MATH0488 – Stochastic Processes

Stochastically perturbed bifurcation

Part 1 of 3: Buckling of perfect beam

Maarten Arnst, Marco Lucio Cerquaglia, and Kavita Goyal

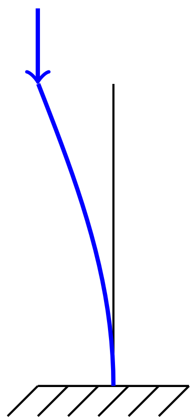
March 17, 2015

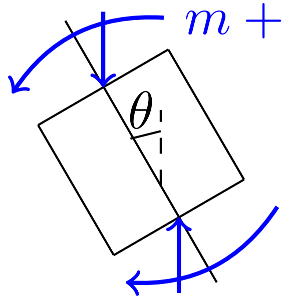
Outline

- Problem formulation.
- Linearized problem.
- Nonlinear problem.
- Finite-difference approximation.
- Assignment.
- References.

Problem formulation

- We consider the following problem setting:





  - Equilibrium: $m - (m + dm) - p\sin(\theta)ds = 0 \implies -\frac{dm}{ds} - p\sin(\theta) = 0$.
  - Constitutive equation: $\frac{d\theta}{ds} = \frac{m}{yj}$.
  - At the one end, the rod is fixed; at the other end, it is subjected to a constant vertical force p.

- Denoting by s the position along the beam, we obtain the following boundary-value problem:

$$\begin{cases} \dfrac{d^2\theta}{ds^2} + \lambda\sin(\theta) = 0 \quad \text{with} \quad \lambda = \dfrac{p}{yj}, \\ \theta(0) = \dfrac{d\theta}{ds}(\ell) = 0 \end{cases}$$

to determine the angle $\theta(s)$ that the tangent vector to the beam makes with the vertical axis as a function of the position s along the beam.

Linearized problem

- The linearized problem "close to $\theta = 0$" reads as follows:

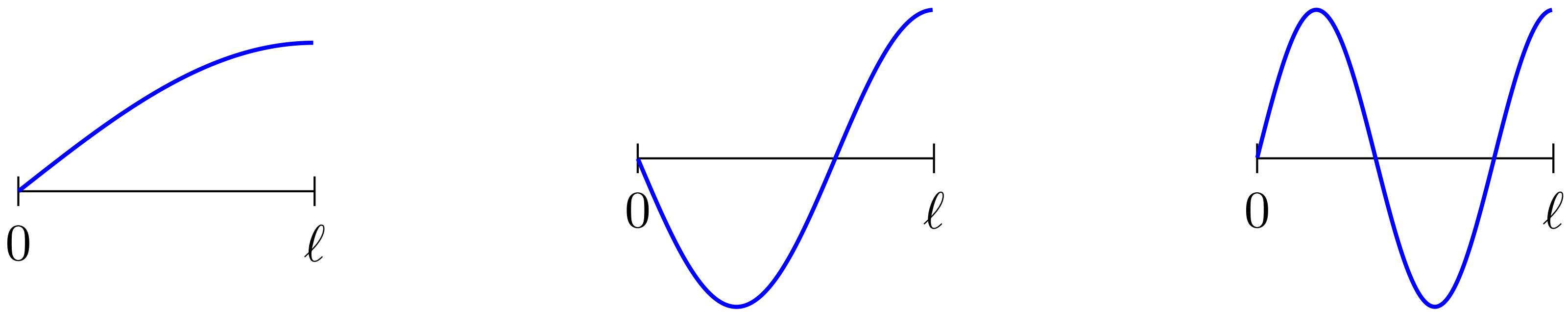
$$\begin{cases} \dfrac{d^2\theta}{ds^2} + \lambda\theta = 0 \quad \text{with} \quad \lambda = \dfrac{p}{yj}, \\ \theta(0) = \dfrac{d\theta}{ds}(\ell) = 0. \end{cases}$$

- The linearized problem admits for any λ in $\mathbb{R}$ the **trivial solution** $\theta = 0$.

- Given a strictly positive value of λ, any linear combination of the two linearly independent elementary solutions $\cos(\sqrt{\lambda}s)$ and $\sin(\sqrt{\lambda}s)$, that is, $\theta(s) = a\cos(\sqrt{\lambda}s) + b\sin(\sqrt{\lambda}s)$, solves the ODE. To satisfy $\theta(0) = 0$, the coefficient a must vanish. From $\frac{d\theta}{ds}(\ell) = 0$, it follows that the linearized problem has a **nontrivial solution** if and only if $\sqrt{\lambda}\cos(\sqrt{\lambda}\ell) = 0$, that is,

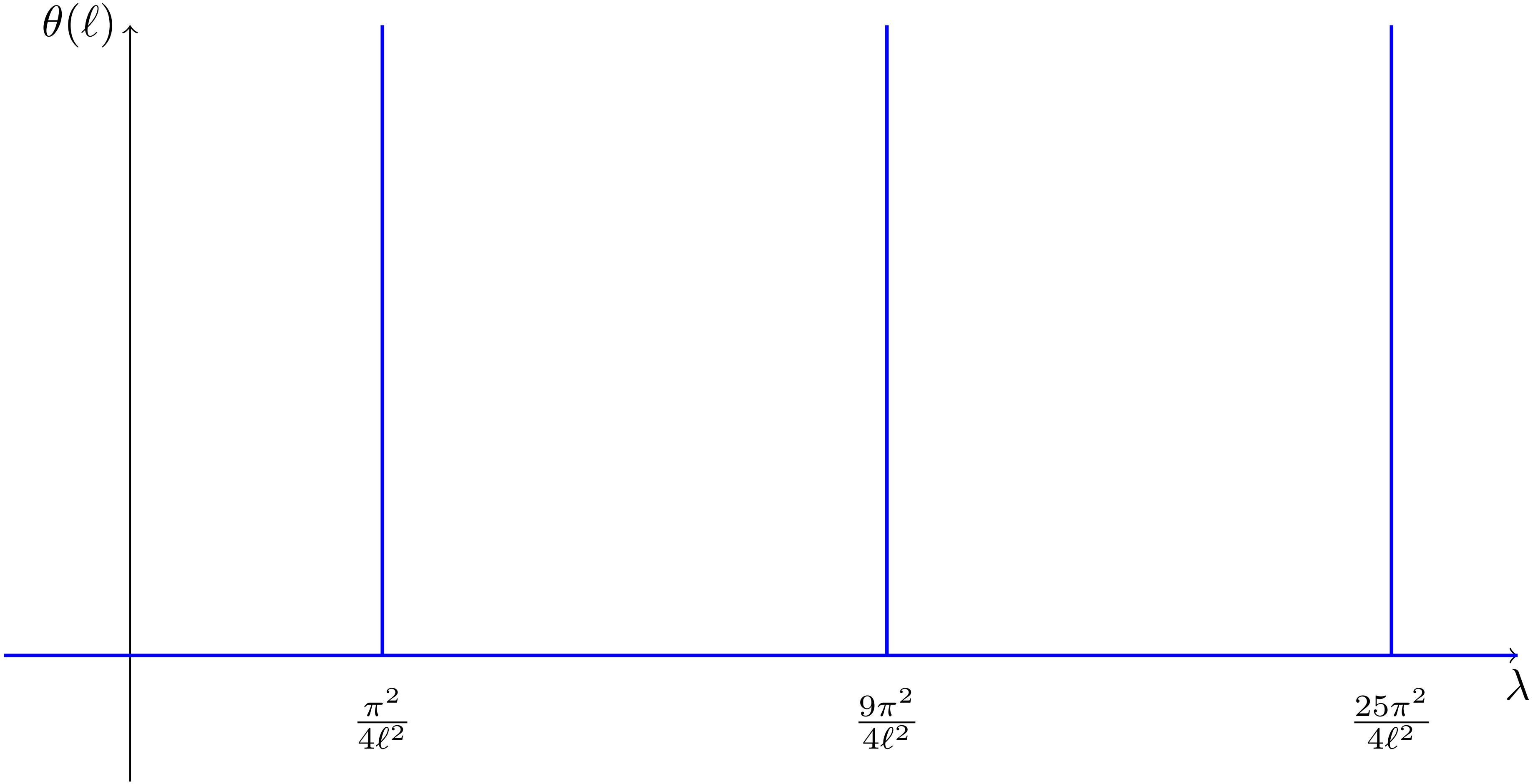
$$\lambda = \lambda_k = \frac{(2k+1)^2\pi^2}{4\ell^2}, \quad \phi_k(s) = \sin(\sqrt{\lambda_k}s), \quad k = 0, 1, \ldots$$

and every solution θ is a constant multiple of ϕ_k.



Linearized problem

- Schematic representation of the solution to the linearized problem:



- For any λ in $\mathbb{R}$, the linearized problem admits the trivial solution $\theta = 0$.
- For $\lambda_k = \frac{(2k+1)^2\pi^2}{4\ell^2}$ with $k = 0, 1, \ldots$, the linearized problem becomes degenerate and admits as nontrivial solution any constant multiple of the corresponding $\phi_k = \sin(\sqrt{\lambda_k} s)$.

Nonlinear problem

- Let us consider again our boundary-value problem

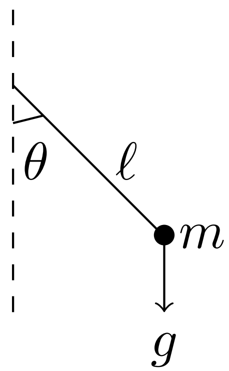
$$\begin{cases} \dfrac{d^2\theta}{ds^2} + \lambda \sin(\theta) = 0 \quad \text{with} \quad \lambda = \dfrac{p}{yj}, \\ \theta(0) = \dfrac{d\theta}{ds}(\ell) = 0. \end{cases}$$

  For nonlinear boundary-value problems, there is often little hope of finding explicit formulas for solutions. For this particular nonlinear boundary-value problem, it turns out that we can gain useful insight into the solutions through a “phase portrait” analysis, as described next.

- This two-dimensional system of first-order ODEs also appears in the study of the pendulum:

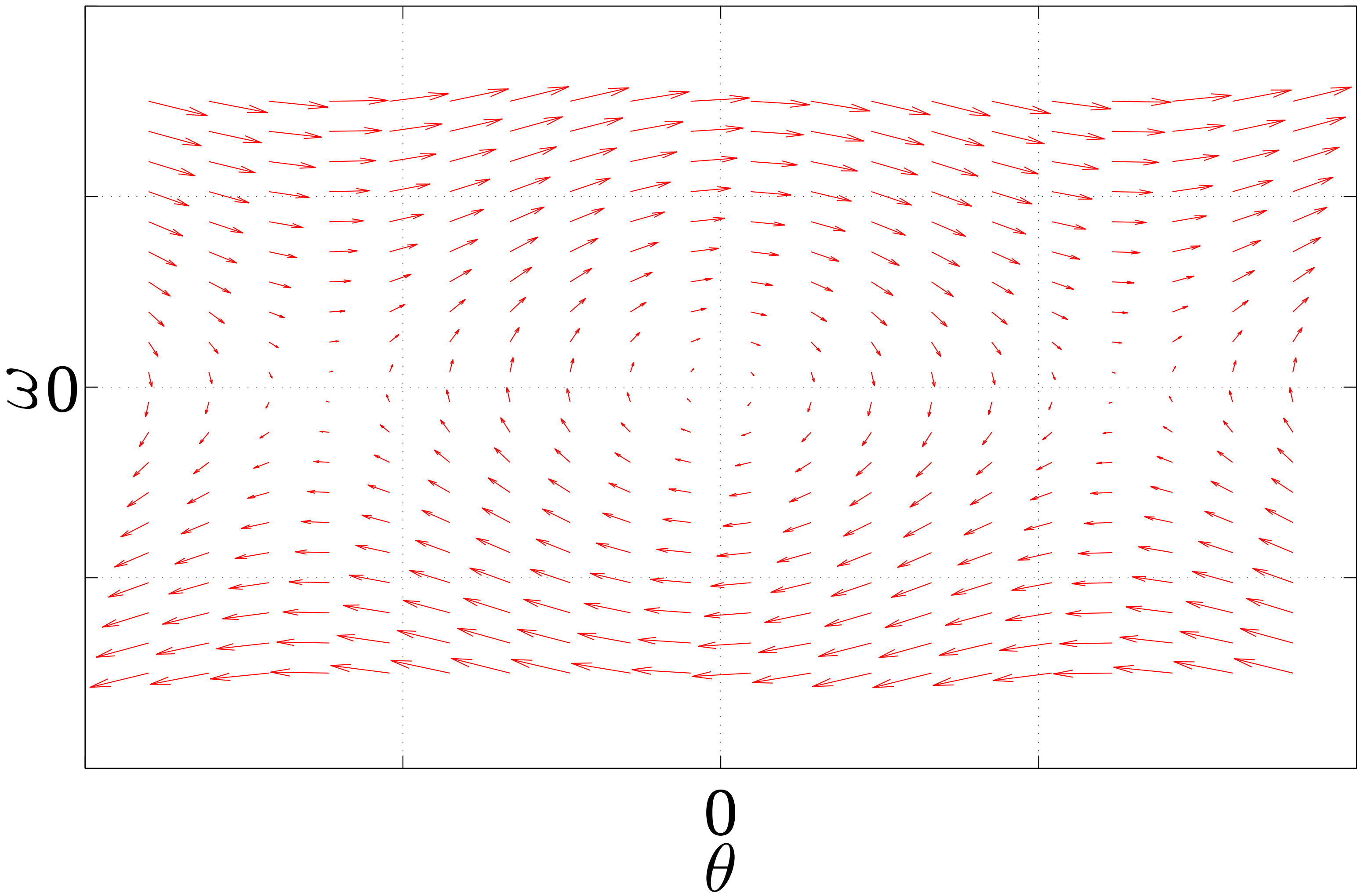
$$m\ell^2 \frac{d^2\theta}{dt^2} + \ell \sin(\theta) mg = 0$$

$$\implies \frac{d^2\theta}{dt^2} + \lambda \sin(\theta) = 0 \quad \text{with} \quad \lambda = \frac{g}{\ell}$$

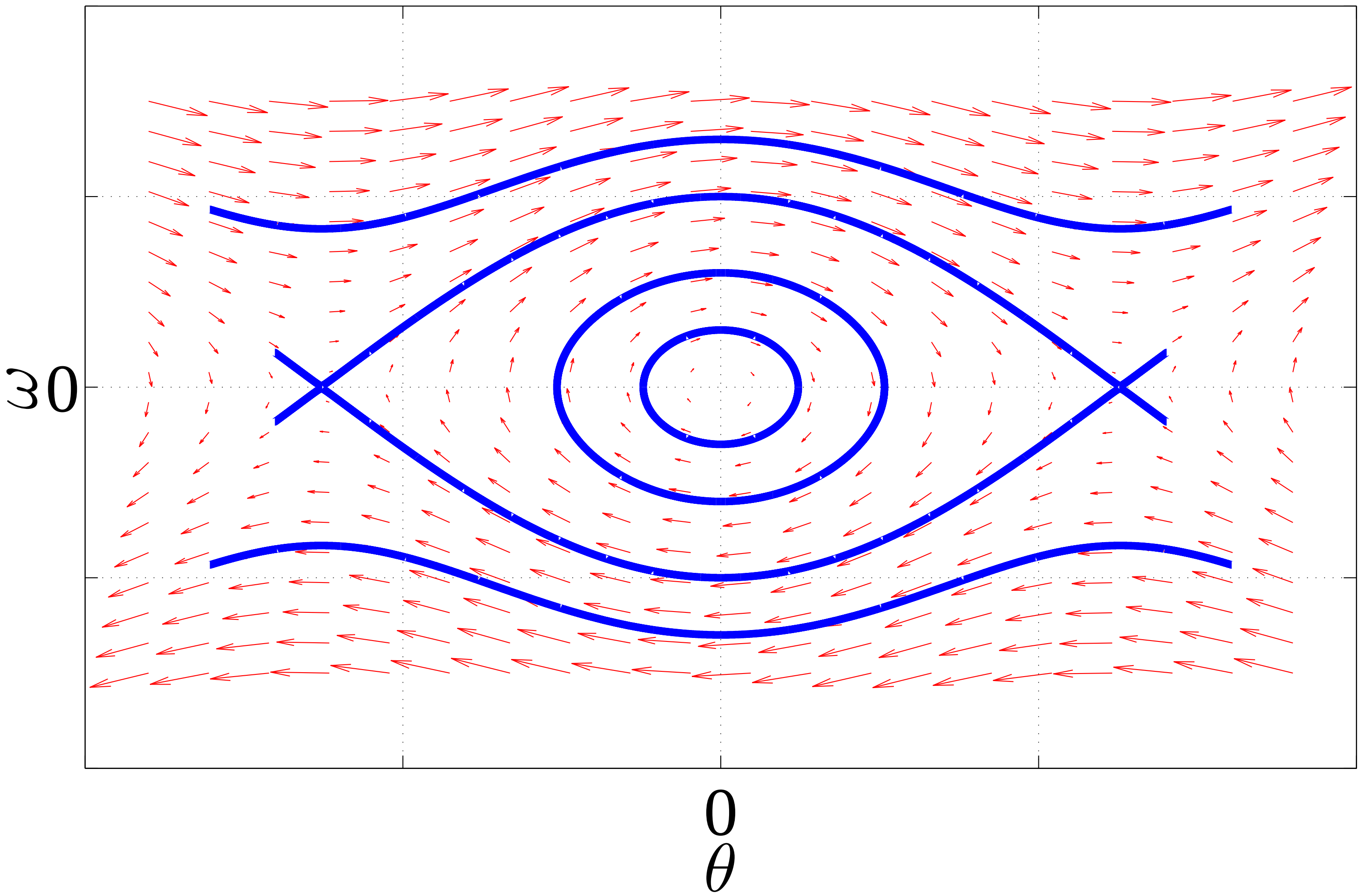


- Introducing the auxiliary variable $\omega = \frac{d\theta}{ds}$, we can write our one-dimensional second-order ODE equivalently as a two-dimensional system of first-order ODEs:

$$\begin{cases} \theta' = \omega, \\ \omega' = -\lambda \sin(\theta). \end{cases}$$

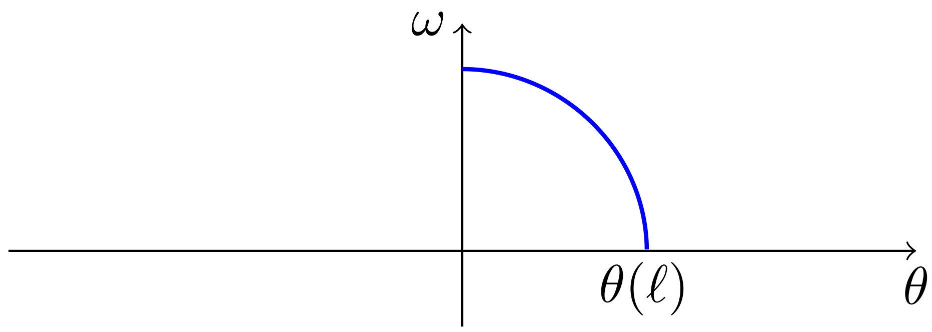


For a grid of points in the $\theta - \omega$ plane, we represented at each point the derivatives θ' and ω' as a vector (θ', ω'). By flowing along the vector field thus obtained, we can trace out solutions $(\theta(s), \omega(s))$.



For a grid of points in the $\theta - \omega$ plane, we represent at each point the derivatives θ' and ω' as a vector (θ', ω'). By flowing along the vector field thus obtained, we can trace out trajectories $(\theta(s), \omega(s))$.

- The boundary conditions in our boundary-value problem imply that we are interested only in trajectories that begin at $s = 0$ on the ω-axis (because $\theta(0)$ must vanish) and cross at precisely $s = \ell$ the θ-axis (because $\theta'(\ell) = \omega(\ell)$ must vanish).



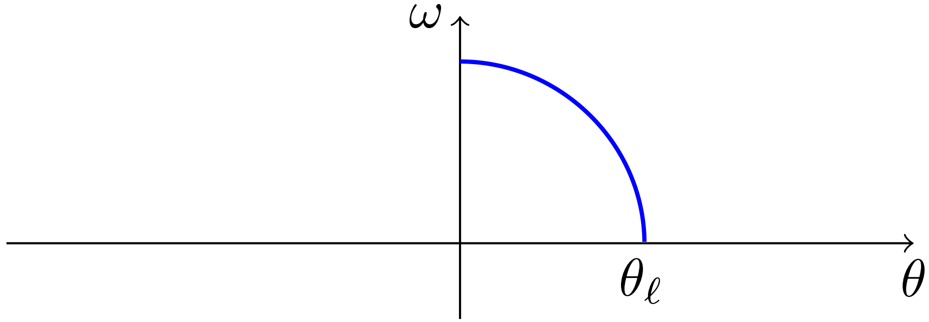
- To determine these trajectories, we proceed as follows. We note that the problem is conservative; specifically, multiplying the ODE with θ' and integrating, we obtain

$$\theta'(\theta'' + \lambda \sin(\theta)) = 0 \implies \frac{1}{2}\omega^2 + \lambda(1 - \cos(\theta)) = \text{constant},$$

that is, along trajectories, the quantity $\frac{1}{2}\omega^2 + \lambda(1 - \cos(\theta))$ is conserved.

Nonlinear problem

- Along a trajectory that begins on the ω-axis and crosses the θ-axis at $(\theta_\ell, 0)$, we thus have



$$\frac{1}{2}\omega(s)^2+\lambda\big(1-\cos(\theta(s))\big) = \lambda\big(1-\cos(\theta_\ell)\big) \implies \frac{d\theta}{ds}(s) = \omega(s) = \sqrt{2\lambda\big(\cos(\theta(s)) - \cos(\theta_\ell)\big)}.$$

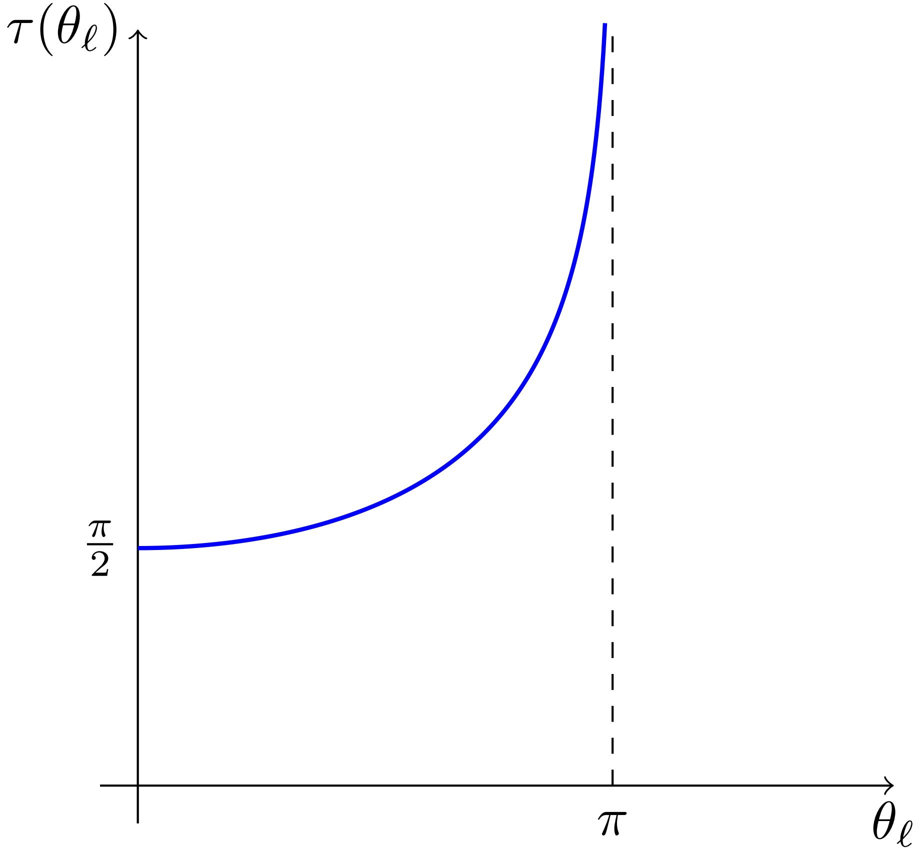
- This allows us to determine the distance traveled along a trajectory that begins on the ω-axis and crosses the θ-axis at $(\theta_\ell, 0)$:

$$s_\ell = \frac{1}{\sqrt{2\lambda}}\int_0^{\theta_\ell} \frac{d\theta}{\sqrt{\cos(\theta(s)) - \cos(\theta_\ell)}} \equiv \frac{\tau(\theta_\ell)}{\sqrt{\lambda}}.$$

- For this trajectory to be a solution to our boundary-value problem, it must begin at $s = 0$ on the ω-axis and cross at precisely $s = \ell$ the θ-axis:

$$\ell = \frac{\tau(\theta(\ell))}{\sqrt{\lambda}} \implies \lambda = \frac{\tau(\theta(\ell))^2}{\ell^2}.$$

- The function τ has the following form:



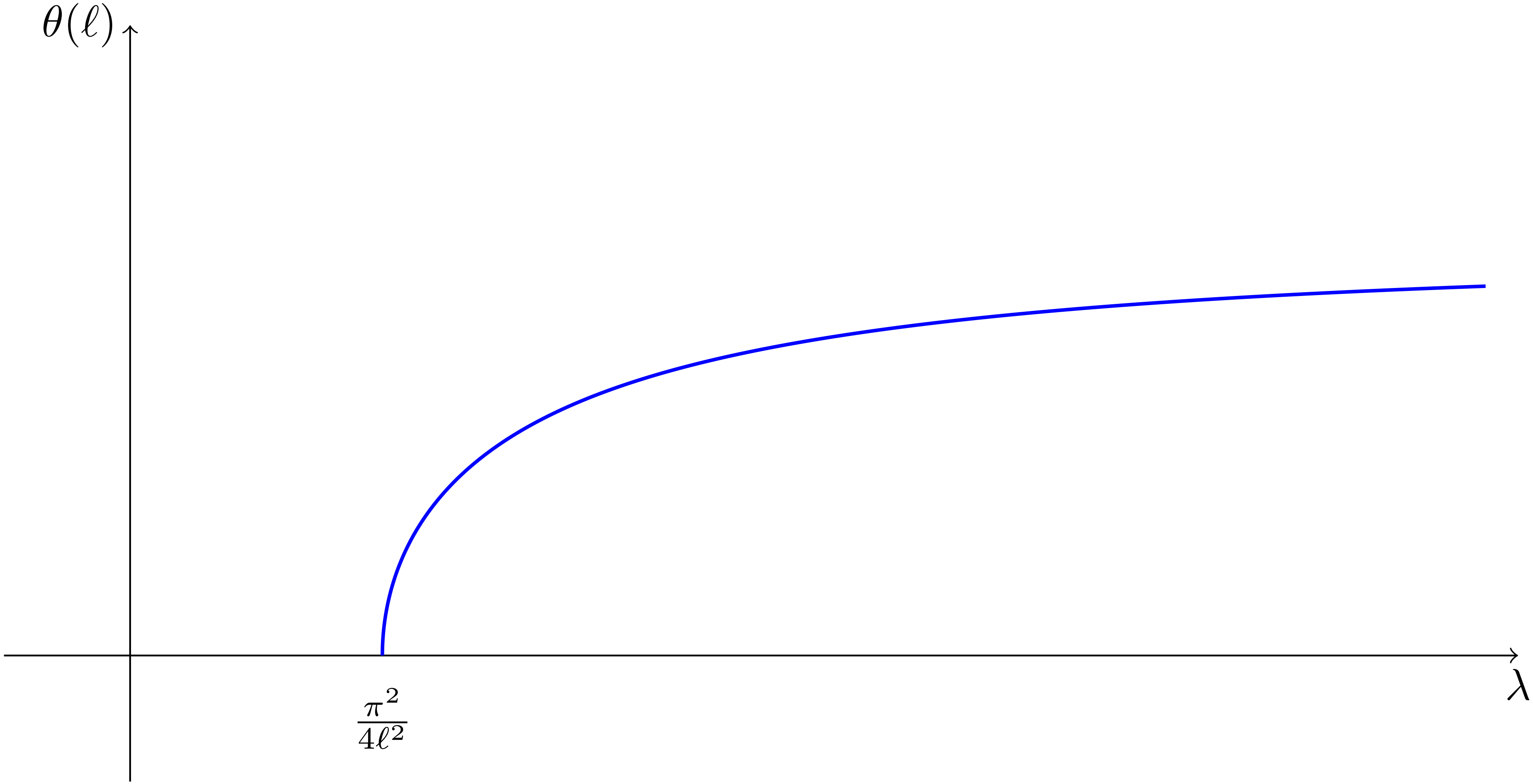
- We have $\tau(0) = \frac{\pi}{2}$. This can be understood from an analysis of the linearized problem:

$$\begin{cases} \theta'' + \lambda\theta = 0, \\ \theta(0) = 0 \quad \text{and} \quad \theta'(0) = \omega_0. \end{cases} \implies \theta(s) = \frac{\omega_0}{\sqrt{\lambda}} \sin(\sqrt{\lambda} s).$$

$$\theta'(s_\ell) = 0 \implies s_\ell = \frac{\pi}{2} \frac{1}{\sqrt{\lambda}}.$$

Nonlinear problem

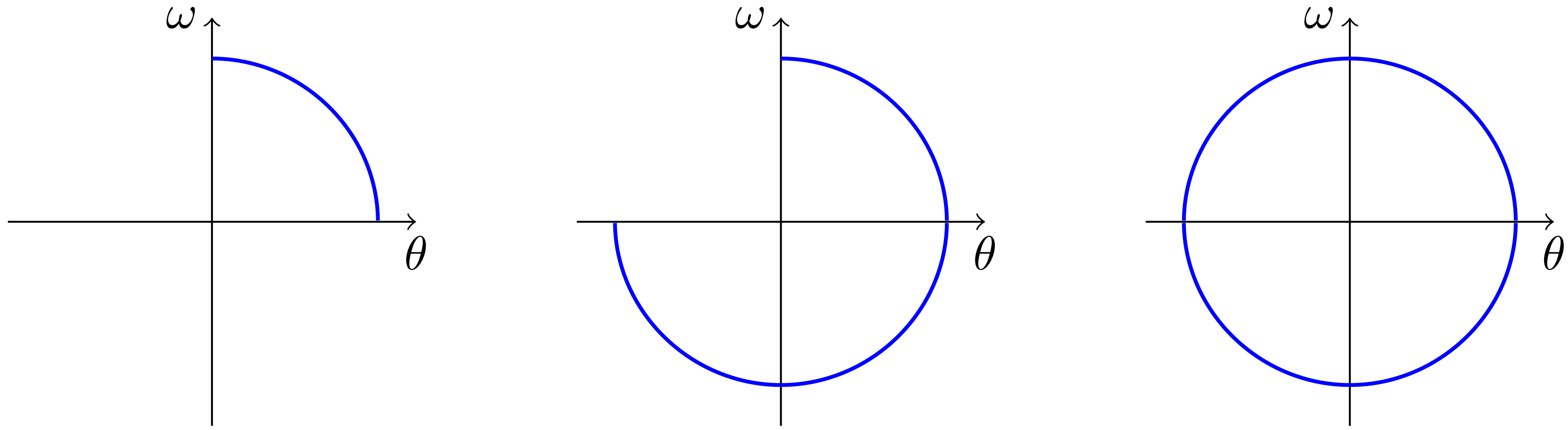
- This leads us to the following schematic representation of the nontrivial solutions found so far:



- For small θ close to the trivial solution $\theta = 0$, the behavior of the nontrivial solution curve obtained for the nonlinear boundary-value problem resembles that obtained for the linearized problem. For larger θ, the effect of the nonlinearity is to bend the nontrivial solution curve.

Nonlinear problem

- Of course, to be a solution to the boundary-value problem, the trajectory may also encircle the origin before it crosses at $s = \ell$ the θ-axis:

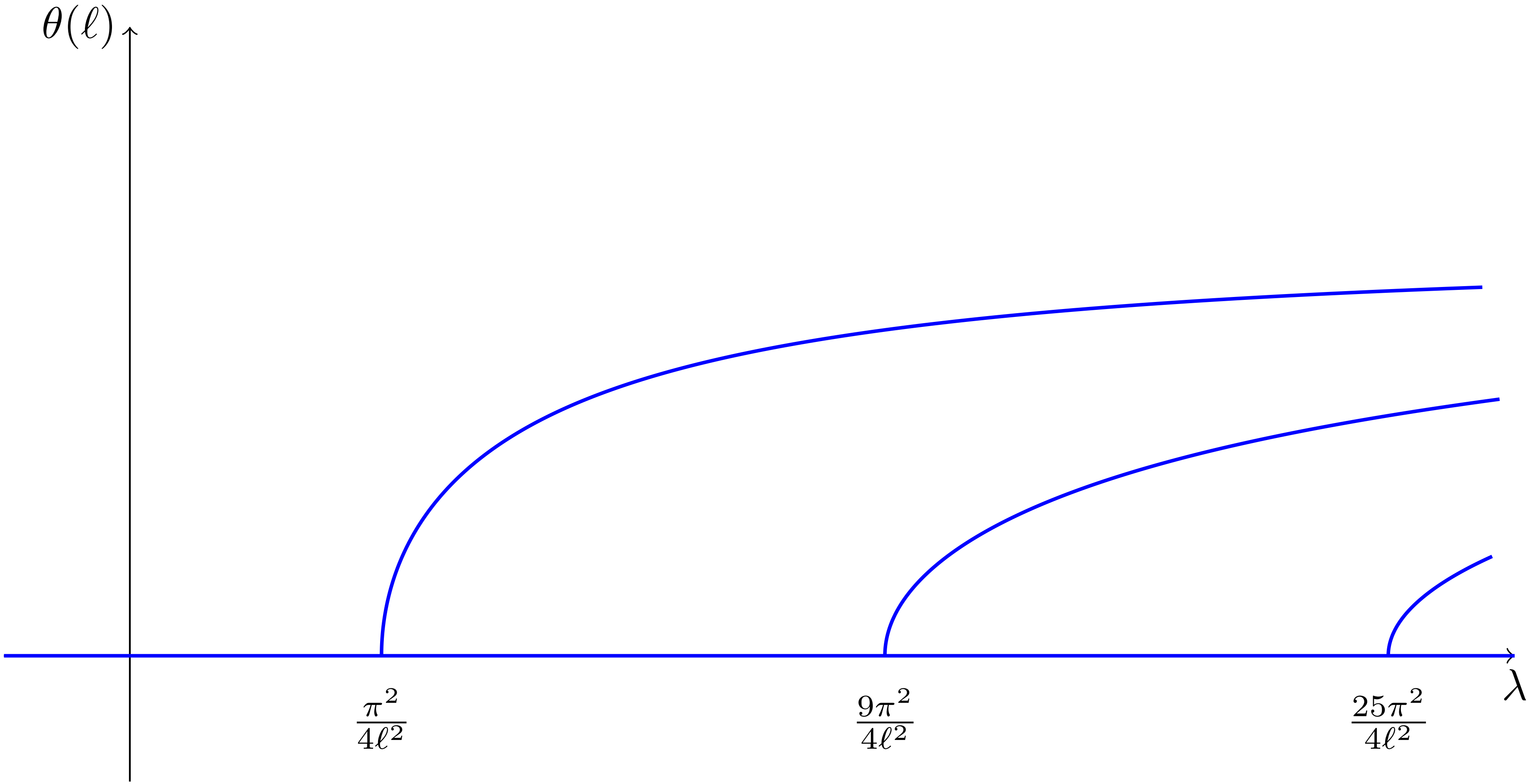


- This leads us to solutions that satisfy

$$\ell = \frac{(2k+1)\tau(\theta(\ell))}{\sqrt{\lambda}} \implies \lambda = \frac{(2k+1)^2\tau(\theta(\ell))^2}{\ell^2} \quad k = 0, 1, \ldots$$

Nonlinear problem

- This leads us to the following schematic representation of the trivial and nontrivial solutions:

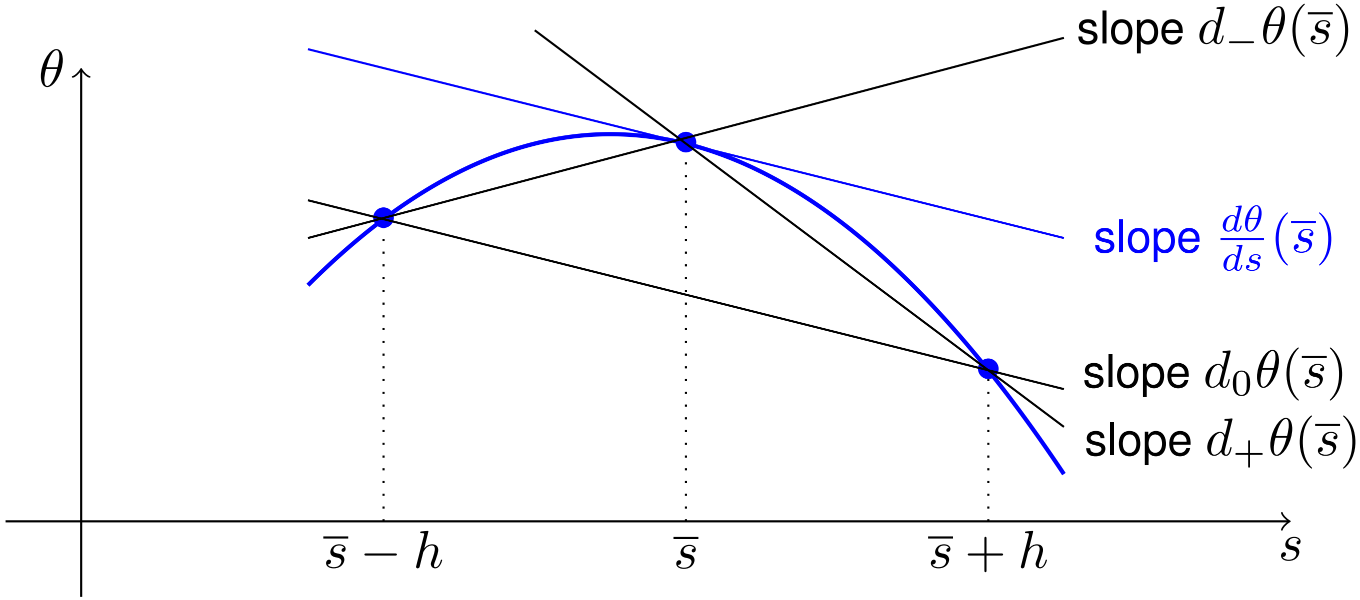


  - We can observe that the trivial and nontrivial solution branches intersect at $\lambda_k = \frac{(2k+1)^2\pi^2}{4\ell^2}$ with $k = 0, 1, \ldots$ We refer to these intersections as **bifurcations**.

Finite-difference approximation

Notion of finite-difference approximation

- Several finite-difference approximations of $\frac{d\theta}{ds}(\overline{s})$:



$$d_+\theta(\overline{s}) = \frac{\theta(\overline{s}+h)-\theta(\overline{s})}{h}, \quad d_-\theta(\overline{s}) = \frac{\theta(\overline{s})-\theta(\overline{s}-h)}{h}, \quad d_0\theta(\overline{s}) = \frac{\theta(\overline{s}+h)-\theta(\overline{s}-h)}{2h}.$$

- Similar finite-difference approximations can be defined for higher order derivatives, for example,

$$\frac{d^2\theta}{ds^2}(\overline{s}) \approx d_0^2\theta(\overline{s}) = \frac{\theta(\overline{s}-h)-2\theta(\overline{s})+\theta(\overline{s}+h)}{h^2}.$$

Finite-difference method for our boundary-value problem

- Let us consider again our boundary-value problem

$$\begin{cases} \theta'' + \lambda \sin(\theta) = 0, \\ \theta(0) = \theta'(\ell) = 0. \end{cases}$$

- We introduce grid points $s_0,\ s_1,\ s_2,\ \ldots,\ s_\mu$ as follows:

The grid spacing is denoted by h; thus, $s_j = jh$ for $j = 0, \ldots, \mu$ with $\mu = \ell/h$.

- A **finite-difference method** is then obtained by computing approximations $\theta_0,\ \ldots,\ \theta_\mu$ of the values $\theta(s_0),\ \ldots,\ \theta(s_\mu)$ taken by the exact solution at the grid points $s_0,\ \ldots,\ s_\mu$ by requiring

$$\begin{cases} \dfrac{\theta_{j-1} - 2\theta_j + \theta_{j+1}}{h^2} + \lambda \sin(\theta_j) = 0 \quad \text{for } j = 1, \ldots, \mu - 1, \\ \theta_0 = \dfrac{\theta_{\mu-2} - 4\theta_{\mu-1} + 3\theta_\mu}{h} = 0. \end{cases}$$

This corresponds to replacing $\frac{d^2\theta}{ds^2}(s_j)$ with $\frac{\theta_{j-1}-2\theta_j+\theta_{j+1}}{h^2}$ in the ODE and $\frac{d\theta}{ds}(s_\mu)$ with $\frac{\theta_{\mu-2}-4\theta_{\mu-1}+3\theta_\mu}{h}$ in the boundary condition at the edge point $s_\mu = \ell$.

Finite-difference method for our boundary-value problem (continued)

- The algebraic problem provided by the aforementioned finite-difference method can be written as

$$\underbrace{\frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1-\frac{1}{3} & -2+\frac{4}{3} \end{bmatrix}}_{[K]} \underbrace{\begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_{\mu-2} \\ \theta_{\mu-1} \end{bmatrix}}_{\boldsymbol{\theta}^h} + \lambda \underbrace{\begin{bmatrix} \sin(\theta_1) \\ \sin(\theta_2) \\ \vdots \\ \sin(\theta_{\mu-2}) \\ \sin(\theta_{\mu-1}) \end{bmatrix}}_{\boldsymbol{f}(\boldsymbol{\theta}^h)} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix};$$

hence, more compactly,

$$[K]\boldsymbol{\theta}^h + \lambda \boldsymbol{f}(\boldsymbol{\theta}^h) = \mathbf{0};$$

Please note that $[K]$ is the above tridiagonal matrix premultiplied with $1/h^2$; by contrast, $\boldsymbol{f}(\boldsymbol{\theta}^h)$ is the above vector without premultiplication with λ.

Linearized problem

- Let us consider again the linearized problem

$$\begin{cases} \theta'' + \lambda\theta = 0, \\ \theta(0) = \theta'(\ell) = 0. \end{cases}$$

- The application of the aforementioned finite-difference method leads to the algebraic problem

$$\underbrace{\frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1-\frac{1}{3} & -2+\frac{4}{3} \end{bmatrix}}_{[K]} \underbrace{\begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_{\mu-2} \\ \theta_{\mu-1} \end{bmatrix}}_{\boldsymbol{\theta}^h} + \lambda \underbrace{\begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_{\mu-2} \\ \theta_{\mu-1} \end{bmatrix}}_{\boldsymbol{\theta}^h} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix};$$

hence, more compactly,

$$[K]\boldsymbol{\theta}^h + \lambda\boldsymbol{\theta}^h = \mathbf{0}.$$

- For any λ in $\mathbb{R}$, the discretized linearized problem admits the trivial solution $\boldsymbol{\theta}^h = \mathbf{0}$. For λ equal to one of the eigenvalues λ_k^h of the eigenproblem $-[K]\boldsymbol{\phi}^h = \lambda_k^h\boldsymbol{\phi}^h$, it becomes degenerate and admits as nontrivial solution any constant multiple of the corresponding eigenvector $\boldsymbol{\phi}_k^h$.

Nonlinear problem

- For a given λ in $\mathbb{R}$, $[K]\boldsymbol{\theta}^h + \lambda \boldsymbol{f}(\boldsymbol{\theta}^h) = \mathbf{0}$ can be solved by using the Newton-Raphson method, an iterative method that constructs a sequence of approximations $\boldsymbol{\theta}_0^h,\ \boldsymbol{\theta}_1^h,\ \boldsymbol{\theta}_2^h, \ldots$ to $\boldsymbol{\theta}^h$. Specifically, given the i-th approximation $\boldsymbol{\theta}_i^h$, the nonlinear algebraic problem is linearized about $\boldsymbol{\theta}_i^h$,
$$[K](\boldsymbol{\theta}_i^h + \Delta\boldsymbol{\theta}_i^h) + \lambda(\boldsymbol{f}(\boldsymbol{\theta}_i^h) + [Z]\Delta\boldsymbol{\theta}_i^h) = \mathbf{0},$$
to determine the next, $(i+1)$-th, approximation $\boldsymbol{\theta}_{i+1}^h$:
$$\boldsymbol{\theta}_{i+1}^h = \boldsymbol{\theta}_i^h + \Delta\boldsymbol{\theta}_i^h \quad \text{where} \quad ([K] + \lambda[Z])\Delta\boldsymbol{\theta}_i^h = -([K]\boldsymbol{\theta}_i^h + \lambda \boldsymbol{f}(\boldsymbol{\theta}_i^h));$$
here, $[Z]$ is the diagonal matrix Diag$(\cos(\theta_j))$. If there exist multiple solutions, the choice of the initial approximation $\boldsymbol{\theta}_0^h$ will determine the one to which the Newton-Raphson method will converge.

- In a computation under "**load control**," one repeats the aforementioned procedure for a sequence of increasing values of λ. To "follow" a particular "branch" of nontrivial solutions, one can systematically choose as initial approximation to the solution for a subsequent value of λ the final approximation to the solution obtained for the previous value of λ.

- If the aforementioned procedure is carried out for a value of λ **slightly larger than one of the eigenvalues** λ_k^h, then one can make the Newton-Raphson converge to a solution on the **corresponding "branch" of nontrivial solutions** by using as **initial approximation** $\boldsymbol{\theta}_0^h$ **a multiple** $\zeta\boldsymbol{\phi}_k^h$ **of the corresponding eigenvector** $\boldsymbol{\phi}_k^h$, where ζ is an "appropriately" chosen constant.

Assignment

- As part 1 of 3 of the project, you are invited to implement the aforementioned finite-difference method and carry out a computation under "load control" that follows the "branch" of nontrivial solutions associated with the smallest magnitude eigenvalue of the linearized problem.

  Please exploit the sparsity structure: `sparse`, `eigs`, `spdiags`, `spy`, `speye`,...

  To solve linear systems, please do not compute the system-matrix inverse, but use the backslash operator (`help \`).

- Please include in your report:
  - a figure analogous to the one on Slide 14,
  - figures that show the solution obtained for several values of λ,
  - a description of how you proceeded to choose the grid spacing h, the number of iterations in the Newton-Raphson method, the value of ζ, and other parameters,
  - a description of the steps that you took to make sure that your results are correct,
  - ...

- If you need some help, Marco Lucio (marcolucio.cerquaglia@ulg.ac.be), Kavita (goyalkavita9@gmail.com), and Maarten Arnst (maarten.arnst@ulg.ac.be) are at your disposal.

References

References consulted to prepare this lecture

- S.-N. Chow and H. Hale. *Methods of bifurcation theory*. Springer, 1982.
- W. Day, A. Karkowski, and G. Papanicolaou. Buckling of randomly imperfect beams. *Acta Applicandae Mathematicae*, 17:269–186, 1989.
- H. Kielhöfer. *Bifurcation theory: An introduction with applications to partial differential equations.* Springer, 2012.
- S. Krantz and H. Parks. *The implicit function theorem*. Springer, 2013.
- R. Leveque. *Finite difference methods for ordinary and partial differential equations*. SIAM, 2007.
- S. Strogatz. *Nonlinear dynamics and chaos*. Perseus Books, 1994.
- W. Wagner and P. Wriggers. A simple method for the calculation of post critical branches. *Engineering Computations*, 5:103–109, 1988.