

A Markov chain model of power indices in corporate structures

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Abstract

This paper proposes to use a game-theoretic framework in analyzing complex corporate networks, notably in measuring the “amount of control” of both direct and indirect shareholders. The values of the indices are defined by complex voting games, composed by interlocked weighted majority games. This paper proposes a characterization of corporate networks in which the notion of “control” can be well defined, as well as an algorithm that consistently estimates the power indices when it is the case.

Keywords: finance, corporate network, game theory

1 Introduction

Economists have been studying the problem of corporate control in several ways. While the structure within corporations, that is, the interaction between shareholders and managers, has been analyzed by contract theorists with principal-agent models, the inter-corporate structure has yet to attract more academic attention. When analyzing the inner structure of a particular corporation, it might seem that whether a shareholder is a individual investor or a corporation is of little importance. Yet the fact is that in complex corporate networks, the two situations can have completely divergent implications.

In its apparent form, the problem of shareholders’ decision can be modeled as a majority voting game. Hence, the notion of control in the game theory can be borrowed to define the “control power” of shareholders (cf. [21]). Notice that the distribution of actual “control power” is often different from the nominal distribution of voting weights, mainly because the voting function is, in its essence, a non-linear function. A trivial example is the case in which there is a dominant voter whose voting weight exceeds the half of the total weight: this voter has absolute voting power, although he might only possess 51% of the total weight.

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Power indices were therefore invented to measure voters' actual power to change the outcome of the vote. The definition of "Shapley value", given by Shapley and Shubik (1954) [21] relies on the concept of cooperative game. Banzhaf (1965) [2] defines the "control power" of a voter as the probability that his or her vote is decisive, i.e. the outcome of the vote is directly determined by his or her vote, assuming that every voter votes "yes" or "no" with equal probability. In fact, both power indices define similar probabilities and their difference remains only in the presumptions: Shapley and Shubik assume the probabilities of having any number of voters voting "yes" are equal, while Banzhaf assumes any individual voter votes independently and randomly "yes" or "no" (with equal probabilities). The literature has also developed several methods for computing power indices. Mann and Shapley (1960) [12] propose using Monte-carlo simulation methods for computing Shapley indices. Mann and Shapley (1962) put forward an alternative computing method using generating functions. Owen (1972, 1975) [19] [20] proposes methods for both exact calculation (by using multilinear extension of a game) and approximation. Leech (2003) [15] combines these methods and presents another approximate method, with a parameter to deal with the trade-off between precision and complexity.

Leech (2002) [14] computes both power indices for large voting bodies in a cross section of British companies and appraises them according to some reasonable criteria. He argues that "the Banzhaf index much better reflects the variations in the power of shareholders between companies as the weights of shareholder blocks vary". It is partly for this reason that the present paper, as well as many others in the literature of corporate vote analysis, adopts Banzhaf indices rather than Shapley values.

However, although the literature has proposed many methods for the computation of power indices for shareholders of one special corporation, little has been said on financial systems which include, by definition, corporations whose shares are possessed by other corporations or dispersed in the hands of public investors. The literature distinguishes two systems of governance in financial market: the Anglo-Saxon outsider system in which corporations rely mainly on public debt and equity market and the majority of their shareholders are individuals (and some investment funds); the insider system in which bank financing, as well as the direct investment of non-financial corporations in shares, plays a more prominent role.[6]

The following figures could help reveal the different features of the two systems. We use graphic tools to show corporate control structure: corporations and investors are represented by vertices, while share ownership is represented by weighted edges. The outsider system is represented by a simple one-layer graph (Figure 1), while the insider system could be as complex as Figure 2.

Gambarelli and Owen (1994) analyze complex corporate networks and introduce the notion of consistent reduction and propose the method of multilinear extension to calculate it. However, this notion of consistent reduction could only solve problems in which there are well-defined investors,

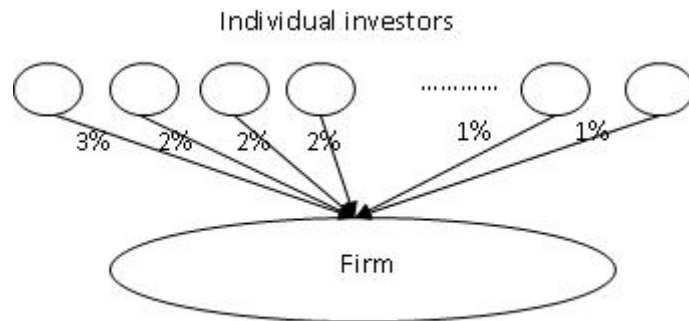


Figure 1: The outsider system: dispersed control

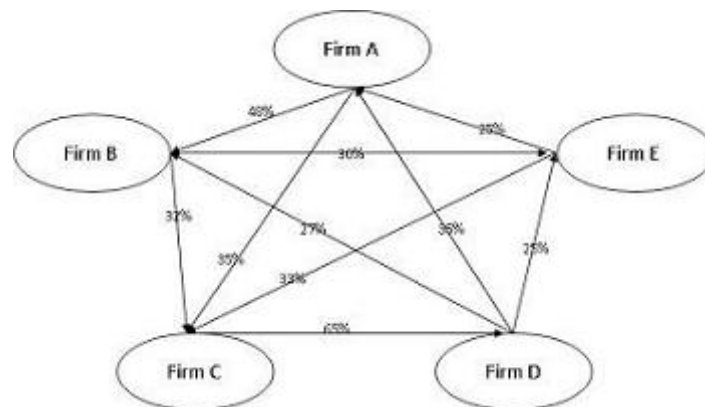


Figure 2: The insider system: mutual control

i.e., firms that can be considered as their own shareholders thus free to make their decisions. But in more complex corporate networks, we might want to tell the “controlling power” of some dependent firms. In addition, they present no algorithms with reasonable complexity to calculate the “consistent reductions” and the passage by multilinear extensions seem to be hard to realize. Hu and Shapley (2003) [10] [11] formulate a notion of equilibrium authority distribution that has much similarity with the invariant measure of a Markov chain. They interpret the authority distribution as some form of individuals’ long-run influence in the network. Yet some of their presumptions seem difficultly applicable to the reality of financial markets. In particular, they implicitly assume the independence of firms when they operate additions and multiplications over probability matrices. Crama and Leruth (2005) [5] make a natural extension of Banzhaf power index in simple voting games to the case of corporate network, and propose a Monte-Carlo simulation algorithm to compute it; yet a formal definition of the power index is missing in this article to concretize their idea and make it more than heuristic.

The purpose of the present paper is to analyze corporate networks in the most general form, define conditions under which Banzhaf power indices can be naturally generalized, as well as to describe an algorithm of estimation.

2 Basic notions

2.1 Game and Graph

We follow some of the terminology and notations used in [5] to describe our formal model of corporate networks. Assume that there are N firms¹; we denote by $V = \{0, 1, \dots, N - 1\}$ the set of firms. For any firm $i \in V$ and another firm $j \in V$, denote by S_{ij} the percentage of firm i ’s shares owned by firm j . S is therefore the shareholding matrix. We have:

$$\forall i, \sum_{j \in V} S_{ij} \leq 1$$

which says that the sum of shares owned by the shareholders of firm i that appear in the network should be less than 1. The equality holds if all shareholders of firm i are included in V . If the equality holds for all banks, that is, if all shareholders of all banks appear in the network, the network is said to be *complete*. The network is *incomplete* if the inequality is strict for some firms.

A firm i is said to be *autonomous* if it satisfies $S_{ii} = 1$. This is in fact a tricky way to represent independent investors as “fictitious firms” whose only shareholders (thus decision makers) are themselves. The model does not need to exclude non-autonomous firms from being their own shareholders,

¹we do not distinguish firms from individual investors: the latter can be regarded as firms who possess all of their own shares

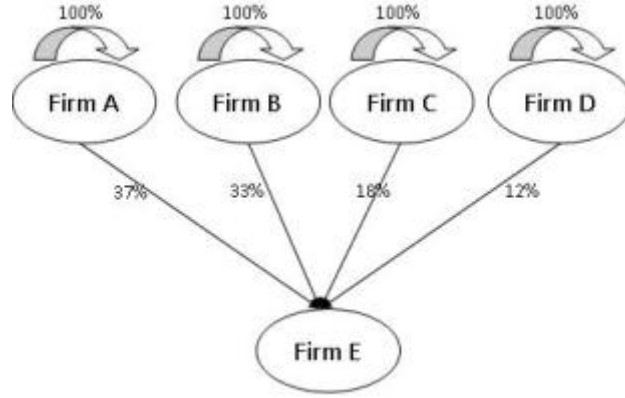


Figure 3: Example of a simple corporate graph

even if we consider this situation is unrealistic.

We use a directed and weighted graph to represent the network.² Every firm is represented by a vertex. For the sake of convenience, we still note the set of vertices by V . There is an edge from firm j to firm i if $S_{ij} > 0$, that is, if j is among i 's shareholders. The edge is associated with the positive weight S_{ij} . Notice that there could be edges going from one firm to itself: this is especially true for autonomous firms. Denote by A the set of edges, and by w the function from A to positive real numbers that associate an edge with its weight. When $(i, j) \in A$, we say i is a *predecessor* (or direct shareholder) of j . $G = (V, A, w)$, therefore, is a complete graph-theoretic description of the network and S is the adjacency matrix of G . The graph G is said to be *complete* if the associated corporate network is complete.

Example 2.1. Figure 3 is the graphic representation of a simple corporate network: there are five firms in this network, four of which, firms A, B, C and D, are autonomous, while they own respectively 37%, 33%, 18% and 12% of firm E's shares. The correspondent shareholding matrix is:

$$S = \begin{pmatrix} & A & B & C & D & E \\ A & 1 & 0 & 0 & 0 & 0 \\ B & 0 & 1 & 0 & 0 & 0 \\ C & 0 & 0 & 1 & 0 & 0 \\ D & 0 & 0 & 0 & 1 & 0 \\ E & 0.37 & 0.33 & 0.18 & 0.12 & 0 \end{pmatrix} \quad (1)$$

²We use the terminology from Bang-Jensen and Gutin [1], for instance

2.2 Banzhaf power index

Consider a simple majority voting game: let i be a firm whose shareholders are all included in G . Denote the set of firm i 's shareholders by Sh_i and its size by n . Let $T \subseteq Sh_i$ be a sub-set of i 's shareholders, define $v_i(T)$ as the characteristic function to judge whether T is a winning coalition for the voting game of firm i , that is, $v_i(T) = 1$ if and only if $\sum_{j \in T} S_{ij} \geq 0.5$ and $v_i(T) = 0$ in the opposite case. The (*non-normalized*) *Banzhaf index* of any of i 's shareholder $j \in Sh_i$ with respect to i is the quantity:

$$Z_i(j) = \frac{1}{2^{n-1}} \sum_{T \subseteq (Sh_i \setminus \{j\})} (v_i(T \cup \{j\}) - v_i(T)) \quad (2)$$

The Banzhaf index of voter j can be usefully interpreted as the probability that j can change the outcome of the vote by changing his own vote from 1 to 0, assuming that all voters vote randomly and are equally likely to vote either 0 or 1. We use the Banzhaf index as a measure of the amount of *a priori* voting power, or control, held by a firm.

Example 2.2. We retake Example 2.1 used in the last paragraph and calculate the Banzhaf power indices of firm A with respect to firm E :

$$\begin{aligned} Z_E(A) &= \frac{1}{8} (v_E(\{A\}) - v_E(\emptyset) + v_E(\{A, B\}) - v_E(\{B\}) + \dots + v_E(\{A, B, C, D\}) - v_E(\{B, C, D\})) \\ &= 0.5 \end{aligned}$$

Similarly, we could compute $Z_E(B) = Z_E(C) = 0.5$ and $Z_E(D) = 0$. This example shows that the control power of shareholders could be considerably different from the distribution of their shares: firm A possesses twice firm E 's shares as firm C does, yet they have exactly the same control power; on the other hand, firm D possesses a non-negligible amount of 12% of E 's shares which gives it no power at all.

2.3 Direct extension to acyclic complete graphs

Let $G = (V, A, w)$ be a corporate network. A *walk* is a sequence of vertices (i_1, i_2, \dots, i_k) such that $(i_r, i_{r+1}) \in A$ for $r = 1, \dots, k-1$. A *cycle* is a walk such that $i_1 = i_k$.³

In order to generalize the notion of Banzhaf power indices to more complex networks, let us first examine the case of acyclic (without cycles) and complete graphs. Let G be such a graph. G must take a pyramidal form similar to Figure 4 (weights on the edges are omitted from the graph).

In such a graph, we define the *layer* of a firm by induction: the autonomous firms are of layer 1; the layer of any other firm is defined to be one plus the maximum layer of its predecessors. The

³We do not allow a path or a cycle to include any edge of type (i, i) even if $(i, i) \in A$.

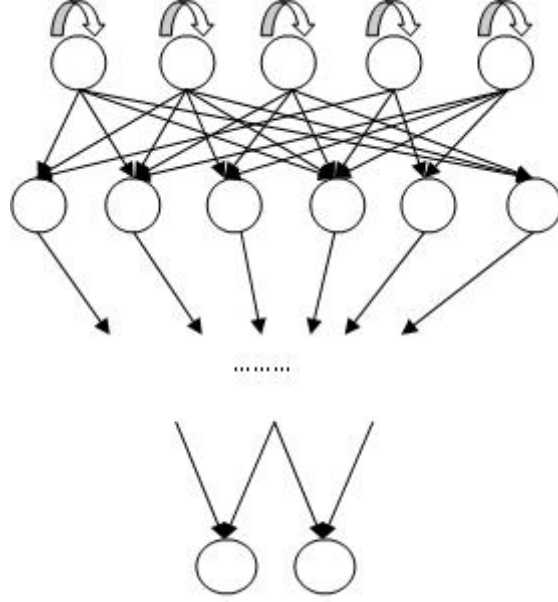


Figure 4: Pyramidal form of an acyclic complete graph

outcome of the vote of a firm in layer k is thus completely determined by the votes of firms in the first $k - 1$ layers. Hence, all firm's votes are utterly determined by autonomous firms which are situated at the top of the pyramid.

Denote by M the set of autonomous firms in G and its size by n . Notice that since votes of firms in $V \setminus M$ are ultimately determined by votes of firms in M , only firms in M should have control power over other firms. Let $t \in V$ be a target firm and $T \subseteq M$ be a sub-set of autonomous firms. Define $v_t(T)$ as the outcome of firm t 's vote if all autonomous firms in T vote 1 and other autonomous firms vote 0. Let $j \in M$ be an autonomous firm, a natural extension of the Banzhaf power index of firm j with respect to a target firm t can be given as:

$$Z_t(j) = \frac{1}{2^{n-1}} \sum_{T \subseteq (M \setminus \{j\})} (v_t(T \cup \{j\}) - v_t(T)) \quad (3)$$

The index can be interpreted as the probability that j can change the “ultimate” outcome of t 's vote by changing his own vote from 1 to 0, assuming that all other autonomous voters vote randomly and are equally likely to vote either 0 or 1. In this case, the non-autonomous firms are of no control power.

2.4 An alternative definition of the Banzhaf power index

In order to describe the results of the present paper in cyclic or incomplete corporate networks, it is useful to consider an alternative definition of the Banzhaf power index in the two cases mentioned above. In fact, the Banzhaf power index of a firm j with respect to another firm t can also be regarded as the probability that j can change the ultimate outcome of t 's vote by changing his own vote, assuming that **all** voters start by voting randomly and equally likely either 0 or 1 and all firms are asked to update (simultaneously) their votes according to their direct predecessors until no further change of vote occurs.

In the acyclic and complete case (the one-layer graph is a special case of this), since the autonomous firms are considered as their own shareholders, their votes should never change in the update process. Therefore, the votes of firms on the first layer should at most be changed once and accordingly, votes of firms on the k -th layer should be changed no more than k times. The update process thus stops after a finite number of times.

Mathematically, let $e \in \{0, 1\}^V$ be firms' votes (for any firm $i \in V$, e_i represents i 's vote). Let t be a target firm and $u_t(e)$ be firm t 's final vote after update processes starting from the state e . An alternative definition of the Banzhaf power index of a firm j with respect to t can be given as: ⁴

$$\begin{aligned} Z_t(j) &= \frac{1}{2^{|V|-1}} \sum_{e \in \{0,1\}^V} \left(\sum_{e_j=1} u_t(e) - \sum_{e_j=0} u_t(e) \right) \\ &= \mathbb{E}[u_t(e)|e_j = 1] - \mathbb{E}[u_t(e)|e_j = 0] \end{aligned} \quad (4)$$

which says, basically, that since non-autonomous firms are of no importance in deciding the results of the voting games, the index does not change if the probability is calculated by assuming that all firms, instead of autonomous firms only, vote randomly.

3 Incomplete networks

Until now, we have assumed the graph is complete. However, real world data in general do have missing shareholder information: financial authorities often do not require small shareholders to report their status. In our model, we define the *float* of a firm i as the quantity:

$$fl_i = 1 - \sum_{j \in V} S_{ij}$$

which is the total share of unidentified shareholders of the firm.

Consider a one-layer graph with float. To correctly define the voting game, one needs apparently specify the voting behavior of the float. Two limiting cases are described in Cubbin and Leech (1983)

⁴ $\mathbb{E}(X)$ stands for the expected value of X . The proof of the equivalence of the two definitions is left to readers.

[7] and Leech (1988 [16], 2002a [14]). In what they call the “*concentrated float*” case, the number of unidentified shareholders is assumed to be as small as possible: since the shares held by each of them must be no larger than the smallest identified shareholder’s portion, denoted by \underline{S}_i , the number of unidentified shareholders of firm i , $n_i^{unidentified}$, should satisfy:

$$n_i^{unidentified} \geq \lceil \frac{fl_i}{\underline{S}_i} \rceil$$

where $\lceil a \rceil$ means the smallest integer larger than or equal to a .

On the contrary, an “*oceanic float*” case is described in Dubbey and Shapley (1979) [8]. An infinite number of unknown shareholders is assumed in this case and each of them holds only an infinitesimal fraction of the shares.

These two models can be regarded as two extreme situations of the float: highly concentrated or largely dispersed. In the light of these models, float or unidentified shareholders can be dealt with in the same way as recognized shareholders are, that is, float can be considered as “fictitious shareholders” that follow the same law as real shareholders. Yet the present paper follows a more general method that avoids the discussion of the modeling of float: let $i \in V$ be a firm with positive float $fl_i > 0$. Let Fl^i be any random variable that takes value in $[0, fl_i]$ with cumulative distribution function F^i and density function f^i . The random variable represents the total weight of unidentified shareholders that vote 1. Thus, if we denote by n the number of identified shareholders of firm i and let j be one of them. Let v_i be the function that associates $T \subseteq Sh_i$, a sub-set of the set of i ’s identified shareholders, with the total shares of shareholders in T . The Banzhaf power index of j with respect to i (for the one-layer situation, with float) can be defined as:⁵

$$\begin{aligned} Z_i(j) &= \frac{1}{2^{n-1}} \sum_{T \subseteq (Sh_i \setminus \{j\})} (\mathbb{P}(Fl^i + v_i(T \cup \{j\}) \geq 0.5) - \mathbb{P}(Fl^i + v_i(T) \geq 0.5)) \\ &= \frac{1}{2^{n-1}} \sum_{T \subseteq (Sh_i \setminus \{j\})} (F^i(0.5 - v_i(T)) - F^i(0.5 - v_i(T \cup \{j\}))) \\ &= \frac{1}{2^{n-1}} \sum_{T \subseteq (Sh_i \setminus \{j\})} \int_{0.5 - v_i(T \cup \{j\})}^{0.5 - v_i(T)} f^i(x) dx \end{aligned} \quad (5)$$

which can be interpreted as the difference of probability that j can change the outcome of a vote by changing his own vote from 1 to 0, assuming that all identified voters vote randomly and are equally likely to vote either 0 or 1, and unidentified voters vote according to the cumulative distribution function F^i .

⁵ $\mathbb{P}(A)$ stands for the probability that A happens.

4 The general model

4.1 Transition matrix

In this section, we intend to solve the problem in its most general form.

The presence of float and cycles reveals the probabilistic aspect of the problem: the Banzhaf power indices have to be defined in a way that combines both the problem of vote updates (cf. Equation 4) and the problem of having random float (cf. Equation 5).

Since a firm can either vote 0 or 1, denote by $E = \{0, 1\}^V$ the space of all possible votes. If e is an element in E , $e(i) \in \{0, 1\}$ describes the vote of firm i for any $i \in V$.

Let $e \in E$ be the “current” state of firms’ votes. If the boards of directors of each firm re-vote again according to shareholders’ current votes, as described by e , the presence of floats could give rise to several possible outcomes (of elements in E), thanks to the existence of floats. Let $f \in E$ be one possible outcome of votes. If we assume furthermore the cumulative distribution functions of floats are known (whatever those distribution functions are), it then follows that we are able to calculate $P(e, f)$, the probability that the outcome of re-votes is f , given the current votes e .

Notice that the huge matrix P , which is of size $2^N * 2^N$ ($N = |V|$), is a transition matrix. That is, $\forall e, f \in E$, $P(e, f) \geq 0$; and $\forall e \in E$, $\sum_{f \in E} P(e, f) = 1$.

For the sake of simplicity, we should label the N corporations from 0 to $N - 1$. Then arrange the 2^N elements of E in a binary way: represent each of the elements by a binary number of exactly N bytes (bytes are also counted from the 0-th to $N - 1$ -th); the k -th binary of an element $e \in E$, denoted by e_k , should represent the vote of the k -th corporation; sort the binary numbers in the increasing order (from 0...0 to 1...1). The matrix P in this basis can therefore be calculated according to the shareholding matrix S . $\forall 0 \leq k, l \leq 2^N - 1$, let a and b be the binary number that equals to k and l . If we take the further assumption that says the floats of different firms vote independently⁶, we have:

$$\begin{aligned}
 P(a, b) &= P_{k, l} \\
 &= \prod_{i=0}^{N-1} \text{Proba}(b_i \text{ be firm } i\text{'s next vote, knowing firms' current votes } a) \\
 &= \prod_{0 \leq i \leq N-1, b_i=1} \text{Proba} \left(Fl^i + \sum_{j=0}^{N-1} a_j s_{ij} \geq 0.5 \right) \prod_{0 \leq i \leq N-1, b_i=0} \text{Proba} \left(Fl^i + \sum_{j=0}^{N-1} a_j s_{ij} < 0.5 \right) \\
 &= \prod_{0 \leq i \leq N-1, b_i=1} \left(1 - F^i \left(0.5 - \sum_{j=0}^{N-1} a_j s_{ij} \right) \right) \prod_{0 \leq i \leq N-1, b_i=0} F^i \left(0.5 - \sum_{j=0}^{N-1} a_j s_{ij} \right)
 \end{aligned}$$

⁶independence is of course a key hypothesis here to give out the formula, yet it is not required to define the transition matrix

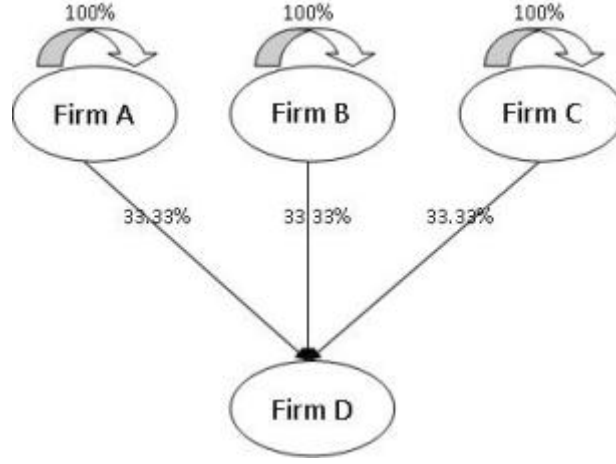


Figure 5: Example 4.1

The following examples illustrate how the shareholding matrix, S , can be transformed into the transition matrix, P .

Example 4.1. Consider 4 firms A,B,C and D (Figure 5). Firms A, B and C are autonomous, while each of them possesses one third of firm D's shares. Their shareholding matrix is as following:

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix} \quad (6)$$

An element of the outcome space can be represented by an binary number $x_a x_c x_b x_d$. For instance, the number 0011 signifies the situation in which both firms A and B vote 1, while firms C and D vote the opposite.

Since there is no float in these firms' shares, we do not need to specify a distribution function of float to give the transition matrix P , which is of size $16 * 16$. In reality, most of the terms in this matrix are zero except exactly 16 of them: the absence of floats implies that the votes are deterministic, which is to say that $\forall i$, there exists an only j such that $P_{ij} \neq 0$. Yet P is a transition matrix, which implies that we have not only $P_{ij} > 0$ for this j but also $P_{ij} = 1$.

Example 4.2. (with float) Consider 3 firms A, B and C (Figure 6). Now firms A and B are autonomous, while both of them controls one third of firm C's shares. The last third of firm C's shares is recognized as float. Therefore, their shareholding matrix is:

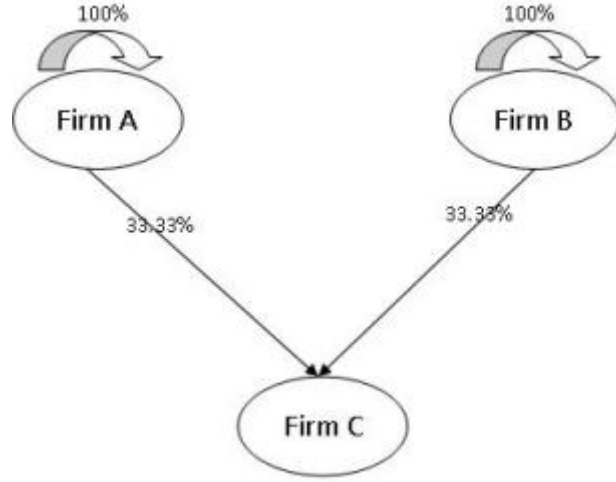


Figure 6: Example 4.2

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix} \quad (7)$$

If we assume that the float of C votes according to a uniform distribution over $[0, \frac{1}{3}]$, their transition matrix is:

$$P = \begin{pmatrix} & 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\ 000 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 001 & 0 & 0.5 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 010 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0.5 & 0 \\ 011 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 100 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 101 & 0 & 0.5 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 110 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0.5 & 0 \\ 111 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (8)$$

Above is a way to build up the transition matrix from a corporate network. However, the existence of transition matrix does not necessarily require a structure of corporate network as what is defined previously, i.e., with well-defined shares and floats. An alternative approach is to define a probabilistic simple game for any firm $t \in V$. The players of this game are all the other firms in the network. Let $T \subset V \setminus \{t\}$ be a sub-set of players, denote by $v_t(T)$ the outcome of the voting game if only players in T vote 1. We define in addition the probabilities $p_t(T) = \mathbb{P}[v_t(T) = 1]$ (the probability that firm t votes 1 given its shareholders vote 1 if and only if they are elements of T).

The transition matrix can then be calculated if, for instance, we assume that the probabilistic simple games are independent one from another.⁷

The following analysis is only based on the existence of such a transition matrix, P , and does not need to specify the origin of this matrix.

4.2 Decomposition into irreducible classes

For any two possible outcomes $e, f \in E$, we shall say e leads to f if $P^k(e, f) > 0$ for some positive integer $k > 0$. We denote this $e \rightarrow f$. If e leads to f and f leads to e , we note $e \leftrightarrow f$. If for any positive integer k , $P^k(e, f) = 0$, we note $e \nrightarrow f$. Notice that although \leftrightarrow is both transitive and symmetric, it is not an equivalence relation because it is not reflexive, which is to say, it does not hold in general that $\forall e \in E, e \leftrightarrow e$.

It is then a well-known property for transition matrices that the elements of E can be classified, in a unique manner, into non-overlapping groups T, C_1, C_2, \dots, C_k such that each C_i is an equivalence class of the relation \leftrightarrow (which means, $\forall e, f \in C_i, e \leftrightarrow f$), said *irreducible classes* of P and $T = \{e \in E | e \rightarrow f \text{ but } f \nrightarrow e \text{ for some } f\}$.⁸ Therefore, P has the form (after some possible re-arrangements) of:

$$P = \begin{bmatrix} P_T & W_1 & \dots & W_k \\ 0 & P_{C_1} & 0 & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & P_{C_k} \end{bmatrix}$$

where P_C is the restriction of P to any sub-set $C \subseteq E$. Furthermore, P_T is such that $\lim_{m \rightarrow \infty} P_T^m = 0$ and k is larger than 1 (there exists at least one equivalence class of \leftrightarrow).

Example 4.3. (with cycles) In Example 4.2, the transition matrix can be re-arranged as:

$$P = \begin{pmatrix} & 011 & 100 & 001 & 101 & 010 & 110 & 000 & 111 \\ 011 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 100 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 001 & 0 & 0 & 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 101 & 0 & 0 & 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 010 & 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 & 0 \\ 110 & 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 & 0 \\ 000 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 111 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

⁷Again, the independence is only one possible sufficient condition for the result.

⁸The readers could refer to [18] for a more thorough description of the theory.

In this example, $T = \{011, 100\}$, $C_1 = \{001, 101\}$, $C_2 = \{010, 110\}$, $C_3 = \{000\}$, $C_4 = \{111\}$.

Notice that in the special case in which there is no float and no cycle in the corporate network, then the transition matrix should have the form of

$$P = \begin{bmatrix} 0 & W \\ 0 & I \end{bmatrix}$$

where I stands for the Identity matrix.

4.3 Formal definition of Banzhaf power index

We have previously seen that, in an acyclic graph without float, the Banzhaf power index of a firm X with respect to a target firm T is obtained by:

- assuming that each of the 2^N votes states is equi-probable to be the starting state
- with a given starting state, calculating the influence “in the long run” on the vote of T if X changes its vote

The difficulty in generalizing this definition stands in how to formalize the notion of firm T ’s vote “in the long run” with a given starting state.

The answer is to first calculate, if they exist, the limits of probabilities of each state “in the long-run”. Inspired from the arborescent case, we will let all firms update again and again their votes, expecting that these votes will converge in a probabilistic sense, that is, the probabilities that each possible outcome happens converge as firms vote and re-vote. It will then be easy to get the expected value of firm T ’s long-term vote by taking the weighted average of its vote in all possible states.

Let $e \in E$ be an arbitrary starting state of votes. For any sub-set A of E and any integer $m \geq 1$, denote by $q(A, m, e)$ the probability that after m tours of re-votes, the state of votes is an element of A . If $\lim_{m \rightarrow \infty} q(\{f\}, m, e)$ exists for every pair $e, f \in E$, the previous discussion allows the extension of Banzhaf power index.

Notice first that it is easy to show that $\lim_{m \rightarrow \infty} q(C_i, m, e)$ exists for any of the irreducible classes C_i , because:

- if $e \in C_i$ for some i , then $\forall m$, $q(C_i, m, e) = 1$ and $q(C_j, m, e) = 0$ if $i \neq j$.

- if $e \in T$, then⁹:

$$\begin{aligned}
q(C_i, m+1, e) &= \sum_{f \in E} q(\{f\}, m, e) \left(\sum_{g \in C_i} P(f, g) \right) \\
&\geq \sum_{C_j} \sum_{f \in C_j} q(\{f\}, m, e) \left(\sum_{g \in C_i} P(f, g) \right) \\
&= \sum_{f \in C_i} q(\{f\}, m, e) \left(\sum_{g \in C_i} P(f, g) \right) \\
&= \sum_{f \in C_i} q(\{f\}, m, e) \\
&= q(C_i, m, e)
\end{aligned}$$

$(q(C_i, m, e))_{m \geq 0}$ is therefore an increasing sequence bounded by 1 and it converges necessarily.

Since P_T is such that $\lim_{m \rightarrow \infty} P_T^m = 0$, it is easy to show that $q(T, m, e)$ converges to 0 for any starting state e . We need the following assumption to conclude the existence of the limits.

Assumption 4.1. *Each equivalence class C_i of \leftrightarrow is aperiodic.*

Theorem 4.1. *Under Assumption 4.1, $\lim_{m \rightarrow \infty} q(\{f\}, m, e)$ exists for any pair $e, f \in E$.*

Proof. The restriction of the transition matrix to any of the equivalence classes of \leftrightarrow is a finite-state irreducible Markov chain. Under the assumption of aperiodicity, it is ergodic, which is to say $\lim_{m \rightarrow \infty} q(\{f\}, m, e)$ exists for any pair of $e, f \in C_i$. For a given $f \in C_i$, the limit is also called the stationary distribution of the irreducible Markov chain, denoted by π_f .

If $f \in C_i$ for some i but $e \in C_j$ with $j \neq i$, then it is clear that $\lim_{m \rightarrow \infty} q(\{f\}, m, e) = 0$.¹⁰

If $f \in C_i$ for some i and $e \in T$, we then have: $\lim_{m \rightarrow \infty} q(\{f\}, m, e) = \pi_f * \lim_{m \rightarrow \infty} q(C_i, m, e)$.

Lastly, if $f \in T$, then $\lim_{m \rightarrow \infty} q(\{f\}, m, e) = 0$ for any $e \in E$. \square

Definition 4.1. *Under the Assumption 4.1, the Banzhaf index of a firm X to a target firm T can be defined as the difference between the expected value of firm t 's votes "in the long run" when firm x changes its vote from 0 to 1. Mathematically, we have*

$$Z_T(X) = \frac{1}{2^{N-1}} \left(\sum_{e \in E, e(X)=1} \sum_{f \in E} f(T) \lim_{m \rightarrow \infty} q(\{f\}, m, e) - \sum_{e \in E, e(X)=0} \sum_{f \in E} f(T) \lim_{m \rightarrow \infty} q(\{f\}, m, e) \right)$$

Corollary 4.1. *In an acyclic corporate network (with float or not), the Banzhaf index is well defined.*

⁹Let $f \in C_i$, $\sum_{g \in C_i} P(f, g) = 1$ by definition of irreducible classes. $P(f, g) = 0$ if $f \in C_i, g \in C_j$ with $i \neq j$.

¹⁰by definition of irreducible classes.

Proof. It suffices to show that any of the equivalence classes in an acyclic corporate network is aperiodic. Let C be one equivalence class in such a network and assume that C is not aperiodic. It implies that for any element $e \in C$, $P(e, e) = 0$.

We use the notion of *layer* defined previously, and for each firm x in the network, denote its layer by $l(x)$.

For any element $e \in C$, let $A(e) = \{f \in C | P(e, f) > 0\}$ be the set of probable outcomes if we let the firms re-vote. If $f \in A(e)$, define $\phi(f, e) = \min(l(x) | \text{firm } x \text{ votes differently in } e \text{ and in } f)$. $\phi(f, e)$ is well defined since $P(e, e) = 0$. Define then $\varphi(e) = \max(\phi(f, e), f \in A(e))$.

Now let $e^* \in C$ be the element that maximizes $\varphi(e)$ and let f^* be the element in $A(e^*)$ that maximizes $\phi(f, e^*)$. Let x^* be the firm with minimum depth that votes differently in e^* and in f^* (if there are several such firms, we choose one of them). Since e^* and f^* are in the same equivalence class, firm x^* should not be that $l(x^*) = 0$ (autonomous firms never change their votes through updates). All firms whose layer is smaller (strictly) than $l(x^*)$, notably its shareholders, should vote the same in e^* as in f^* (by the definition of $\phi(f^*, e^*)$). The fact that $f^* \in A(e^*)$ shows that, if we let firms re-vote from situation f^* , there is a non-zero probability that all these firms whose layers are smaller than $l(x^*)$ will get the same votes, and so does the firm x^* . But this is true for all firms having the same layer with x^* . That is to say, there must be some $g \in A(f^*)$ such that $\phi(g, f^*) > l(x^*)$ (because it might be that none of the firms whose layer is weakly smaller than $l(x^*)$ changes its vote). Therefore, $\varphi(f^*) \geq \phi(g, f^*) > l(x^*) = \varphi(e^*)$ which contradicts the hypothesis that e^* maximizes φ . \square

Banzhaf power index is defined in, but not restricted to, acyclic graphs, as shown in the following example.

Example 4.4. Consider four firms A, B, C and D (Figure 7). Only firm A is autonomous. Firms B, C and D are in symmetric positions: firm B 's shares are controlled by firm A (33.33%) and firm D (33.33%), with a float of 33.33%; firm C 's shares are controlled by firm A (33.33%) and firm B (33.33%), with a float of 33.33%; similarly, firm D 's shares are controlled by firm A (33.33%) and firm C (33.33%), with the same amount of float.

The shareholding matrix, S , is as following:

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 \end{pmatrix} \quad (9)$$

If we represent a possible state of votes by the binary number $x_d x_c x_b x_a$ and assume that all floats follow uniform law, the transition matrix takes the following form (Figure 8), which could be

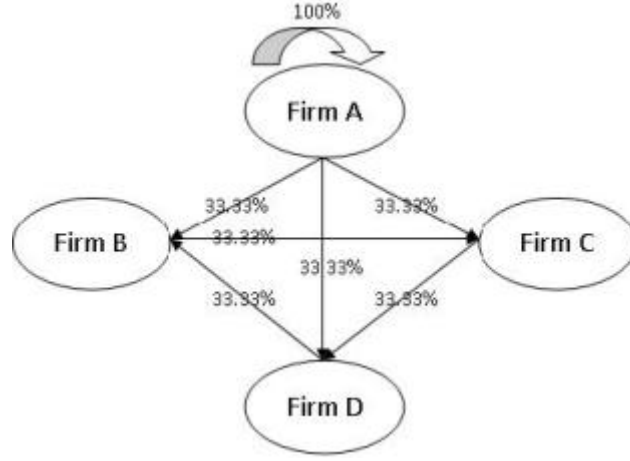


Figure 7: Example 4.4

	0000	0001	0010	0011	0100	0101	0110	0111	1000	1001	1010	1011	1100	1101	1110	1111
0000	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0001	0	1/8	0	1/8	0	1/8	0	1/8	0	1/8	0	1/8	0	1/8	0	1/8
0010	1/2	0	0	0	1/2	0	0	0	0	0	0	0	0	0	0	0
0011	0	0	0	0	0	1/4	0	1/4	0	0	0	0	0	1/4	0	1/4
0100	1/2	0	0	0	0	0	0	0	1/2	0	0	0	0	0	0	0
0101	0	0	0	0	0	0	0	0	0	1/4	0	1/4	0	1/4	0	1/4
0110	1/4	0	0	0	1/4	0	0	0	1/4	0	0	0	1/4	0	0	0
0111	0	0	0	0	0	0	0	0	0	0	0	0	0	1/2	0	1/2
1000	1/2	0	1/2	0	0	0	0	0	0	0	0	0	0	0	0	0
1001	0	0	0	1/4	0	0	0	1/4	0	0	0	1/4	0	0	0	1/4
1010	1/4	0	1/4	0	1/4	0	1/4	0	0	0	0	0	0	0	0	0
1011	0	0	0	0	0	0	0	1/2	0	0	0	0	0	0	0	1/2
1100	1/4	0	1/4	0	0	0	0	0	1/4	0	1/4	0	0	0	0	0
1101	0	0	0	0	1/2	0	0	0	0	0	0	0	0	0	0	1/2
1110	1/8	0	1/8	0	1/8	0	1/8	0	1/8	0	1/8	0	1/8	0	1/8	0
1111	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Figure 8: Transition matrix in Example 4.4

re-arranged into (Figure 9):

One can see from the re-arranged matrix that there are only two irreducible classes: $\{0000\}$ and $\{1111\}$. Other states of votes have a non-zero possibility of becoming either 0000 or 1111. Therefore, in the long-run, the state of votes converges in probability to 0000 or to 1111, depending on the initial vote of (only) firm A.

Hence, the Banzhaf power indices of firm A with respect to other firms are $Z_B(A) = Z_C(A) = Z_D(A) = 1$, and all other Banzhaf power indices are zero. Despite having only one third of shares of each of the other firms, firm A has total control over the group of firms B, C and D.

Remark 4.1. Cyclic corporate network could be periodic with period larger than 1. An easiest (but probably unrealistic) case consists of two firms A and B who possess more than 50% of each other's

	1110	0001	0010	0011	0100	0101	0110	0111	1000	1001	1010	1011	1100	1101	0000	1111
1110	1/2	0	1/2	0	1/2	0	1/2	0	1/2	0	1/2	0	1/2	0	1/2	0
0001	0	1/2	0	1/2	0	1/2	0	1/2	0	1/2	0	1/2	0	1/2	0	1/2
0010	0	0	0	0	1/2	0	0	0	0	0	0	0	0	0	1/2	0
0011	0	0	0	0	0	1/4	0	1/4	0	0	0	0	0	1/4	0	1/4
0100	0	0	0	0	0	0	0	0	1/2	0	0	0	0	0	1/2	0
0101	0	0	0	0	0	0	0	0	0	1/4	0	1/4	0	1/4	0	1/4
0110	0	0	0	0	1/4	0	0	0	1/4	0	0	0	1/4	0	1/4	0
0111	0	0	0	0	0	0	0	0	0	0	0	0	0	1/2	0	1/2
1000	0	0	1/2	0	0	0	0	0	0	0	0	0	0	0	1/2	0
1001	0	0	0	1/4	0	0	0	1/4	0	0	0	1/4	0	0	0	1/4
1010	0	0	1/4	0	1/4	0	1/4	0	0	0	0	0	0	0	1/4	0
1011	0	0	0	0	0	0	0	1/2	0	0	0	0	0	0	0	1/2
1100	0	0	1/4	0	0	0	0	0	1/4	0	1/4	0	0	0	1/4	0
1101	0	0	0	0	1/2	0	0	0	0	0	0	0	0	0	0	1/2
0000	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
1111	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Figure 9: Re-arranged transition matrix

shares. In this case, there are 3 equivalence classes in the transition matrix which are $\{00\}$, $\{11\}$ and $\{01, 10\}$. The last one is periodic with period 2.

Remark 4.2. The previous extension of Banzhaf power index does not allow us to define it in cases where there periodic equivalent classes. However, the author of the present paper considers it not as a flaw of the developed theory, but as a shortage of the voting process. In principle, we rely on the voting process to deal with any divergence of opinions among shareholders. However, in some extreme cases (which are supposed to be “rare” in real life) such as the one mentioned in the last paragraph, voting does not allow the two firms to solve their difference. This is part of the reason why we need negotiation and bargaining in real life, in addition to polls. The Banzhaf power index, as a measure of the influence of one participator in a voting game, does not and should not go out of the principles of voting and that is why it is not defined under some circumstances.

5 Algorithm for estimating Banzhaf power indices

Always under the Assumption 4.1, the definition of the Banzhaf Index of a firm X to a target firm T can be formulated in a slightly different way. If $e \in E$, let $\bar{e}(X)$ be the voting situation in which all firms but firm X vote the same as in e . The Banzhaf Index can then be rewritten as:

$$Z_T(X) = \mathbb{E} \left| \sum_{f \in E} f(T) \lim_{m \rightarrow \infty} q(\{f\}, m, e) - \sum_{f \in E} f(T) \lim_{m \rightarrow \infty} q(\{f\}, m, \bar{e}(X)) \right|$$

where \mathbb{E} stands for the expected value and e be a random variable uniformly distributed over E .

Therefore, when both n and m increase to infinity, the following algorithm generates a consistent sequence of estimators of $BZ(x, t)$:

- **(Counter)** $aux_0 \leftarrow 0; aux_1 \leftarrow 0$
 $num_0 \leftarrow 0; num_1 \leftarrow 0$
- **(Initialization)** Choose randomly an element e in E
- **(Propagation)** Starting from e , let the network vote and re-vote m times, let the final state be f
 $aux_{e(x)} \leftarrow aux_{e(x)} + f(t)$
 $num_{e(x)} \leftarrow num_{e(x)} + 1$
- **(Repetition)** Repeat **Initialization** and **Propagation** n times.
- $BZ(x, t) = \frac{aux_1}{num_1} - \frac{aux_0}{num_0}$

6 Conclusion

In this paper, we have extended the notion of Banzhaf power index to complex corporate networks. We have shown that under a technical condition, the well-defined Banzhaf power index is capable to measure the “control power” of one corporation with respect to another. It has also been shown that the condition is always satisfied in acyclic corporate networks. We have, in addition, proposed a Monte-Carlo simulation algorithm for the computation of the indices in case of existence. The approach is based on a formal game-theoretical framework.

Major questions for future research would be associated with the technical condition. It would be a significant improvement if we could characterize all the networks that satisfy (or not satisfy) it. It might also be interesting to consider, when the condition is not satisfied, the possibility to define other similar notion of control power.

The algorithm developed in the present paper could also be served to calculate Banzhaf indices in other similar situations, with probably slight modifications. For instance, we believe that this work could open the way for further application of power indices to the analysis of interaction among banks, say, to the measurement of dependency of one bank on the banking network.

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