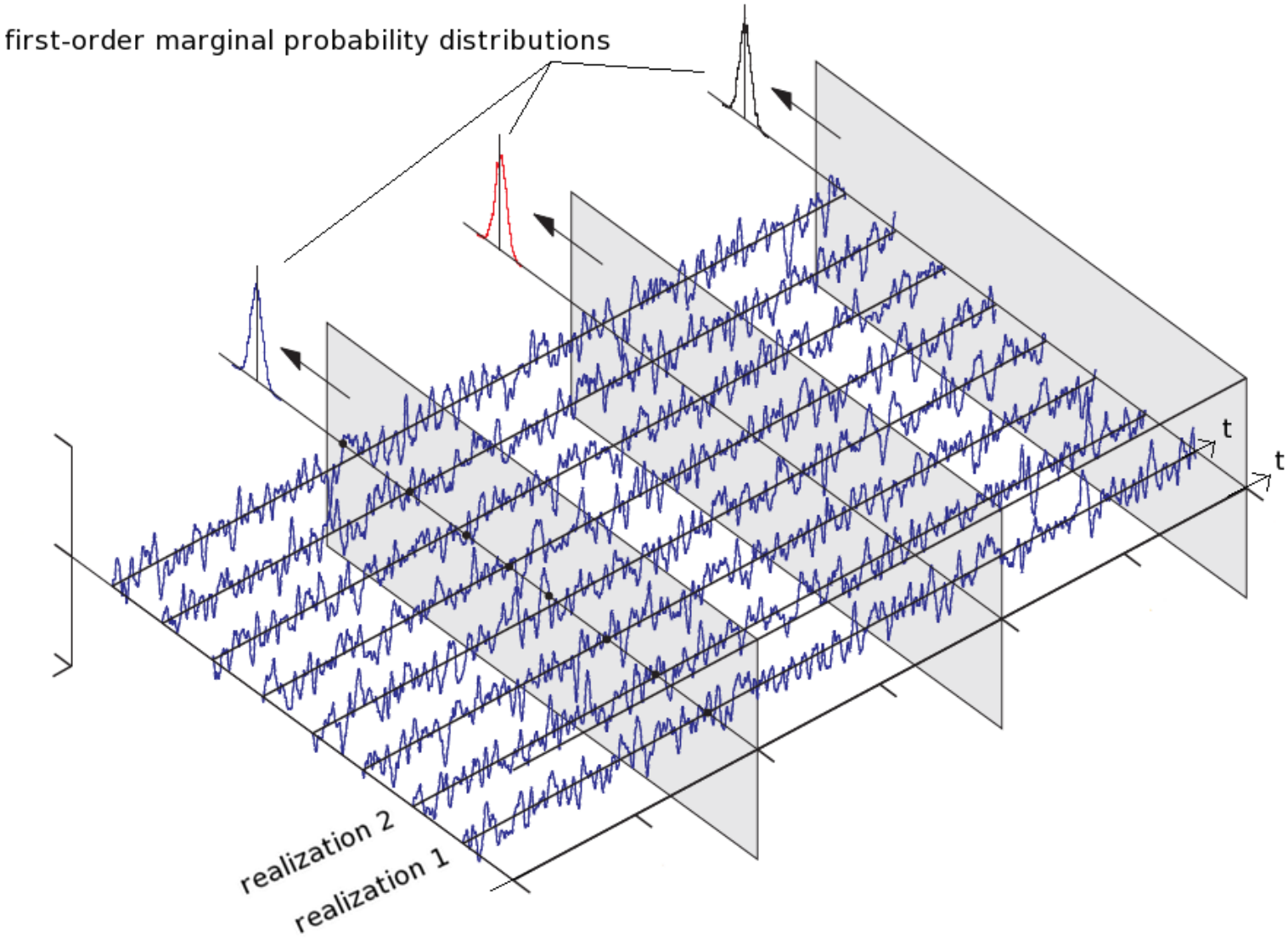
- A **stochastic process** $\{Z(t),\ t \in \mathcal{T}\}$ indexed by a subset $\mathcal{T}$ of $\mathbb{R}$ and with values in $\mathbb{R}$ is a collection of random variables $Z(t)$ with values in $\mathbb{R}$ indexed by t in $\mathcal{T}$.

- For any nonempty finite subset $\{t_1, \ldots, t_m\}$ of $\mathcal{T}$, where m denotes the number of elements in this subset, the joint probability distribution $P_{(Z(t_1),\ldots,Z(t_m))}$ of $(Z(t_1), \ldots, Z(t_m))$ is called a **(m-th order) marginal probability distribution** of the stochastic process $\{Z(t),\ t \in \mathcal{T}\}$.

  The collection of all the marginal probability distributions of a stochastic process is called the **system of marginal probability distributions**.

- A stochastic process $\{Z(t),\ t \in \mathcal{T}\}$ indexed by $\mathcal{T}$ and with values in $\mathbb{R}$ is **Gaussian** if each probability distribution in its system of marginal probability distributions is Gaussian.

Review: Stochastic processes



Review: Stochastic processes

- A stochastic process $\{Z(t),\ t \in \mathcal{T}\}$ indexed by $\mathcal{T}$ with values in $\mathbb{R}$ is of the **second order** if

$$E\big(Z(t)^2\big) = \int_{\mathbb{R}} z^2 \rho_{Z(t)}(z) dz < +\infty, \quad \forall t \in \mathcal{T}.$$

- The **mean function** of a second-order stochastic process $\{Z(t),\ t \in \mathcal{T}\}$ indexed by $\mathcal{T}$ with values in $\mathbb{R}$ is the function $\overline{z}$ from $\mathcal{T}$ into $\mathbb{R}$ such that

$$\overline{z}(t) = E\big(Z(t)\big) = \int_{\mathbb{R}} z \rho_{Z(t)}(z) dz.$$

- The **autocorrelation function** of a second-order stochastic process $\{Z(t),\ t \in \mathcal{T}\}$ indexed by $\mathcal{T}$ with values in $\mathbb{R}$ is the function r_Z from $\mathcal{T} \times \mathcal{T}$ into $\mathbb{R}$ such that

$$r_Z(t,\tilde{t}) = E\big(Z(t)Z(\tilde{t})\big) = \int_{\mathbb{R}\times\mathbb{R}} z\tilde{z} \rho_{(Z(t),Z(\tilde{t}))}(z,\tilde{z}) dz d\tilde{z}.$$

- The **covariance function** of a second-order stochastic process $\{Z(t),\ t \in \mathcal{T}\}$ indexed by $\mathcal{T}$ with values in $\mathbb{R}$ is the function c_Z from $\mathcal{T} \times \mathcal{T}$ into $\mathbb{R}$ such that

$$c_Z(t,\tilde{t}) = E\Big(\big(Z(t)-\overline{z}(t)\big)\big(Z(\tilde{t})-\overline{z}(\tilde{t})\big)\Big) = \int_{\mathbb{R}\times\mathbb{R}} \big(z-\overline{z}(t)\big)\big(\tilde{z}-\overline{z}(\tilde{t})\big) \rho_{(Z(t),Z(\tilde{t}))}(z,\tilde{z}) dz d\tilde{z}.$$

- Please note that $c_Z(t,\tilde{t}) = E\Big(\big(Z(t)-\overline{z}(t)\big)\big(Z(\tilde{t})-\overline{z}(\tilde{t})\big)\Big) = r_Z(t,\tilde{t}) - \overline{z}(t)\overline{z}(\tilde{t})$.

Review: Stochastic processes

- A second-order stochastic process $\{Z(t),\ t \in \mathbb{R}\}$ indexed by $\mathbb{R}$ with values in $\mathbb{R}$ is **mean-square stationary** if $\overline{z}(t) = \overline{z}$ is independent of t and $r_Z(t, \tilde{t}) = r_Z(t - \tilde{t})$ depends on only $t - \tilde{t}$.

- The **power spectral density function** of a zero-mean, mean-square stationary, second-order stochastic process $\{Z(t),\ t \in \mathbb{R}\}$ indexed by $\mathbb{R}$ with values in $\mathbb{R}$, if it exists, is the function s_Z from $\mathbb{R}$ into $\mathbb{R}^+$ such that

$$r_Z(t - \tilde{t}) = \frac{1}{2\pi} \int_{\mathbb{R}} s_Z(\omega) \exp\big(i\omega(t - \tilde{t})\big) d\omega.$$

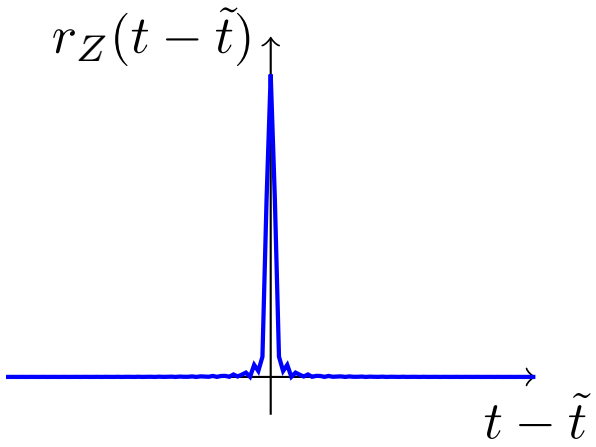
  The power spectral density function s_Z has the following properties:

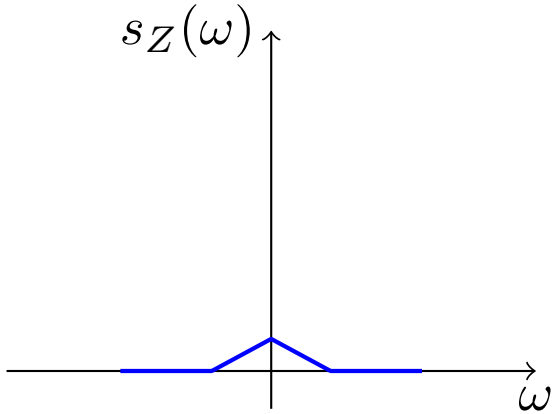
  - it is even because of the evenness of r_Z,
  - it is positive owing to Bochner's theorem,
  - it is integrable because $E\big(Z(t)^2\big) = r_Z(0) = \frac{1}{2\pi} \int_{\mathbb{R}} s_Z(\omega) d\omega < +\infty$.

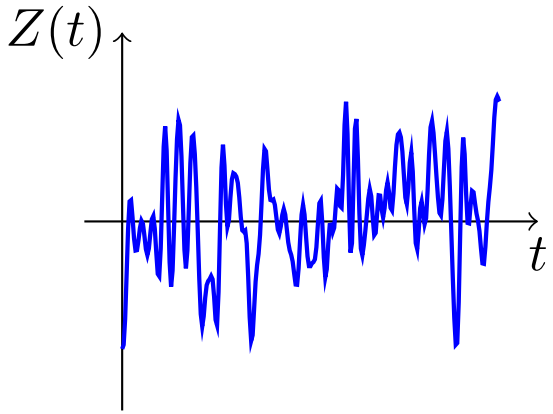
- If α is an integer, the **spectral moment of order α**, denoted by m_α, if it exists, is the integral

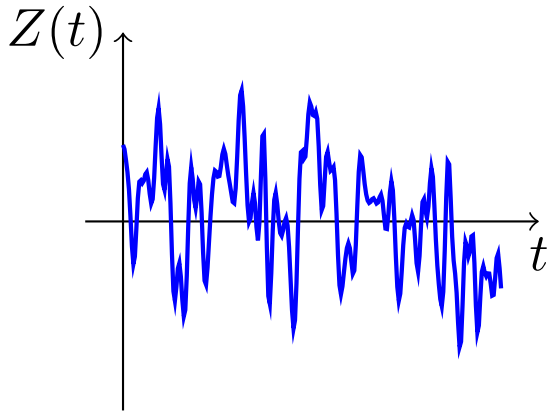
$$m_\alpha = \frac{1}{2\pi} \int_{\mathbb{R}} \omega^\alpha s_Z(\omega) d\omega.$$

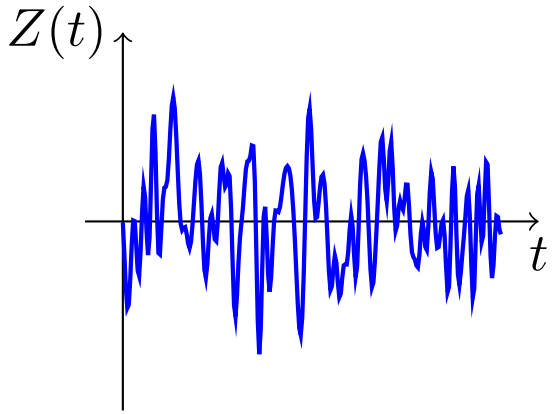
Review: Stochastic processes



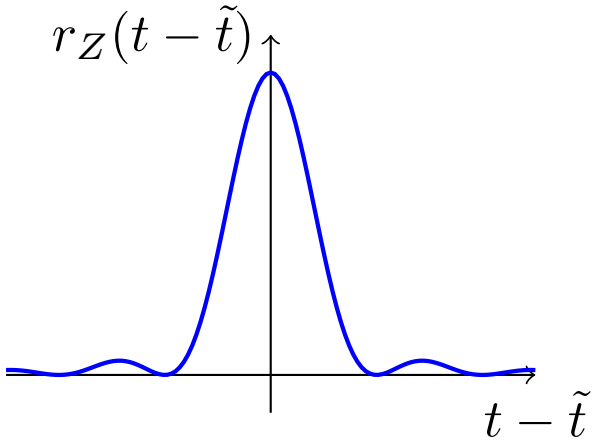


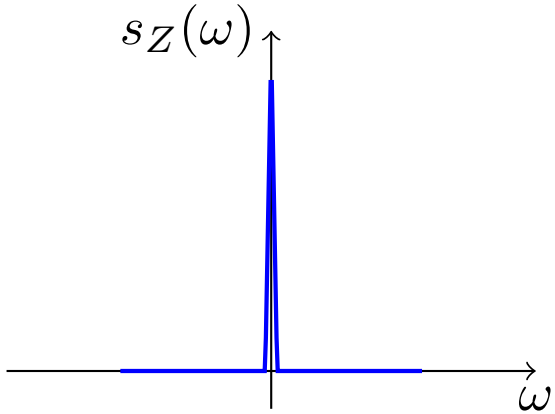


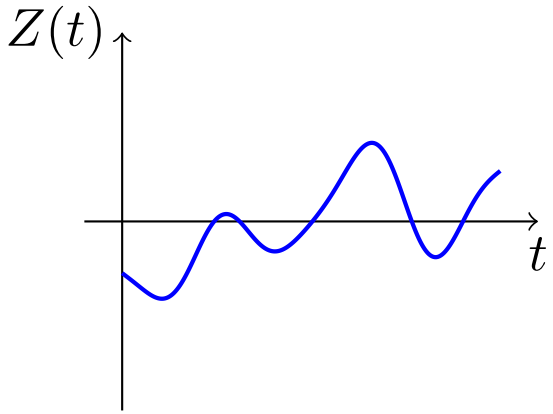


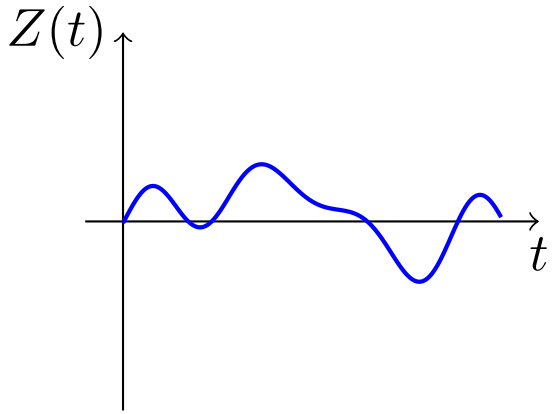


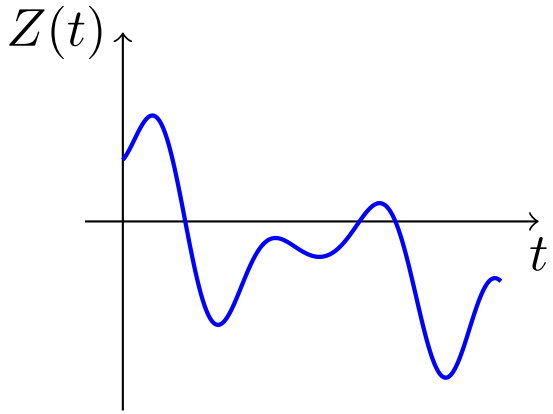
Review: Stochastic processes











Buckling of imperfect beam

- We model the random imperfection as a stochastic process $\{\sigma\theta_\epsilon(s),\ 0 \leq s \leq \ell\}$ indexed by $[0, \ell]$ with values in $\mathbb{R}$, Gaussian, zero mean, that is, $\sigma\overline{\theta}_\epsilon = 0$, and with autocorrelation function

$$r_{\sigma\theta_\epsilon}(s, \tilde{s}) = r_{\sigma\theta_\epsilon}(s - \tilde{s}) = \sigma^2 \,\mathrm{sinc}\left(\frac{\pi(s - \tilde{s})}{2\epsilon}\right)^2 .$$

- This autocorrelation function corresponds to the power spectral density function

$$s_{\sigma\theta_\epsilon}(\xi) = \sigma^2\, 2\epsilon \triangle\left(\frac{\xi\epsilon}{\pi}\right),$$

where $\triangle$ is the triangle function such that $\triangle(-\frac{\xi\epsilon}{\pi}) = \triangle(\frac{\xi\epsilon}{\pi})$ and $\triangle(\frac{\xi\epsilon}{\pi}) = 1 - \frac{\xi\epsilon}{\pi}$ if $0 \leq \frac{\xi\epsilon}{\pi} \leq 1$ and $\triangle\frac{\xi\epsilon}{\pi} = 0$ otherwise.

- This stochastic model depends on two parameters: σ^2 is the variance of the stochastic process and ϵ determines its spatial correlation.

Sampling method

Sampling method

- To numerically simulate a realization of the stochastic process, we apply a numerical method based on the spectral representation of this stochastic process. Fully describing this numerical method is beyond the scope of this lecture; for details, please refer to [Poirion and Soize, 1995].

- This numerical method allows a realization of the stochastic process, in particular, its values at μ grid points $s_j = jh,\ 0 \leq j \leq \mu - 1$ with $h < \epsilon$, to be computed as follows:

$$\sigma\theta_\epsilon^\mu(s_j) = \sqrt{2\Delta\xi}\ \mathrm{Re}\left(\sum_{k=1}^{\mu} \sqrt{\frac{1}{2\pi} s_{\sigma\theta_\epsilon}(\xi_k)}\ \zeta_k\ \exp(i s_j \xi_k + i\phi_k) \right);$$

  - $\xi_k = -\xi_\mathsf{L} + (k - 1/2)\Delta\xi, 1 \leq k \leq \mu$ with $\xi_\mathsf{L} = \pi/h$ and $\Delta\xi = 2\xi_\mathsf{L}/\mu$;
  - $\{\phi_k,\ 1 \leq k \leq \mu\}$ are μ independent realizations of a uniform r.v. valued in $[0, 2\pi]$;
  - $\{\zeta_k,\ 1 \leq k \leq \mu\}$ are such that $\zeta_k = \sqrt{-\log(\psi_k)}, 1 \leq k \leq \mu$, where the values $\{\psi_k,\ 1 \leq k \leq \mu\}$ are μ independent realizations of a uniform r.v. valued in $[0, 1]$.

- We can observe that the previous equation is consistent with the interpretation of the power spectral density function as indicating the distribution of the variance of the stochastic process over harmonic components of different wavenumbers. In fact, the previous equation provides an approximation of a realization of the stochastic process as a linear combination of harmonic components of different wavenumbers, each with a random phase shift and a random amplitude proportional to the square root of the value taken by the power spectral density function at the corresponding wavenumber.

Sampling method

- The previous equation can be written equivalently as follows:

$$\sigma\theta_\epsilon^\mu(s_j) = \sqrt{2\Delta\xi}\ \mathrm{Re}\left(\exp\left(ij\pi\left(-1+\frac{1}{\mu}\right)\right) \right.$$
$$\left. \sum_{k=1}^{\mu} \sqrt{\frac{1}{2\pi} s_{\sigma\theta_\epsilon}(\xi_k)}\ \zeta_k \exp\left(i\frac{2\pi}{\mu}(k-1) + i\phi_k \right) \exp\left(i(j-1)\frac{2\pi}{\mu}(k-1) \right) \right),$$

  thus allowing the summation to be computed by means of the discrete Fourier transform, hence, if μ is a power of two, by means of the fast Fourier transform algorithm (FFT/IFFT).

- This numerical simulation can be implemented in Matlab as follows:

```
xi=-pi/(ell/mu)+([0:mu-1]+0.5)*2*pi/ell;
s=sigma^2*2*epsilon*tripuls(xi*epsilon/pi/2);

thetaepsilon=sqrt(2*2*pi/ell)*real(mu*exp(i*[0:mu-1]*pi*(-1+1/mu)).*...
ifft(sqrt(s/2/pi).*sqrt(-log(rand(1,mu))).*exp(i*2*pi/mu*[0:mu-1]+i*2*pi*rand(1,mu))));
```

Assignment

- Please begin part 2 of 3 of the project by carrying out a few checks to make sure that realizations provided by the aforementioned sampling method can be considered to be good approximations to realizations of the stochastic process $\{\sigma\theta_\epsilon(s),\ 0 \leq s \leq \ell\}$.

- Then, please discuss the impact of σ and ϵ on the shape of the samples of the stochastic process, as well as on that of the autocorrelation function and the power spectral density function.

- Subsequently, for fixed values of σ and ϵ, please use the aforementioned sampling method to sample a realization of the stochastic process $\{\sigma\theta_\epsilon(s),\ 0 \leq s \leq \ell\}$, and consider the boundary-value problem for the imperfect beam determined by this sample. Implement the finite-difference method and carry out a computation under "load control" that follows the "branch" of solutions that starts at $\lambda = 0$. Systematically check the eigenvalues of the Jacobian matrix and as a bifurcation is "detected," make the algorithm "switch" to the new "branch."

- Finally, please try to gain some insight into the impact of the random imperfection on the first bifurcation by repeating the previous question for several more realisations of the stochastic process $\{\sigma\theta_\epsilon(s),\ 0 \leq s \leq \ell\}$. You could also consider using different values of σ and ϵ.

References

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