Invariant games and non-homogeneous Beatty sequences

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joint work with
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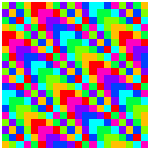


Combinatorial games

Wythoff's game defined by a set of moves (or rules)

- 2 players play alternatively,
- first player unable to move loses (normal condition),
- 2 piles of tokens:
  - remove a positive number of tokens from one pile,
  - remove the same positive number of tokens from both piles.

$$\mathcal{M}_W := \{(i,0) \mid i > 0\} \cup \{(0,j) \mid j > 0\} \cup \{(k,k) \mid k > 0\}.$$

Definition

A game is *invariant* if the same moves can be played from every position (the only restriction is that enough tokens are available).

Invariant take-away games are Golomb's vector subtraction games (1966).

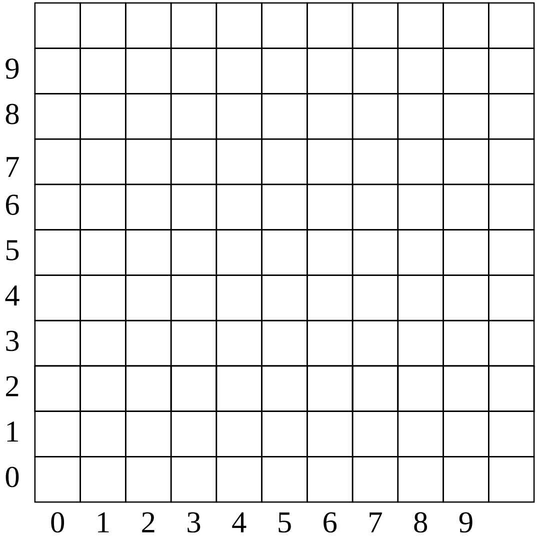
Variant rulesets (example)

- Remove an even number of tokens from one pile whenever the total number of tokens is even;
- remove an odd number of tokens from one pile, otherwise.

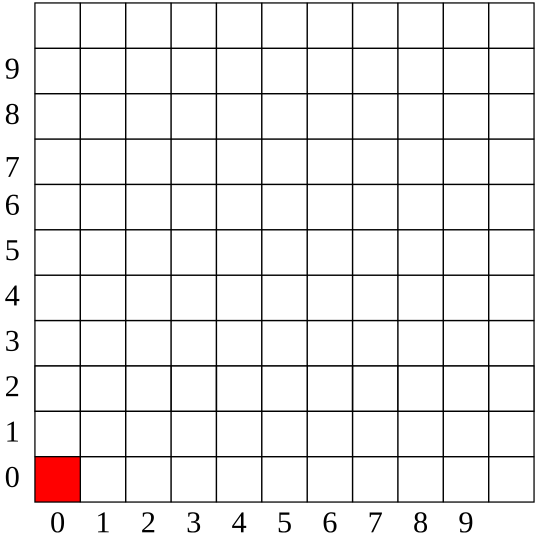
Other examples of games usually defined with a variant ruleset

- Rat and mouse game, Fraenkel
- Raleigh game, Fraenkel'07
- Tribonacci game, Duchêne-R.'08
- Pisot cubic games, Duchêne-R.'08
- Flora game, Fraenkel'10
- ...

Wythoff's game or "catching the queen"



Wythoff's game or "catching the queen"



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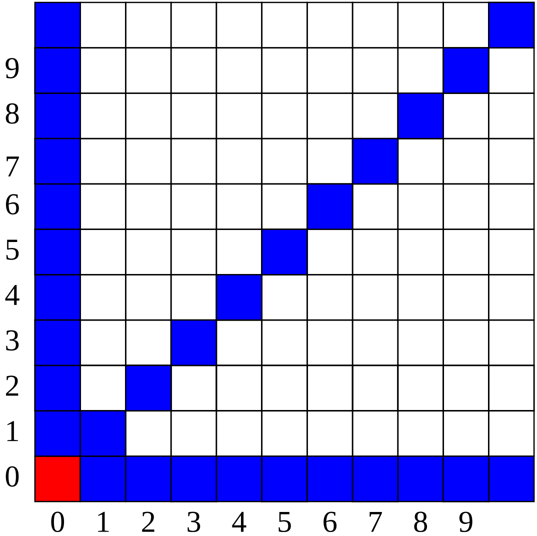

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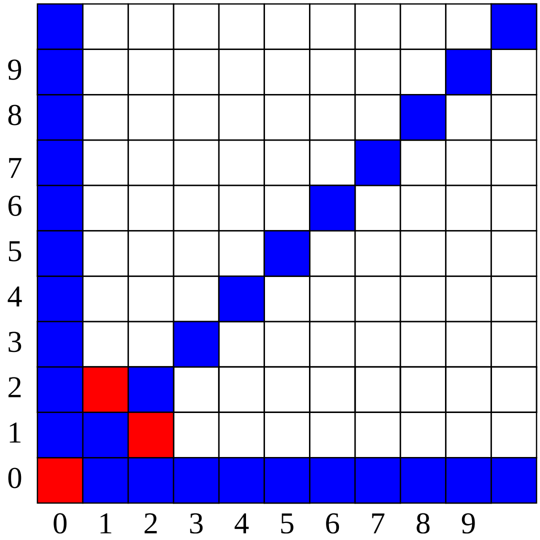

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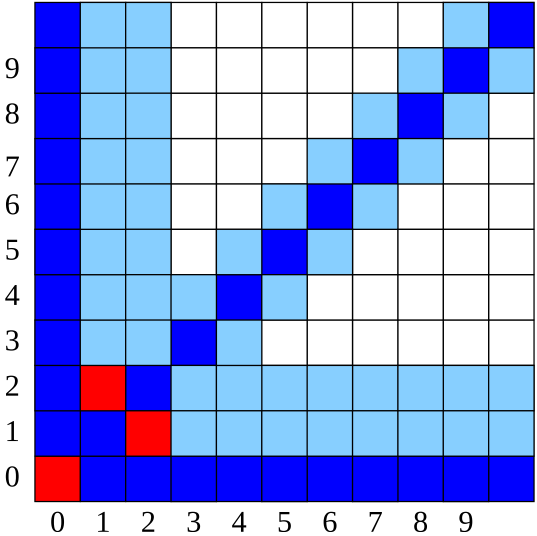

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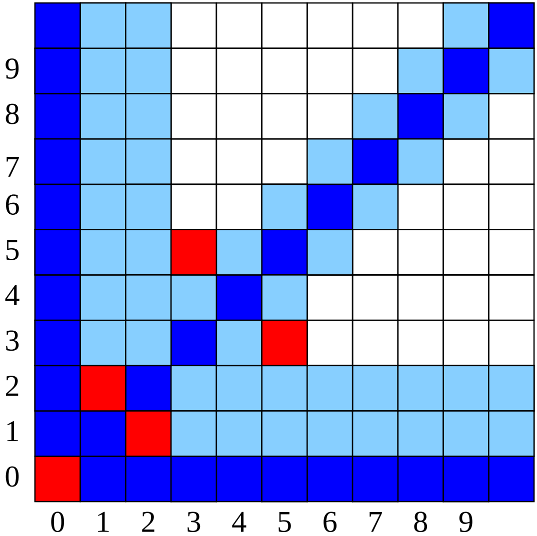

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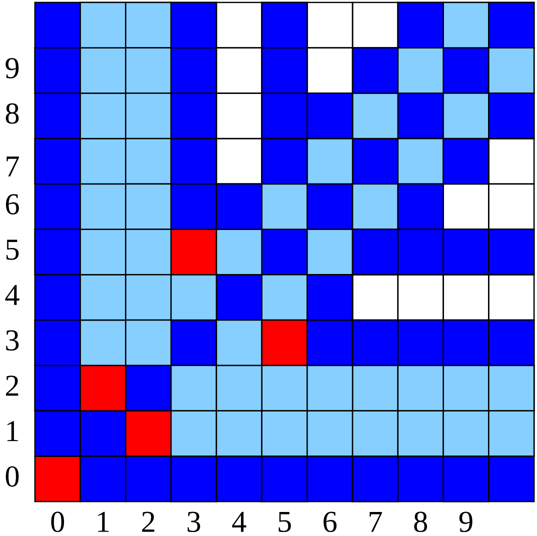

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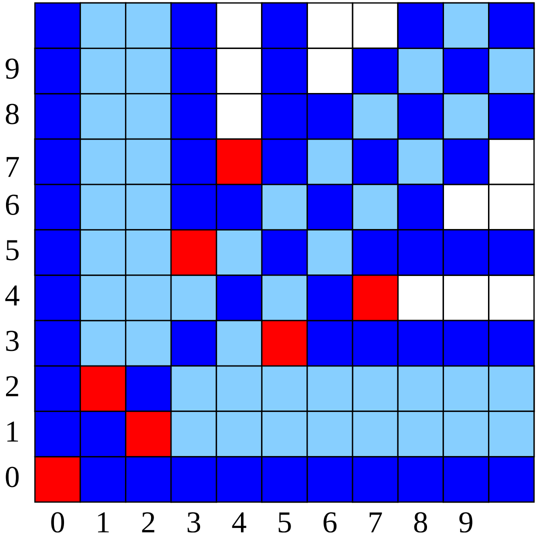

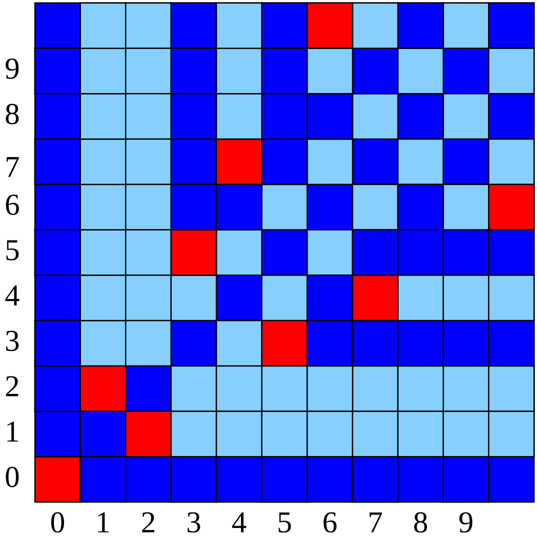
9
8
7
6
5
4
3
2
1
0
0 1 2 3 4 5 6 7 8 9

Combinatorial games

The first few P-positions (up to symmetry)

$$(0,0),\ (1,2),\ (3,5),\ (4,7),\ (6,10),\ (8,13),\ldots$$

Definition of P-positions

A P-position is a position q from which the *previous* player (moving to q) can eventually force a win.

Well-known characterizations of the P-positions of Wythoff's game (recursive, morphic, syntactical properties of Fibonacci expansions, ...)

Theorem

The P-positions of the Wythoff's game are exactly the pairs

$$(\lfloor n\tau \rfloor, \lfloor n\tau^2 \rfloor), \quad n \geq 0$$

where τ is the golden ratio $(1+\sqrt{5})/2$.

Combinatorial games

Remarks

- To a game, i.e., a set of rules, corresponds a set of P-positions.
- Several games may have the same set of P-positions.
- Hence, a given set of P-positions can be associated with invariant as well as variant games.
- One can define the notion of invariant subset of $\mathbb{N}^p$.

N.B. Ho, two variants of Wythoff's game...

Adjoining a move removing k tokens from the smaller pile
(or any pile if the two piles have the same size)
and ℓ tokens from the other pile where $\ell < k$.

Beatty sequences

Rayleigh or Beatty's theorem

Let $\alpha, \beta > 1$ be irrational numbers such that $\alpha^{-1} + \beta^{-1} = 1$.
The sequences $(\lfloor n\alpha \rfloor)_{n>0}$ and $(\lfloor n\beta \rfloor)_{n>0}$ partition $\mathbb{N}_{>0}$.

Two such sequences are *complementary (homogeneous) Beatty sequences*.

Example

For the golden ratio, $\tau^2 = \tau + 1$ thus $1 = \tau^{-1} + (\tau^2)^{-1}$.
Hence, the set of P-positions of Wythoff's game is derived from a pair of complementary homogeneous Beatty sequences.
`A000201` and `A001950` in OEIS.

Beatty sequences

Question

Given a pair $(\lfloor n\alpha \rfloor)_{n>0}$ and $(\lfloor n\beta \rfloor)_{n>0}$ of homogeneous Beatty sequences, does there exist an invariant game having

$$\{(0,0)\} \cup \{(\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor), (\lfloor n\beta \rfloor, \lfloor n\alpha \rfloor) \mid n > 0\}$$

as set of P-positions?

- τ has c.f.-expansion $(1; \overline{1})$, Wythoff
- if α has c.f.-expansion $(1; \overline{k})$, see Fraenkel'82.
- if α has c.f.-expansion $(1; \overline{1, k})$, see Duchêne-R'10.

We conjectured that the above question always has a positive answer.

Beatty sequences

A sequence $(B_n)_{n\geq 0}$ is B_1-*superadditive* if, for all $m, n > 0$,

$$B_m + B_n \leq B_{m+n} < B_m + B_n + B_1.$$

Theorem (Larsson, Hegarty, Fraenkel'11)

Let $(A_n, B_n)_{n\geq 0}$ be a pair of complementary sequences with $A_0 = B_0 = 0$. (*Not* necessarily Beatty sequences.) If the sequence $(B_n)_{n\geq 0}$ is B_1-superadditive, then

$$\{(A_n, B_n), (B_n, A_n) \mid n \geq 0\}$$

is the set of P-positions of an invariant game.

$\star$ Every pair of homogeneous Beatty sequences satisfy the above condition, thus our conjecture holds (Larsson et al.).

Beatty sequences

Question 1

What about non-homogeneous Beatty sequences that realize the set of P-positions of an invariant game?

$$A_n = \lfloor n\alpha + \gamma \rfloor, \quad B_n = \lfloor n\beta + \delta \rfloor$$

where $\gamma, \delta \in \mathbb{R}$. We set $A_0 = B_0 = 0$.

We want that $\{A_n \mid n \geq 1\}$ and $\{B_n \mid n \geq 1\}$ partition $\mathbb{N}_{>0}$, this means that we look for an extension of Nim, i.e.,

$$\mathcal{M} = \{(i, 0) \mid i \geq 1\} \cup \{(0, i) \mid i \geq 1\} \cup \cdots .$$

Question 2 (Larsson et al.)

The superadditivity is a sufficient condition to get a set of P-positions of an invariant game. Is it also a necessary condition?

Beatty sequences

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Beatty sequences

Theorem (Fraenkel'69)

Let $\alpha, \beta > 1$ be irrational numbers such that

(1) $\alpha^{-1} + \beta^{-1} = 1$.

Then $\{\lfloor n\alpha + \gamma \rfloor \mid n \geq 1\}$ and $\{\lfloor n\beta + \delta \rfloor \mid n \geq 1\}$ partition $\mathbb{N}_{>0}$ if and only if

(2) $\frac{\gamma}{\alpha} + \frac{\delta}{\beta} = 0$ and

(3) $n\beta + \delta \notin \mathbb{Z}$, for all $n \geq 1$.

We assume moreover that

(4) $A_1 = 1$ and $B_1 \geq 3$.

also see, K. O'Bryant (Integers 2003)

Our main result

Roughly stated

We can characterize the 4-tuples $(\alpha, \beta, \gamma, \delta)$ such that the corresponding set

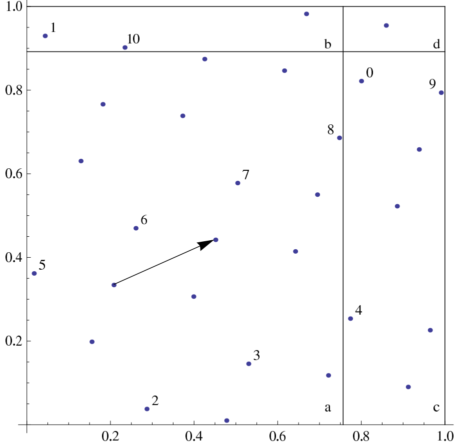
$$\{(0,0)\} \cup \{(\lfloor n\alpha + \gamma \rfloor, \lfloor n\beta + \delta \rfloor), (\lfloor n\alpha + \gamma \rfloor, \lfloor n\beta + \delta \rfloor) \mid n > 0\}$$

is the set of P-positions of an invariant game.

The correct expression involves *combinatorial properties of some infinite word* derived from the two Beatty sequences (direct product of two mechanical words, except maybe for the first symbol).

Our main result

$\beta = 3.99 + \sqrt{5}/2 \simeq 5.108$, $\gamma = -0.2$, $\alpha = \beta/(\beta - 1) \simeq 1.243$, $\delta = -\beta\gamma/\alpha \simeq 0.821$



iterating $R_{\alpha,\beta}$: translation of $(\{\alpha\}, \{\beta\})$ over $\mathbb{T}^2$ starting from $(\{\gamma\}, \{\delta\})$, product of two Sturmian words

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$A_{n+1} - A_n$	1	1	1	1	2	1	1	1	1	2	1	1	1	2	1	1	1	2	1
$B_{n+1} - B_n$	5	6	5	5	5	5	5	5	5	5	6	5	5	5	5	5	5	5	5
$\mathbf{w}_-$	a	b	a	a	c	a	a	a	a	c	b	a	a	c	a	a	a	c	a

Test

Our combinatorial condition reduces to

Take two intervals $I, J \neq \emptyset$ over $[0, 1)$ interpreted as intervals over the unit circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$.
For a given 4-tuple $(\alpha, \beta, \gamma, \delta)$ of real numbers, we ask, whether or not there exists some i such that $R^i_{\alpha,\beta}(\gamma, \delta) \in I \times J$?

extension of the density theorem of Kronecker

The set $\{R^i_{\alpha,\beta}(\gamma, \delta) = (\{i\alpha + \gamma\}, \{i\beta + \delta\}) \in \mathbb{T}^2 \mid i \in \mathbb{N}\}$
is dense in $\mathbb{T}^2$ if and only if $\alpha, \beta, 1$ are rationally independent.

$\alpha, \beta, 1$ are *rationally independent* (i.e., linearly independent over $\mathbb{Q}$), if whenever there exist integers p and q such that $p\alpha + q\beta$ is an integer, then $p = q = 0$.

Test

So, if $\alpha, \beta, 1$ are rationally independent, then there exist infinitely many i such that $R^i_{\alpha,\beta}(\gamma, \delta) \in I \times J$.

Remark

If α and β are irrational numbers satisfying $\alpha^{-1} + \beta^{-1} = 1$ which are not both algebraic numbers of degree 2, then $\alpha, \beta, 1$ are rationally independent.

Test

If $\alpha, \beta, 1$ are rationally dependent, since α and β are irrational numbers, there exist integers p, q, r with $p, q \neq 0$ such that $p\alpha + q\beta = r$.

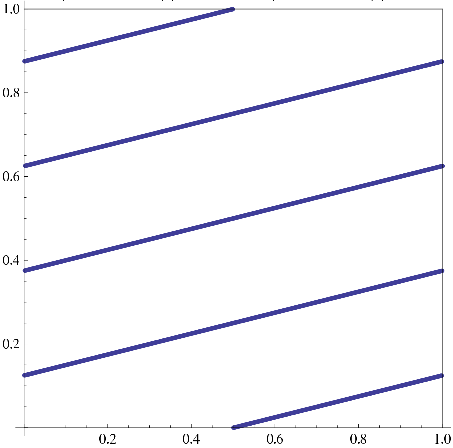
We deduce that $q\beta^2 + (p - q - r)\beta + r = 0$, i.e., β is thus an algebraic number of degree 2. The same holds for α.

The set of points $\{R^n_{\alpha,\beta}(\gamma, \delta) \mid n \in \mathbb{N}\}$ is dense on a straight line in $\mathbb{T}^2$ with rational slope.

The initial question is reduced to determine whether or not a line intersect a rectangle.

Test

$\alpha = (3+\sqrt{17})/2$, $\beta = (7+\sqrt{17})/8$, $\alpha = 4\beta - 2$, rational slope $1/4$



Superadditivity

Thanks to our main result, we can produce examples as the following one.

Counter-example to Question 2

The 4-tuple given by

$$\beta = 1.99 + \frac{\sqrt{5}}{2}, \quad \alpha = \frac{\beta}{\beta - 1}, \quad \gamma = -0.2 \text{ and } \delta = -\frac{\beta\gamma}{\alpha}$$

satisfies the characterization given by our main result, i.e., leads to a set of P-positions of an invariant game, but the sequence is not superadditive.