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Analysis of P Cygni line profiles: generalization of the nth order moment W_n

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Summary. The generalization of the nth order moment W_n of a P Cygni line profile leads to a new approach of deriving the various physical quantities which characterize the phenomenon of mass-loss around stars, quasars, etc.

Considering first the case of rapidly expanding atmospheres (Sobolev approximation), we present in two types of different diagrams the results of calculations of W_n ($n = 0$, 1, 2, and 3) for realistic velocity ($v(r)$) and opacity ($\tau^r_{12}(X')$) distributions. In the first so-called "$\log(W_n) - \log(W^0_n)$" diagrams, we show that for unsaturated P Cygni profiles there exists a linear relation, irrespective of the choice of $v(r)$ and $\tau^r_{12}(X')$, between the observed moment W_n and the physical parameter W^0_n. For $n = 0$, 1, 2 and 3, we establish that W^0_n is a quantity directly related to the column density N_1, to the mass-loss rate $\dot{M}\bar{n}^{(1)}$ (level), to the column impulsion and to (twice) the column kinetic energy, respectively, of the relevant species in the flow. A consistent definition of the average fractional abundance $\bar{n}^{(n)}$(level) of an ion is given. The values of $\bar{n}^{(n)}$ (level) are found to be little dependent on the value of the order n.

When the Sobolev-type approximations used for the transfer of line radiation are no longer fulfilled, we find that only the relation $W_1 \propto \dot{M}\bar{n}^{(1)}$(level) remains valid and that the relation $W_0 \propto N_1$ still holds provided that the expanding envelope is much larger than the stellar core and that the macroscopic velocity $v(r)$ of the flow is greater everywhere than the chaotic (thermal and turbulent) velocities $u(r)$ of the ions.

For values of $W_0 > 0.31$, $W_1 > 0.24$, $W_2 > 0.17$ or $W_3 > 0.15$, the corresponding moment W_n ceases to provide an accurate estimate of the physical parameter W^0_n. It is then only possible to assign a lower limit to the value of W^0_n. By just locating observed values of the moments W_n ($n = 1$, 2 and 3) and of W_0 in combined "$\log(W_n) - \log(W_0)$" diagrams, we propose a new way of determining the types of opacity and velocity distributions characterizing the flow. Of course, this only applies to the non-saturated region of the "$\log(W_n) - \log(W_0)$" diagrams. Our numerical applications do also clearly show that the moments W_n of an observed P Cygni line profile are very dependent on $\tau^r_{12}(X')$ and that they are definitely less sensitive to $v(r)$.

Key words: lines: formation – lines: profile – radiation transfer-stars: mass loss

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1. Introduction

The first concept of the nth order moments W_n of a P Cygni line profile has been given by Castor et al. (1981) who have developed a theory particularly well adapted to the interpretation of IUE and other low resolution spectra. In the framework of the Sobolev approximation (Sobolev, 1947, 1957, 1958; Castor, 1970) and for the case of optically thin lines, these authors have established a linear relation between the first order moment W_1 and the quantity $\dot{M}n$(level), where n(level) is the fractional abundance of an ion in the lower atomic level associated with the given line transition and $\dot{M}$ is the mass-loss rate of the central source. Whereas this relation has been set up by Castor et al. for particular opacity ($\tau^r_{12}(X') \propto (1 - X')$) and velocity ($X' = 1 - 1/L$) distributions, Surdej (1982) has shown that this result was in fact independent of the choice taken for these distributions and that, furthermore, it was irrespective of the Sobolev-type approximations used for describing the transfer of line radiation (Surdej, 1983b).

We are then naturally led to wonder about the possibility of deriving the additional – all remaining? – physical parameters characterizing a P Cygni line profile on the basis of the other moments W_n ($n \neq 1$). It is easy to establish that for even values of the order $n(n = 0$, 2, etc.), the expression of the moments W_n remains a complicated function of these physical parameters. A slightly modified definition of W_n allows one to overcome these difficulties (Sect. 2). In the context of rapidly expanding atmospheres (Sect. 3), we derive the general expression of W_n in terms of well known quantities such as $\tau^r_{12}(X')$, X' and W^0_1. For unsaturated P Cygni line profiles, we establish the linear relations existing between W_n and the physical parameter W^0_n.

Adopting realistic expressions for the velocity and opacity distributions (cf. Castor and Lamers, 1979; Garmany et al., 1981), we present (Sect. 4) and discuss (Sect. 5) the results of calculations of the moments W_n in two types of different diagrams: the "$\log(W_n) - \log(W^0_n)$" and "$\log(W_n) - \log(W_0)$" diagrams. The physical representation of the parameters W^0_n is then given in Sect. 6. In Sect. 7, we test and generalize the validity of some of the linear relations $W_n \propto W^0_n$ for the case of slowly expanding (i.e. non-Sobolev type), optically thin atmospheres. Discussion and conclusions form the last Section.

2. Generalization of the nth order moment W_n of a P Cygni line profile

Considering a two-level atom model interacting with line photons (conservative scattering) in a spherical envelope that is accelerated radially outward around a central source, let us generalize the expression of the nth order moment of a P Cygni line profile (see Castor et al., 1981) as follows:

$$W_n = \frac{\lambda_{12}}{\lambda_{\min}(\lambda_{12}-\lambda_{\min})} \int_{\lambda_{\min}}^{+\infty} \left(\frac{E(\lambda)}{E_c} - 1\right) \left|\frac{\lambda - \lambda_{12}}{\lambda_{12} - \lambda_{\min}}\right|^n \cdot \operatorname{sign}(\lambda - \lambda_{12}) \left(\frac{\lambda_{\min}}{\lambda}\right)^{n+2} d\lambda, \quad (1)$$

with $E(\lambda)/E_c$ being the normalized line profile function and where λ_{12} (resp. $\lambda_{\min}$) denotes the rest (resp. most blueshifted) wavelength of the line profile in the frame of the observer. It is then convenient to make use of the dimensionless frequency

$$X = -(\nu - \nu_{12})/(\nu_{\max} - \nu_{12}), \quad (2)$$

such that the previous expression of W_n reduces to

$$W_n = \int_{-1}^{1} \left(\frac{E(X)}{E_c} - 1\right) \operatorname{sign}(X)|X|^n \, dX. \quad (3)$$

Let us immediately note that the moments W_n are defined irrespective of the possible redshift – or blueshift – of the astronomical object under study (quasar, star, etc.).

3. Rapidly expanding atmospheres

Using at first Sobolev-type approximations for the transfer of line radiation in rapidly expanding atmospheres (cf. Surdej, 1979; referred to below as Paper I) and assuming – for the sake of simplicity – that the central core, having a radius R^*, emits a flat continuum with no photospheric absorption line and with no limb darkening ($\Psi(\mu^*) = 1$), let us follow a similar reasoning as in Surdej (1982; Paper II) in order to transform Equation (3) into (compare Eqs. (1) and (4) with (II.7) and (II.70)[1])

$$W_n = \int_{-X_{\min}}^{1} \tau^r_{12}(X') 4L^2(X') X'^n \gamma_n(1 - 2W(L(X')), 1) \cdot (1 - \beta^3_{12}(X')/\beta^1_{12}(X'))\, dX', \quad (4)$$

where

$$\gamma_n(a,b) = \frac{1}{2}\int_a^b \mu^n \frac{(1 - \exp(-\tau_{12}(X',\mu)))}{\tau_{12}(X',\mu)}\, d\mu. \quad (5)$$

We briefly recall that:

– $X' = v(r)/v_\infty$ denotes the frequency at which a stellar photon is likely to be scattered in the medium at a radial distance r where the macroscopic velocity is $v(r)$. v_∞ stands for the asymptotic velocity $v(r)$ as $r \to \infty$.
– $L(X') = r/R^*$,
– $W(L(X'))$ is the geometrical dilution factor at a distance $L(X')$ (cf. Eq. (II.53)).
– The frequency $X_{\min}$ is such that $L(-X_{\min}) = 1$.
– β^3_{12} $(=\gamma_0(1 - 2W(L(X')), 1))$ and β^1_{12} $(=\gamma_0(-1, 1))$ are the well known escape probabilities of a line photon in an expanding atmosphere.
– $\tau^r_{12}(X')$ and $\tau_{12}(X',\mu)$ are fictitious opacities evaluated at $L(X')$ along a direction making an angle $\theta = 0$ and $\theta = \arccos(\mu)$ with respect to the radial direction (cf. Eqs. (II.19) and (II.20)).

[1] i.e. Eqs. (7) and (70) in Paper II

A more complete description of the above quantities may be found in Paper II.

Because the "point-like" star approximation is essentially a good one, and is very useful in order to understand the asymptotic behaviours of more general solutions given by Eq. (4), we merely state the result when $R^* \to 0$

$$W^*_n = \int_{-X_{\min}}^{1} X'^n(1 - \exp(-\tau^r_{12}(X')))\, dX'. \quad (6)$$

Let us mention that with the old definition of the moments W_n (cf. Castor et al., 1981 and Paper II), no such simple relations (Eqs. (4) and (6)) could be derived for even values of n. Within a good approximation, one can show that for even values of n, $W^{\text{old}}_n \sim (1/(n+1) - 1)W_n$ such that $W^{\text{old}}_0 \sim 0$ and $|W^{\text{old}}_n| < |W_n|$ for $n > 0$.

3.1. The optically thin case

For the case of optically thin lines (i.e., $\tau_{12}(X',\mu) < 1$) and recalling the expression of the fictitious radial opacity (cf. Eq. (II.19))

$$\tau^r_{12}(X') = K \cdot \dot{M} n(\text{level}) \frac{A(el)}{v^2_\infty} \frac{d(1/L)}{X'\, dX'}, \quad (7)$$

with the constant

$$K = \frac{\Pi e^2}{mc} f_{12}\lambda_{12}/(4\Pi\bar{\mu}M_{\text{amu}}R^*), \quad (8)$$

and where:

– $\dot{M}$ denotes the mass-loss rate of the central object,
– n(level) is the fractional abundance of the relevant ion in the lower atomic level 1 of the given line transition,
– $A(el)$ is the abundance of the given element,
– f_{12} is the oscillator strength of the line transition,
– $\bar{\mu}$ is the mean atomic weight of the nuclei and
– M_{amu} is the unit of atomic mass.

Equation (4) now takes the form

$$W^0_n = K\dot{M}\overline{X'^{n-1}}\bar{n}^{(n)}(\text{level})q^{cn}_{(\infty)} \frac{A(el)}{v^2_\infty}, \quad (9)$$

with the average quantities being defined as

$$\overline{X'^{n-1}} = \frac{\int_1^\infty X'^{n-1} n(\text{level})(1 - (1 - 1/L^2)^{(n+1)/2})(1 + \sqrt{1 - 1/L^2})\, dL}{\int_1^\infty n(\text{level})(1 - (1 - 1/L^2)^{(n+1)/2})(1 + \sqrt{1 - 1/L^2})\, dL}, \quad (10)$$

$$\bar{n}^{(n)}(\text{level}) = \frac{\int_1^\infty n(\text{level})(1 - (1 - 1/L^2)^{(n+1)/2})(1 + \sqrt{1 - 1/L^2})\, dL}{\int_1^\infty (1 - (1 - 1/L^2)^{(n+1)/2})(1 + \sqrt{1 - 1/L^2})\, dL} \quad (11)$$

and with the constant $q^{cn}_{(\infty)}$ given by

$$q^{cn}_{(\infty)} = -\frac{1}{(n+1)} \int_1^\infty (1 - (1 - 1/L^2)^{(n+1)/2})(1 + \sqrt{1 - 1/L^2})\, dL, \quad (12)$$

leading to the values: $q^{c0}_{(\infty)} = -1.00000$, $q^{c1}_{(\infty)} = -0.89271$, $q^{c2}_{(\infty)} = -0.81737$ and $q^{c3}_{(\infty)} = -0.76029$.

Considering the "point-like" star approximation, we easily find that Eq. (9) should be replaced by

$$W_n^{*0} = -K\dot{M}\overline{X'^{n-1}}\bar{n}(\text{level})\frac{A(el)}{v_\infty^2}, \tag{13}$$

with the average quantities now defined as

$$\overline{X'^{n-1}} = \frac{\int_1^\infty X'^{n-1}n(\text{level})/L^2\,dL}{\int_1^\infty n(\text{level})/L^2\,dL}, \tag{14}$$

$$\bar{n}(\text{level}) = \frac{\int_1^\infty n(\text{level})/L^2\,dL}{\int_1^\infty 1/L^2\,dL}. \tag{15}$$

Let us note here that the average fractional abundance $\bar{n}$(level) is no longer dependent on the order n of the considered moment.

3.2. The optically thick case

If the expanding atmosphere gets optically thick ($\tau_{12}(X',\mu) > 1$) to the spectral line radiation, Eq. (4) can be easily transformed into

$$W_n^t = \int_{-X_{\min}}^{1} L^2(X')X'^n\left\{\left(1-\frac{d\ln L}{d\ln X'}\right)\frac{1}{(n+3)}\left(1-\left(1-\frac{1}{L^2}\right)^{(n+3)/2}\right) + \frac{d\ln L}{d\ln X'}\frac{1}{(n+1)}\left(1-\left(1-\frac{1}{L^2}\right)^{(n+1)/2}\right)\right\}\left(1+\sqrt{1-\frac{1}{L^2}}\right)dX', \tag{16}$$

assuming that $\beta_{12}^3(X')/\beta_{12}^1(X') \sim W(L(X'))$. We conclude that the asymptotic values W_n^t are only velocity field dependent.

If the dimensions of the central object are negligible with respect to that of the moving envelope, Eq. (16) should be replaced by

$$W_n^{*t} = \frac{1-(-X_{\min})^{n+1}}{n+1}. \tag{17}$$

For $X_{\min} = -0.01$, we have $\log_{10}(W_n^{*t}) = -0.00$, -0.30, -0.48 and -0.60 for $n = 0$, 1, 2 and 3, respectively.

4. Numerical applications

Combining Eqs. (7) and (9) for $n = 1$, the expression of the fictitious radial opacity may be rewritten as

$$\tau_{12}^r(X') = \frac{W_1^0 n(\text{level})\,d(1/L)}{q_{(\infty)}^{c1}\bar{n}^{(1)}(\text{level})X'\,dX'}. \tag{18}$$

For any specified velocity and opacity distributions, it is then straightforward to compute the moments W_n (cf. Eq. (4)) as well as W_n^0 (see Eq. (9)) as a function of the parameter W_1^0. Such calculations have already been performed and discussed for the case $n = 1$ (see Surdej, 1983a; Paper III). Adopting the 3 velocity fields and 6 opacity distributions discussed in that paper (see Table 1), we have illustrated in Figs. 1–3 the 18 resulting "$\log_{10}(W_n) - \log_{10}(W_n^0)$" model calculations for $n = 0$, 2 and 3. In Figs. 4–6 we show the 18 "$\log_{10}(W_n) - \log_{10}(W_0)$" curves for $n = 1$, 2 and 3, respectively. In order to construct each single curve in Figs. 1–6, 50 equally spaced values of $\log_{10}(W_1^0)$ have been chosen in the range $[-3, 3]$. As in Paper III, we have taken the values $X_{\min} = -0.01$ and $L_{\max} = 1000$ (i.e., $L_{\max} \gg 1$).

Table 1. Adopted velocity and opacity distributions for calculating the moments W_n of P Cygni line profiles. The value of the constants $k_\alpha, \ldots, k_\eta$ essentially depends on the choice of the value of the parameter W_1^0 [see Eq. (18)]

(A)	$X' = -X_{\min} + (1 + X_{\min})(1 - 1/\sqrt{L})$
(B)	$X' = -X_{\min} + (1 + X_{\min})(1 - 1/L)$
(C)	$X' = \sqrt{1 - (1 - X_{\min}^2)/L}$
(α)	$\tau_{12}^r(X') = k_\alpha\Big/\left(\frac{X'\,dX'}{d(1/L)}\right)$
(β)	$\tau_{12}^r(X') = k_\beta(1 - X')$
(γ)	$\tau_{12}^r(X') = k_\gamma$
(δ)	$\tau_{12}^r(X') = k_\delta\sqrt{1 - X'}$
(ε)	$\tau_{12}^r(X') = k_\varepsilon(1 - X')^2$
(η)	$\tau_{12}^r(X') = k_\eta/X'$

5. Discussion of the results illustrated in Figs. 1–6

On the basis of "$\log(W_1) - \log(W_1^0)$" diagrams, we have shown in Paper III how it is possible to derive the value of the parameter W_1^0 – a quantity that is proportional to $\dot{M}\bar{n}^{(1)}$(level) – from the measurement of the first order moment W_1 of an observed P Cygni line profile. Similarly, using "$\log(W_n) - \log(W_n^0)$" diagrams (see Figs. 1–3) it is possible to estimate the values of the parameters W_n^0 ($n = 0$, 1, 2, etc.) from the measurement of the moments W_n. The physical representation of the parameters W_0^0, W_2^0 and W_3^0, defined in Sect. 3.1, shall be discussed later (see Sect. 6).

Let us immediately point out that for the case of unsaturated profiles ($\tau_{12}^r(X') < 1$), we have the linear relation (cf. Sect. 3.1)

$$W_n = W_n^0, \tag{19}$$

irrespective of the velocity and/or opacity distributions used for modelling the expanding envelope.

For large values of W_1^0, and correspondingly of W_n^0, the atmosphere gets optically thick and the P Cygni profiles become saturated. Using "$\log(W_n) - \log(W_n^0)$" diagrams, it is then only possible to derive a lower limit for the value of the parameter W_n^0. As $\log(W_n^0) \to \infty$, $\log(W_n)$ tends towards the asymptotic value $\log(W_n^t)$ (see Sect. 3.2). If we assume that the "$\log(W_n) - \log(W_n^0)$" curves illustrated in Figs. 1–3 encompass most of the realistic solutions, we can then estimate from the observed dispersion of these curves the lowest value W_n^l such that, due to the uncertainty in the choice of a model, the relative error affecting the determination of W_n^0 is roughly equal to 100%. We find that $W_0^l = 0.31$, $W_1^l = 0.24$ (see Paper III), $W_2^l = 0.17$ and $W_3^l = 0.15$.

We can still notice in Figs. 1–3 that the "$\log(W_n) - \log(W_n^0)$" curves calculated for a same opacity distribution $\tau_{12}^r(X')$ are very little dependent on the type used for the velocity field $v(r)$ (e.g. models ($A.\gamma$), ($B.\gamma$) and ($C.\gamma$)). The results are directly accounted for by the greater sensitivity of the line profile function $E(X)/E_c$ – and of the resulting moments W_n, W_n^* – onto the choice of

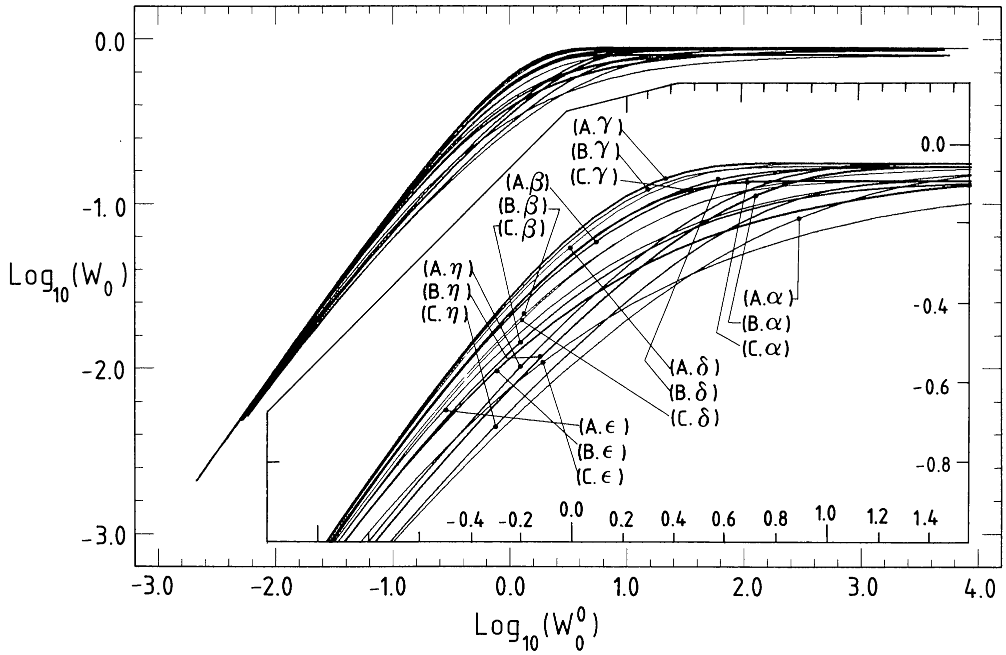

Fig. 1. "$\mathrm{Log}_{10}(W_0) - \log_{10}(W_0^0)$" curves for the eighteen possible models derived from Table 1

the opacity distribution rather than on the choice of the velocity field. Considering, for instance, the case of a "point-like" star, we directly see from Eq. (6) that the moment W_n^* essentially relies on the distribution $\tau_{12}^r(X')$. The "$\log(W_n) - \log(W_n^0)$" curves pertaining to the models labelled $(A.\alpha)$, $(B.\alpha)$ and $(C.\alpha)$ in Figs. 1–3 do appear to be very different from one another. Indeed, we recall that these model calculations are characterized by the assumption of mass conservation in the flow (see Table 1) and that consequently there is a distinct opacity distribution which corresponds to each velocity field.

All these considerations do also apply to the "$\log(W_n) - \log(W_0)$" curves displayed in Figs. 4–6. The very slight dependence of these curves versus the choice of the velocity field is particularly well seen here. In principle, these "$\log(W_n) - \log(W_0)$" curves can be used in the following manner: measurement of the moments W_n ($n = 1, 2$, etc.) and W_0 from an observed line profile

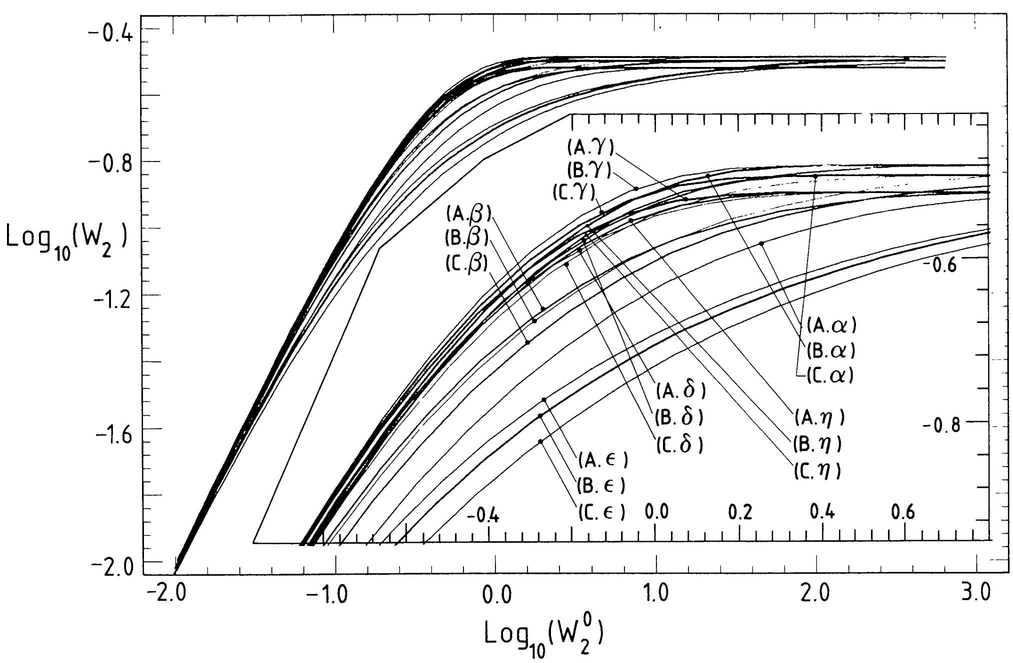

Fig. 2. "$\mathrm{Log}_{10}(W_2) - \mathrm{Log}_{10}(W_2^0)$" curves for the eighteen possible models derived from Table 1

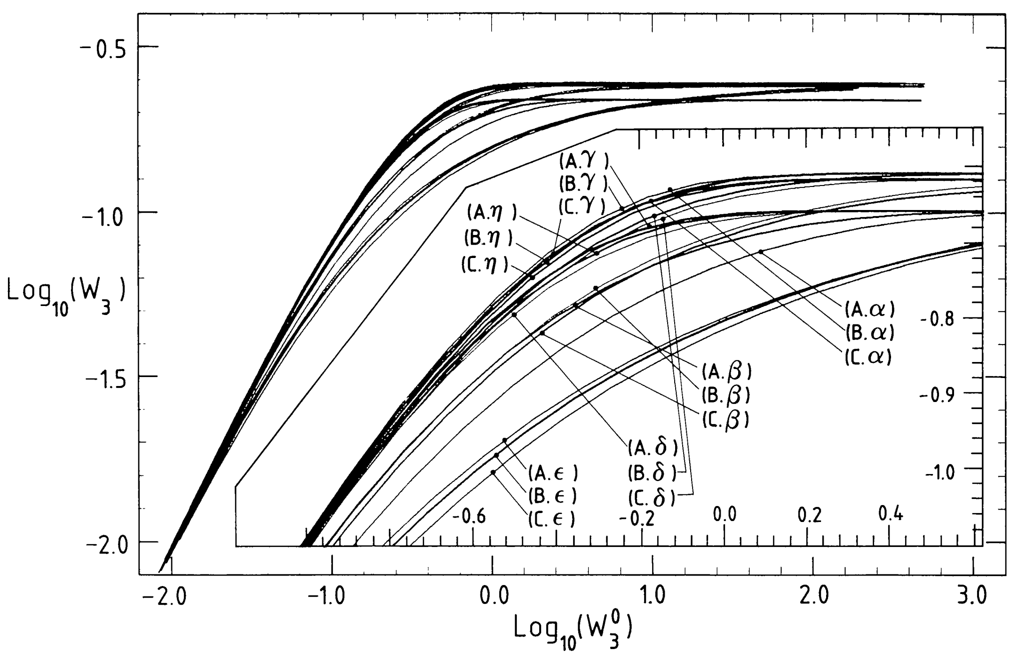

Fig. 3. "$\text{Log}_{10}(W_3) - \text{Log}_{20}(W_3^0)$" curves for the eighteen possible models derived from Table 1

should allow one to determine the type of opacity and velocity distributions characterizing that particular P Cygni profile by just locating the measured "$\log(W_n) - \log(W_0)$" points in Figs. 4–6. Let us remark that for unsaturated P Cygni profiles the relation between $\log(W_n)$ and $\log(W_0)$ is essentially linear (the corresponding W_n^0/W_1^0 ratios for $n = 0$, 2 and 3 are listed in Table 2). As the profiles get saturated, there results a crowding effect between all curves: in that region of the "$\log(W_n) - \log(W_0)$" diagrams, it is no longer possible to distinguish between the different models! As $W_1^0 \to \infty$, the different curves tend towards the asymptotic values "$\log(W_n^t) - \log(W_0^t)$" which are only velocity field dependent (cf. Sect. 3.2).

We note in Figs. 4–6 that for opacity distributions of the type

$$\tau_{12}^r(X') \propto (1 - X')^\gamma, \tag{20}$$

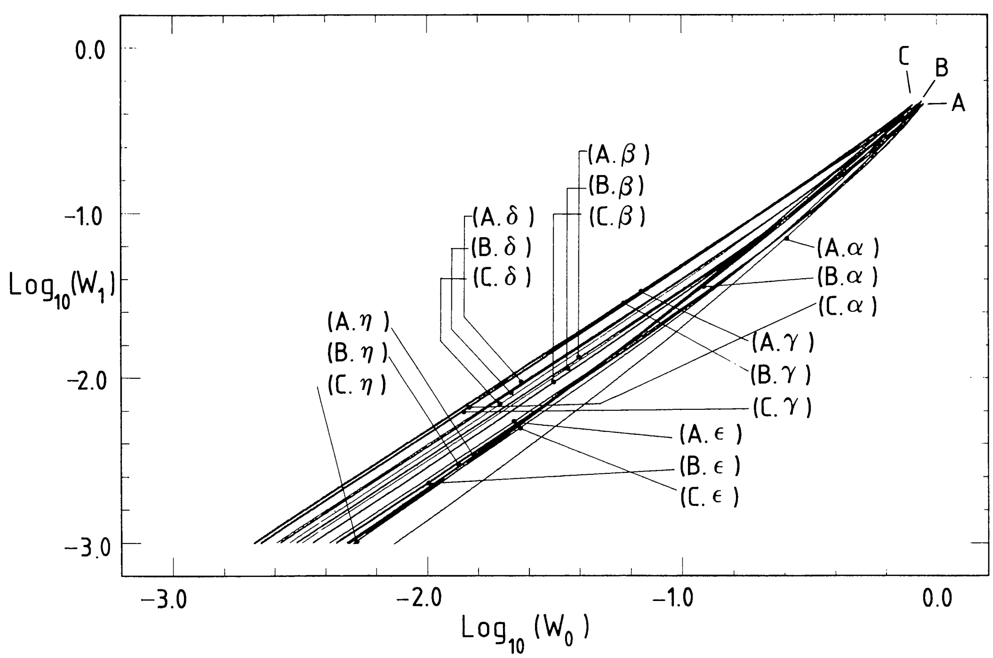

Fig. 4. "$\text{Log}_{10}(W_1) - \text{Log}_{10}(W_0)$" curves for the eighteen possible models derived from Table 1

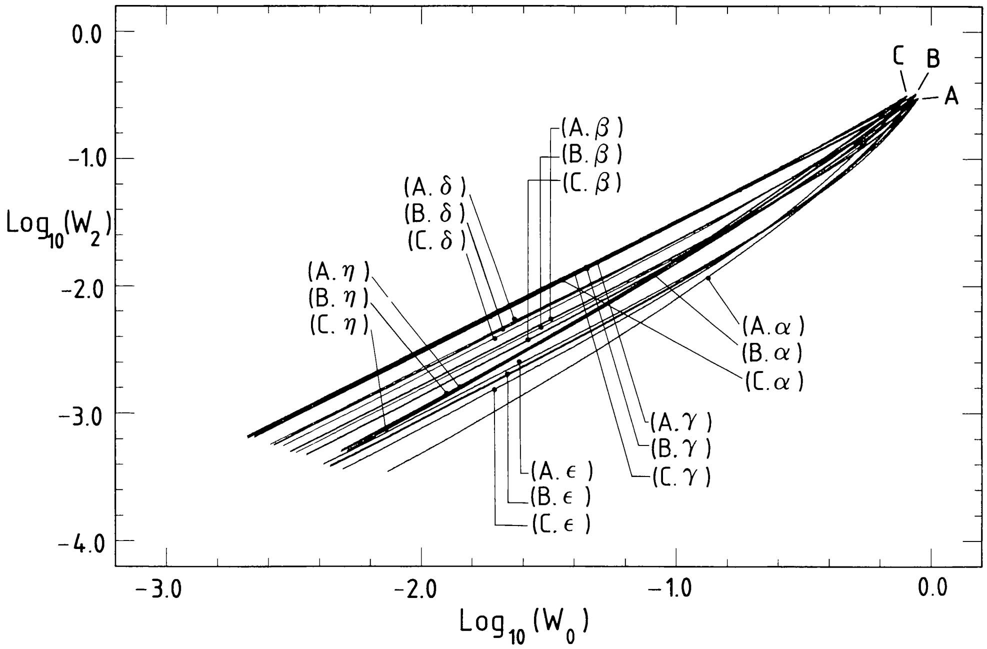

Fig. 5. "$\mathrm{Log}_{10}(W_2) - \mathrm{Log}_{10}(W_0)$" curves for the eighteen possible models derived from Table 1

there is an increasing shift between the nearly parallel "$\log(W_n) - \log(W_0)$" curves as the parameter y increases ($y = 0, \frac{1}{2}$, 1 and 2 for the models labelled $(*.\gamma)$, $(*.\delta)$, $(*.\beta)$ and $(*.\varepsilon)$, respectively, in Figs. 4–6).

Although it is unlikely that a single measured "$\log(W_1) - \log(W_0)$" point would suffice to derive both the unknown opacity and velocity distributions, we think that the simultaneous use of different "$\log(W_n) - \log(W_0)$" diagrams will enable one to derive these two important physical quantities characterizing an observed line profile.

6. Physical representation of the parameters W_n^0 through the related quantities Q_n, R_n, S_n and T_n

Adopting the following relation between Q_n and W_n^0

$$Q_n = \frac{W_n^0 v_\infty^{n+1}}{K q_{(\infty)}^{cn}}, \tag{21}$$

we find by means of Eq. (9) that

$$Q_n = \dot{M} \overline{v^{n-1} \bar{n}^{(n)}(\text{level}) A(el)}. \tag{22}$$

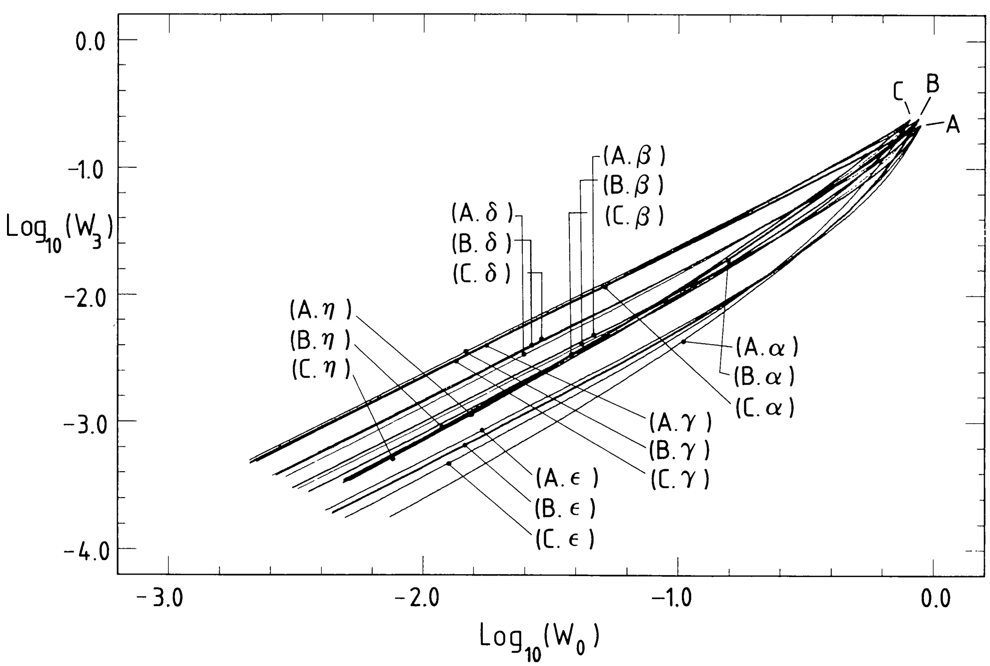

Fig. 6. "$\mathrm{Log}_{10}(W_3) - \mathrm{Log}_{10}(W_0)$" curves for the eighteen possible models derived from Table 1

Table 2. The ratios W_n^0/W_1^0, $\bar{n}^{(n)}$(level)/$\bar{n}^{(1)}$(level) and the average quantity $\overline{X'^{n-1}}$ ($n = 0$, 2 and 3) for the velocity laws (A)–(C) and the opacity distributions (α)–(η) (see Table 1)

	n	(A)			(B)			(C)		
		W_n^0/W_1^0	$\bar{n}^{(n)}/\bar{n}^{(1)}$	$\overline{X'^{n-1}}$	W_n^0/W_1^0	$\bar{n}^{(n)}/\bar{n}^{(1)}$	$\overline{X'^{n-1}}$	W_n^0/W_1^0	$\bar{n}^{(n)}/\bar{n}^{(1)}$	$\overline{X'^{n-1}}$
(α)	0	8.2720	1.0000	7.3845	5.2156	1.0000	4.6560	2.2196	1.0000	1.9815
	2	0.3505	1.0000	0.3828	0.5146	1.0000	0.5620	0.6554	1.0000	0.7158
	3	0.1792	1.0000	0.2104	0.3483	1.0000	0.4090	0.4917	1.0000	0.5773
(β)	0	3.0787	0.9326	2.9470	3.2368	0.9823	2.9416	3.6072	1.0947	2.9416
	2	0.5010	1.0520	0.5201	0.4919	1.0063	0.5339	0.4752	0.9323	0.5567
	3	0.3019	1.0942	0.3240	0.2950	1.0069	0.3440	0.2816	0.8805	0.3755
(γ)	0	2.0870	0.9116	2.0438	2.0847	0.9388	1.9824	2.2196	1.0000	1.9815
	2	0.6486	1.0726	0.6604	0.6635	1.0459	0.6929	0.6554	1.0000	0.7158
	3	0.4732	1.1348	0.4896	0.4985	1.0824	0.5408	0.4917	1.0000	0.5773
(δ)	0	2.5637	0.9216	2.4833	2.6525	0.9612	2.4635	2.8962	1.0496	2.4633
	2	0.5687	1.0626	0.5845	0.5661	1.0251	0.6031	0.5530	0.9631	0.6271
	3	0.3772	1.1150	0.3972	0.3780	1.0423	0.4258	0.3670	0.9339	0.4614
(ε)	0	4.1551	0.9549	3.8845	4.4391	1.0203	3.8840	5.1001	1.1722	3.8841
	2	0.4007	1.0310	0.4245	0.3874	0.9747	0.4341	0.3665	0.8853	0.4521
	3	0.2023	1.0537	0.2254	0.1923	0.9486	0.2380	0.1775	0.8022	0.2598
(η)	0	5.1023	0.9547	4.7710	5.2156	1.0000	4.6560	5.7626	1.1053	4.6542
	2	0.5032	1.0372	0.5299	0.5146	1.0000	0.5620	0.5108	0.9406	0.5931
	3	0.3299	1.0691	0.3623	0.3483	1.0000	0.4090	0.3476	0.9013	0.4528

For the particular cases of $n = 1$, 2 and 3, Q_n merely represents the mass-loss rate, an average impulsion rate and (twice) an average kinetic energy rate carried out by the relevant species in the envelope.

Let us point out that the dependence of the average fractional abundance $\bar{n}^{(n)}$(level) versus the order n is only small. For the eighteen model calculations illustrated in Figs. 1–6 we have reported in Table 2 the ratios $\bar{n}^{(n)}$(level)/$\bar{n}^{(1)}$(level) for $n = 0$, 2 and 3. In no case does the relative error between $\bar{n}^{(n)}$(level) and $\bar{n}^{(1)}$(level) exceed 20%. We have also indicated in Table 2 the values of $\overline{X^{n-1}}$ ($n = 0$, 2 and 3) calculated for the eighteen models. As expected, the highest values of $\overline{X^{n-1}}$ ($n = 2, 3$) – and correspondingly, of Q_n – do occur for the steepest velocity field and the smoothest opacity distribution listed in Table 1 (i.e. model ($C.\alpha$) equivalent to ($C.\gamma$)).

Another interpretation of the parameter W_n^0 may also be obtained in a different way. Let us define the quantity R_n such as

$$R_n = \frac{W_n^0 v_\infty^{n+1}(n+1)}{\dfrac{\Pi e^2}{mc} f_{12}\lambda_{12}}. \tag{23}$$

Combination of Eqs. (9)–(12) and (II.16)–(II.18) for the expression of the mass-loss rate $\dot{M}$ leads to the interesting result

$$R_n = \int_{R^*}^{\infty} n_1(r)v(r)^n \left(1 - \left(1 - \left(\frac{R^*}{r}\right)^2\right)^{(n+1)/2}\right) \times \left(1 + \sqrt{1 - \left(\frac{R^*}{r}\right)^2}\right)\left(\frac{r}{R^*}\right)^2 dr, \tag{24}$$

with n_1 being the volume density of the relevant ion in the lower atomic level. For $n = 0$, 1, 2, etc., the quantity R_n thus represents the column density, the column velocity, (twice) the column square velocity, etc., of the species under consideration. We have the obvious relation

$$Q_n/R_n = 4\Pi\bar{\mu}M_{\rm amu}R^*/(q_{(\infty)}^{cn}(n+1)). \tag{25}$$

It is then natural to define the additional quantities

$$S_n = Q_n/(\bar{n}^{(n)}(\text{level})A(el)), \tag{26}$$

and

$$T_n = R_n\bar{\mu}M_{\rm amu}/(\bar{n}^{(n)}(\text{level})A(el)), \tag{27}$$

such that for $n = 1$, 2, 3, etc., S_n represents the mass-loss rate, an average impulsion rate, (twice) an average kinetic energy rate, etc., and T_n represents the column mass, the column impulsion, (twice) the column kinetic energy, etc., of all species that have been ejected by the star. We have also the relation

$$S_n/T_n = 4\Pi R^*/(q_{(\infty)}^{cn}(n+1)). \tag{28}$$

The estimate of $\bar{n}^{(n)}$(level) remains the only true limitation to the precise determination of the quantities S_n and T_n.

Whereas the values of the moments W_n – and W_n^0 – depend on the accurate determination of the maximum velocity v_∞($W_n \propto v_\infty^{-(n+1)}$) on an observed line profile, it is interesting to note that none of the quantities Q_n, R_n, S_n or T_n are sensitive to this.

Let us finally recall that for saturated P Cygni line profiles, it is only possible to infer lower limits for the values of the physical quantities Q_n, R_n, S_n and T_n.

7. Slowly expanding atmospheres

For the case of optically thin lines ($\tau_{12}^r(X') < 1$), we have already established that the relation existing between the first order moment W_1 and the quantity $\dot{M}\bar{n}^{(1)}$(level) (cf. Eq. (9) and Sect. 6) holds irrespective of any Sobolev-type approximations used for the transfer of line radiation (Surdej, 1983b; Paper IV).

Considering such slowly expanding and optically thin atmospheres, we have investigated the possible generalization of the previous result to any order $n = 0, 1, 2$, etc. Our conclusions are summarized below.

For a star having finite dimensions, it can be shown (cf. Appendix A) that for even values of n ($n = 0, 2, 4, \ldots$) there is no simple relation between the nth order moment – denoted hereafter W_n^s – of a P Cygni line profile and the parameter W_n^0 (see Eq. (9)). As expected, for $n = 1$, we have

$$W_1^s = \frac{W_1^0}{\left(1 + \frac{u(L_{max})}{v_\infty}\right)^2}, \tag{29}$$

with $u(L_{max})$ being the maximum chaotic (thermal, turbulent, etc.) velocity of the ions at L_{max}. For $n = 3$, we find that

$$W_3^s = \frac{W_3^0}{\left(1 + \frac{u(L_{max})}{v_\infty}\right)^4} \cdot \left(1 + \left(\frac{\bar{u}(\bar{L})}{v_\infty}\right)^2 \frac{W_1^0}{W_3^0}\right), \tag{30}$$

where $\bar{u}(\bar{L})$ represents a typical average thermal velocity in the wind.

Within the "point-like" star approximation, and assuming that the macroscopic velocity $v(r)$ is everywhere greater than the local maximum chaotic velocity $u(L)$ of the ions, we obtain for $n = 0$ and $n = 1$ the interesting results (cf. Appendix B)

$$W_0^{*s} = \frac{W_0^{*0}}{\left(1 + \frac{u(L_{max})}{v_\infty}\right)}, \tag{31}$$

and

$$W_1^{*s} = \frac{W_1^{*0}}{\left(1 + \frac{u(L_{max})}{v_\infty}\right)^2}, \tag{32}$$

where W_0^{*0} and W_1^{*0} are given by Eq. (13).

For $n = 2$ and $n = 3$, we find that

$$W_2^{*s} = \frac{W_2^{*0}}{\left(1 + \frac{u(L_{max})}{v_\infty}\right)^3} \left(1 + \frac{1}{3}\left(\frac{\bar{u}(\bar{L})}{v_\infty}\right)^2 \frac{W_0^{*0}}{W_2^{*0}}\right), \tag{33}$$

and

$$W_3^{*s} = \frac{W_3^{*0}}{\left(1 + \frac{u(L_{max})}{v_\infty}\right)^4} \left(1 + \left(\frac{\bar{u}(\bar{L})}{v_\infty}\right)^2 \frac{W_1^{*0}}{W_3^{*0}}\right). \tag{34}$$

Since the ratios W_0^{*0}/W_2^{*0} and W_1^{*0}/W_3^{*0} are of the order $(v_\infty/\bar{v})^2$, Eqs. (33) and (34) reduce to those obtained in the context of rapidly expanding atmospheres whenever the typical average maximum chaotic velocity $\bar{u}$ is negligible with respect to the average macroscopic velocity $\bar{v}$. Let us finally remark that the additional correcting factor $1/(1 + u(L_{max})/v_\infty)^{n+1}$ appearing in Eqs. (29)–(34) just accounts for the fact that radiation contributing to the observed frequency ν_{max} (cf. Eq. (2)) is due to line photons emitted – or absorbed – radially at L_{max} with a local frequency $\nu_{12} + \Delta\nu(L_{max})/2$. Because in a rapidly expanding atmosphere, the width $2u = (\Delta\nu/\nu_{12})c$ of the line transition is negligible when compared to the maximum velocity v_∞ (Sobolev approximation), the correcting factor is essentially equal to unity.

8. Discussion and conclusions

The main conclusions of the present work have already been summarized in the abstract. We shall not repeat them here. In the remainder, we just aim at discussing two different points: i) some remarks on the correct use of the average fractional abundance $\bar{n}^{(n)}$(level); ii) a comparison between the "line profile fitting" technique and the use of the moments W_n when analyzing P Cygni line profiles.

In view of the results obtained in Sect. 3, we wish to warn the reader against some possible misuse of the average fractional abundance $\bar{n}^{(n)}$(level) – also referred to in the literature as the mean ionization fraction $\bar{q}_i$ – of an ion in the expanding atmosphere. Indeed, several authors make the implicit or explicit wrong assumption that the quantity $\overline{X'^{-1}}\,\bar{n}$(level) (see Eq. (13)) is equivalent to $\bar{q}_i \int_1^\infty 1/(X'L^2)\,dL$. From this, they infer an unjustified relation between the column density N_i of the relevant ion and the mass-loss rate $\dot{M}$, namely

$$\bar{q}_i \propto N_i/\dot{M}, \tag{35}$$

where the constant of proportionality only depends on the adopted velocity field $v(r)$ (cf. Lamers et al., 1980; Garmany et al., 1981; Gathier et al., 1981). Using Eqs. (13) and (23), we rigorously find that

$$\bar{n}(\text{level}) \propto N_i/(\dot{M}\overline{X'^{-1}}), \tag{36}$$

where the average quantity $\overline{X'^{-1}}$ is essentially dependent on the chosen opacity distribution. Considering for instance the model calculations performed in Sect. 4, a look at Table 2 clearly indicates that for a given velocity field, the quantity $\overline{X'^{-1}}$ may vary by as much as ~260%, thus invalidating relation (35).

Let us now compare the two major techniques used for the determination of the physical parameters characterizing a P Cygni line profile. In the first of these techniques, the types of velocity $v(r)$ and opacity $\tau_{12}^r(X')$ distributions are obtained by matching observed profiles with theoretical ones, using for instance the atlas of Castor and Lamers (1979). When doing this, an additional parameter T (equivalent to W_0^0, see Eq. (9)) is scaled in order to achieve the best fitting. Alike this procedure, we have shown that it was possible to derive the same quantities by just locating the measured moments W_n of a P Cygni profile in "$\log(W_n) - \log(W_n^0)$" and "$\log(W_n) - \log(W_0)$" diagrams. In our case, the additional scaling factor that we have used is the parameter W_1^0 (see Eq. (18)).

Whereas in the "line profile fitting" approach there is no direct way of testing the unicity of the solution (i.e. the unicity of $\tau_{12}^r(X')$, $X'(L)$ and W_0^0), we have seen that realistic error estimates of the derived parameters W_n^0 could be assigned on the basis of "$\log(W_n) - \log(W_n^0)$" diagrams. Furthermore, by locating the measured moments W_n in "$\log(W_n) - \log(W_0)$" diagrams, it is possible to infer the most likely opacity distribution and – to a lesser extent – the velocity field which suit best the observations

Recalling that the theory of the moments W_n is particularly well suited to the analysis of underresolved line profiles (see Castor et al., 1981), there is no doubt that a realistic confrontation between the moments W_n of observed P Cygni profiles and the calculations presented here will shed more light on the correctness of all our assumptions.

For the sake of completeness, we rewrite hereafter in practical units the useful relations between the physical quantities Q_n, R_n and the parameter W_n^0 (see Eqs. (21) – (24)). Remembering that for $n = 1, 2, 3$, etc., Q_n represents the mass-loss rate, an average impulsion rate and (twice) an average kinetic energy rate, etc., carried out by the relevant species, we have

$$Q_n(M_\odot\,\mathrm{yr}^{-1}\,(\mathrm{km\,s}^{-1})^{n-1}) = 8.673\,10^{-22}\,v_\infty\,(\mathrm{km\,s}^{-1})^{n+1}\,\frac{R^*(R_\odot)\bar{\mu}W_n^0}{f_{12}\lambda_{12}\,(10^3\,\text{Å})\,q_{i(\infty)}^{cn}}, \tag{37}$$

and

$$R_n\,(\mathrm{cm}^{-2}(\mathrm{km\,s}^{-1})^n) = 3.768\,10^{11}\,\frac{(n+1)v_\infty(\mathrm{km\,s}^{-1})^{n+1}\,W_n^0}{f_{12}\lambda_{12}\,(-10^3\,\text{Å})} \tag{38}$$

such that for $n = 0$, 1 and 2, R_n represents the column density, the column velocity and (twice) the column square velocity of the ion under consideration.

Appendix A

In order to establish the expression of the generalized moment W_n for the case of a slowly expanding and optically thin atmosphere around a stellar core having a radius R^*, let us follow the same reasoning as in Paper IV. Let us first replace Eqs. (IV.3) and (IV.5) by the more precise relation

$$v = v_L\left(1 + \frac{v(L)}{c}\,\mu\right), \tag{A.1}$$

which is such that $v_{\max} = (v_{12} + \Delta v(L_{\max})/2)(1 + v_\infty/c)$. It naturally follows that Eqs. (IV.7), (IV.9), (IV.21) – (IV.25), (IV.36) and (IV.43) will undergo slight modifications due to the possible presence of the correcting factor $1/(1 + u(L_{\max})/v_\infty)$. Replacing now W_1 by $W_n^{s(2)}$ and the factor X by $\mathrm{sign}(X)|X|^n$ in Eqs. (IV.26), (IV.28), (IV.35) and (IV.42), we directly find that in the absence of limb darkening (i.e. $\Psi(\mu^*) = 1$)

$$W_n^s = 2R^*\int_0^1 \mu^*\,d\mu^* \int_1^{L_{\max}} (W(L) - 1)\,\frac{n_1 \frac{\Pi e^2}{mc} f_{12}}{\mu(\mu^*, L)}\,I_n(L)\,dL, \tag{A.2}$$

with

$$I_n(L) = \int_{-\Delta v/2}^{\Delta v/2} \mathrm{sign}\left(X_L - \frac{lc}{v_{12}v_\infty}\right)\left|X_L - \frac{lc}{v_{12}v_\infty}\right|^n \Phi_L(l)\,dl\Big/\left(\left(1 + \frac{u(L_{\max})}{v_\infty}\right)^{n+1} v_{12}\frac{v_\infty}{c}\right). \tag{A.3}$$

The dimensionless frequency X_L is here defined by

$$X_L = -X'\mu, \tag{A.4}$$

where μ denotes the cosine of the angle between the line-of-sight and the radial direction.

Under the assumption that the redistribution function $\Phi_L(l)$ is symmetric, we derive for $n = 1$

$$I_1(L) = X_L\Big/\left(\left(1 + \frac{u(L_{\max})}{v_\infty}\right)^2 \frac{v_{12}v_\infty}{c}\right). \tag{A.5}$$

With the help of Eqs. (II.16) – (II.18), we then straightforwardly find the result

$$W_1^s = K\dot{M}\bar{n}^{(1)}(\text{level})q_{i(\infty)}^{c1}\,\frac{A(el)}{v_\infty^2}\cdot\frac{1}{\left(1 + \frac{u(L_{\max})}{v_\infty}\right)^2}, \tag{A.6}$$

which for $L_{\max} \to \infty$ is equivalent to Eq. (29).

For $n = 3$, we similarly obtain

$$I_3(L) = \left(X_L^3 + X_L\left(\frac{\bar{u}(L)}{v_\infty}\right)^2\right)\Big/\left(\left(1 + \frac{u(L_{\max})}{v_\infty}\right)^4 v_{12}\frac{v_\infty}{c}\right), \tag{A.7}$$

where we have used the approximation

$$\bar{u}(L)^2 = 3\int_{-\Delta v/2}^{\Delta v/2} l^2\Phi_L(l)\,dl/(v_{12}/c)^2. \tag{A.8}$$

Inserting (A.7) into (A.3), we find that

$$W_3^s = \left(K\dot{M}\overline{X'^2}\bar{n}^{(3)}(\text{level})q_{i(\infty)}^{c3}\,\frac{A(el)}{v_\infty^2} + W_1^0\,\frac{\bar{u}(\bar{L})}{v_\infty}\right)\Big/\left(1 + \frac{u(L_{\max})}{v_\infty}\right)^4, \tag{A.9}$$

which for $L_{\max} \to \infty$ is equivalent to Eq. (30).

For even values of $n (n = 0, 2$, etc.), Eq. (A.3) fails to reduce to any tractable result. This is because for some values of L and μ (e.g. $\mu = 0$), the quantity $X_L - lc/(v_{12}v_\infty)$ may take positive values when $l \in [-\Delta v/2, 0]$. Since within the "point-like" star approximation we have $\mu = 1$, we do expect interesting results whenever $v(L) > u(L)$ everywhere in the atmosphere (see Appendix B).

Appendix B

Within the "point-like" star approximation, let us evaluate the expression of the moments W_n^{*s} assuming that the optically thin atmosphere is slowly expanding. Using the results of Appendix A and imposing the condition $R^* \to 0$, it is straightforward to establish that Eqs. (A.2) and (A.3) transform into

$$W_n^{*s} = -R^*\int_1^{L_{\max}} n_1\,\frac{\Pi e^2}{mc}\,f_{12}I_n^*(L)\,dL, \tag{B.1}$$

and

$$I_n^*(L) = -\int_{-\Delta v/2}^{\Delta v/2}\left|\frac{lc}{v_{12}v_\infty} - X_L\right|^n \Phi_L(l)\,dl\Big/\left(\left(1 + \frac{u(L_{\max})}{v_\infty}\right)^{n+1} v_{12}\frac{v_\infty}{c}\right), \tag{B.2}$$

respectively, provided that the condition $v(L) > u(L)$ is fulfilled everywhere in the expanding envelope.

Particularizing first to the case $n = 0$, we find that

$$I_0^*(L) = -1\Big/\left(\left(1 + \frac{u(L_{\max})}{v_\infty}\right)v_{12}\frac{v_\infty}{c}\right), \tag{B.3}$$

[2] s referring to "slowly expanding and optically thin atmospheres".

and

$$W_0^{*s} = -K\dot{M}\overline{X'^{n-1}}\,\bar{n}(\text{level})\,\frac{A(el)}{v_\infty^2}\,\frac{1}{\left(1+\dfrac{u(L_{\max})}{v_\infty}\right)}. \quad \text{(B.4)}$$

Letting $L_{\max} \to \infty$; this latter result is equivalent to Eq. (31).
For $n = 1$

$$I_1^*(L) = X_L \Big/ \left(\left(1+\frac{u(L_{\max})}{v_\infty}\right)^2 v_{12}\frac{v_\infty}{c}\right), \quad \text{(B.5)}$$

and from this, it is obvious to recover Eq. (32).
For $n = 2$ and $n = 3$, we have successively

$$I_2^*(L) = -\left(X_L^2 + \frac{1}{3}\left(\frac{\bar{u}(L)}{v_\infty}\right)^2\right) \Big/ \left(\left(1+\frac{u(L_{\max})}{v_\infty}\right)^3 v_{12}\frac{v_\infty}{c}\right), \quad \text{(B.6)}$$

and

$$I_3^*(L) = \left(X_L^3 + X_L\left(\frac{\bar{u}(L)}{v_\infty}\right)^2\right) \Big/ \left(\left(1+\frac{u(L_{\max})}{v_\infty}\right)^4 v_{12}\frac{v_\infty}{c}\right), \quad \text{(B.7)}$$

where $\bar{u}(L)^2$ has been defined in Eq. (A.8). Inserting Eqs. (B.6) and (B.7) into (B.1), it is straightforward to recover the results (33) and (34), when $L \to \infty$.

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