Published in: ***IEEE TRANSACTIONS ON MARGNETICS, VOL. 44, NO. 4, APRIL 2008.***
Status: Postprint (Author´s version)
Digital Object Identifier: 10.1109/TMAG.2007.916146

An Iterative Finite Element Perturbation Method for Computing Electrostatic Field Distortions

Mohamed Boutaayamou[1], Ruth V. Sabariego[1] and Patrick Dular[1,2]

Department of Electrical Engineering and Computer Science, [1]University of Liège, Belgium

[2]Fonds de la Recherche Scientifique - FNRS, Belgium

mboutaayamou@ulg.ac.be

Abstract—**A finite element perturbation method is developed for computing electrostatic field distortions and the ensuing charges and forces on moving conductive regions subjected to fixed potentials. It is based on the subsequent solution of an unperturbed problem in a complete domain, where conductive regions have been extracted, and of perturbation problems in sub-domains restricted to the surroundings of the added conductive regions. The solution of the unperturbed problem serves as source (with a very reduced support) for the perturbation sub-problems. For every new position of the conductors, the solution of the unperturbed problem does not vary and is thus re-used, only the perturbation sub-problems have to be solved. Further, this approach allows for the use of independent and well-adapted meshes. An iterative procedure is required if the conductive regions are close to the sources.**

Index Terms—**Electrostatic field distortions and forces, finite element method, perturbation method.**

1. Introduction

THE finite element (FE) modeling of an electrostatically driven moving conductor needs successive computations for each new position. This may be computationally expensive specially when dealing with 3D models. It is worth then benefiting from previous computations instead of starting a completely new FE solution for every new position.

The perturbation method has been shown to be of interest for computing field distortions and induced effects in magnetodynamic [1][2] and electrostatic problems [3]. An unperturbed problem is first solved in a large domain taking advantage of any symmetry and excluding additional regions and thus avoiding their mesh. Its solution is applied as a source to the further computations of the perturbed problems when some regions are added. The method benefits from the use of different subproblem-adapted meshes, what increases the computational efficiency when the size of each sub-problem diminishes [2]. For some positions where the coupling between regions is significant, an iterative procedure is required to obtain an accurate solution. In [3], the perturbation method is applied to consider the introduction in a given domain of conductors subjected to a floating potential. A global-to-local method for static electric field calculations is presented in [4]. The mesh of the local domain is there included in that of the global domain. In the proposed perturbation method, however, the meshes of the perturbing regions are independent of the meshes of the unperturbed domain.

In this paper, the considered conductors are supposed to move with small speed in the absence of any magnetic field in order to satisfy the static field assumption.

The perturbation method is developed herein for solving electrostatic problems with an electric scalar potential FE formulation and computing the ensuing quantities, i.e. the electric field, electric charges and forces. The method is

Published in: ***IEEE TRANSACTIONS ON MARGNETICS, VOL. 44, NO. 4, APRIL 2008.***
Status: Postprint (Author´s version)
Digital Object Identifier: 10.1109/TMAG.2007.916146

extended to account for the addition of conductors subjected to fixed potentials.

By way of validation, a capacitor with an added conductor between its plates is considered.

2. Strong Formulations

A. Unperturbed and Perturbed Electrostatic Problems

We consider an electrostatic problem in a domain Ω, with boundary $\Gamma = \Gamma_e \cup \Gamma_d$, of the 2-D or 3-D Euclidean space. The conductive parts of Ω are denoted Ω_c, with boundary Γ_c. The governing differential equations and constitutive law in Ω are

$$\operatorname{curl} \boldsymbol{e} = 0, \qquad \operatorname{div} \boldsymbol{d} = q, \qquad \boldsymbol{d} = \varepsilon\, \boldsymbol{e}, \tag{1a-b-c}$$

with associated boundary conditions (BCs)

$$\boldsymbol{n} \times \boldsymbol{e} \,|_{\Gamma_e} = 0, \qquad \boldsymbol{n} \cdot \boldsymbol{d} \,|_{\Gamma_d} = 0, \tag{2a-b}$$

where $\boldsymbol{e}$ is the electric field, $\boldsymbol{d}$ is the electric flux density, q is the electric charge density, ε is the electric permittivity and $\boldsymbol{n}$ is the unit normal exterior to Ω. According to (1a), the electric field $\boldsymbol{e}$ can be derived from an electric scalar potential v, i.e. $\boldsymbol{e} = -\operatorname{grad} v$.

Hereafter, the subscripts u and p refer to the unperturbed and perturbed quantities and associated domains, respectively. A so-called unperturbed electrostatic problem is first defined in Ω without considering the conductive region $\Omega_{c,p}$. Using the solution of the unperturbed problem as a source, the perturbed problem is solved in a domain Ω_p, with boundary $\Gamma_p = \Gamma_{e,p} \cup \Gamma_{d,p}$, including the so-called perturbing region $\Omega_{c,p}$ and its neighborhood. Indeed, electrostatic field distortions appear when the perturbing region $\Omega_{c,p}$ is added to the initial configuration.

At the discrete level, two independent meshes are used. A mesh of the whole domain in the absence of some additional conductive regions and a mesh of the perturbing region. The electrostatic problem defined in each domain asks for mesh refinement of different regions [3].

B. Perturbation Problem

Particularizing (1a-b-c) and (2a-b) for both the unperturbed and perturbed quantities, and subtracting the unperturbed equations from the perturbed ones, a perturbation problem is defined in Ω_p (initially in Ω) [1]-[3]. The obtained equations are expressed in terms of the field distortions $\boldsymbol{e} = \boldsymbol{e}_p - \boldsymbol{e}_u$ and $\boldsymbol{d} = \boldsymbol{d}_p - \boldsymbol{d}_u$ as follows

$$\operatorname{curl} \boldsymbol{e} = 0, \quad \operatorname{div} \boldsymbol{d} = 0, \quad \boldsymbol{d} = \varepsilon_p\, \boldsymbol{e} + (\varepsilon_p - \varepsilon_u)\, \boldsymbol{e}_u, \tag{3a-b-c}$$

$$\boldsymbol{n} \times \boldsymbol{e} \,|_{\Gamma_{e,p}} = \boldsymbol{n} \times \boldsymbol{e}_s, \quad \boldsymbol{n} \cdot \boldsymbol{d} \,|_{\Gamma_{d,p}} = \boldsymbol{n} \cdot \boldsymbol{d}_s. \tag{4a-b}$$

In (3b), we assume that no volume charge density exists in Ω_p. BCs (4a-b) on the outer boundary of Ω_p are first defined as homogeneous, i.e. with the given sources $\boldsymbol{n} \times \boldsymbol{e}_s = 0$ and $\boldsymbol{n} \cdot \boldsymbol{d}_s = 0$, to neglect the field distorsion at a certain distance from $\Omega_{c,p}$.

Note that in regions where $\varepsilon_p \neq \varepsilon_u$, an additional source term given by the unperturbed solution $(\varepsilon_p - \varepsilon_u)\, \boldsymbol{e}_u$ is considered in (3c). Because the added region $\Omega_{c,p}$ is a perfect conductor, ε_p and ε_u are equal and denoted ε. Actually, this region is extracted from Ω_p and treated via a BC of type (4a) on its boundary $\Gamma_{c,p}$ (thus added to $\Gamma_{e,p}$), with

$$\boldsymbol{n} \times \boldsymbol{e}_s \,|_{\Gamma_{c,p}} = -\boldsymbol{n} \times \boldsymbol{e}_u \,|_{\Gamma_{c,p}} \tag{5}$$

Published in: ***IEEE TRANSACTIONS ON MARGNETICS, VOL. 44, NO. 4, APRIL 2008.***
Status: Postprint (Author´s version)
Digital Object Identifier: 10.1109/TMAG.2007.916146

or

$$v\,|_{\Gamma_{c,p}} = -v_s \quad \text{with} \quad v_s = v_u\,|_{\Gamma_{c,p}}\ . \tag{6}$$

3. Weak Formulations

A. Unperturbed electric scalar potential formulation

The unperturbed field distribution is first calculated in Ω as the solution of an electric scalar potential formulation, obtained from the weak form of the Laplace equation, i.e. $\mathrm{div}(-\varepsilon\,\mathrm{grad}\,v_u) = 0$, as

$$(-\varepsilon\,\mathrm{grad}\,v_u, \mathrm{grad}\,v')_\Omega - \langle \boldsymbol{n}\cdot\boldsymbol{d}_u, v'\rangle_{\Gamma_d} = 0, \forall v' \in F(\Omega), \tag{7}$$

where $(\cdot,\cdot)_\Omega$ and $\langle\cdot,\cdot\rangle_\Gamma$ denote a volume integral in Ω and a surface integral on Γ of the product of its arguments; F(Ω) is the function space defined on Ω containing the basis functions for v as well as for the test function v' [5]. At the discrete level, F(Ω) is approximated with nodal FEs. The surface integral term in (7) can be associated with a global quantity or used for fixing a natural BC (usually homogeneous for a tangent electric field constraint) on a portion Γ_d of the boundary of Γ.

B. Perturbation electric scalar potential formulation

The source of the perturbation problem v_s (6) is determined in the new added region $\Omega_{c,p}$ through a projection method [6]. Given the conductive nature of the perturbing region, the projection of v_u from its original mesh to that of $\Omega_{c,p}$ is limited to $\Gamma_{c,p}$. It reads

$$\langle \mathrm{grad}\,v_s, \mathrm{grad}\,v'\rangle_{\Gamma_{c,p}} - \langle \mathrm{grad}\,v_u, \mathrm{grad}\,v'\rangle_{\Gamma_{c,p}} = 0, \forall v' \in F(\Gamma_{c,p}), \tag{8}$$

where the function space $F(\Gamma_{c,p})$ contains v_s and its associated test function v'. At the discrete level, v_s is discretized with nodal FEs and is associated to a gauge condition fixing a nodal value in $\Gamma_{c,p}$.

Further, the projection is to be extended to the whole domain $\Omega_{c,p}$ in case of a dielectric perturbing region. We choose to directly project $\mathrm{grad}\,v_u$ in order to assure a better numerical behaviour in the ensuing equations where the involved quantities are also gradients.

The perturbation electrostatic weak formulation in Ω_p is still of the form of (7) and reads

$$(-\varepsilon\,\mathrm{grad}\,v, \mathrm{grad}\,v')_{\Omega_p} - \langle \boldsymbol{n}\cdot\boldsymbol{d}, v'\rangle_{\Gamma_{d,p}} = 0, \forall v' \in F(\Omega_p), \tag{9}$$

with non-homogeneous Dirichlet BC (6) and BCs (4a-b) in homogeneous forms.

When the unperturbed electric field $\boldsymbol{e}_u = -\mathrm{grad}\,v_u$ is needed in the layer of FEs touching $\Gamma_{c,p}$ in $\Omega_p\backslash\Omega_{c,p}$, denoted $\Omega_{l,p}$, the projection (8) of v_u has to be extended only to this transition layer. This way, the computational effort of the projection is also reduced. Having access to $\boldsymbol{e}_u$ and $\boldsymbol{e}$ in this layer allows to compute there the perturbed electric field $\boldsymbol{e}_p$. Both charges and electric forces can thus be calculated on $\Gamma_{c,p}$.

C. Iterative sequence of perturbation problems

For close relative positions where the coupling between the source and perturbing regions is significant, an accurate solution can be obtained via an iterative procedure that calculates successive perturbations not only from the initial source region to the added conductor but also from the latter to the former. Each region gives a suitable correction as a perturbation with an accuracy dependent of the fineness of its mesh.

For each iteration i ($i = 0, 1, ...$), we determine the electric scalar potential v_{2i} in Ω, with $v_0 = v_u$. The projection of this solution from its original mesh to that of the added conductor $\Omega_{c,p}$ gives a source $v_{s,2i+1}$ for a perturbation

Published in: ***IEEE TRANSACTIONS ON MARGNETICS, VOL. 44, NO. 4, APRIL 2008.***
Status: Postprint (Author´s version)
Digital Object Identifier: 10.1109/TMAG.2007.916146

problem. This way, we obtain a potential v_{2i+1} and an electric charge on $\Gamma_{c,p}$ that counterbalance the potential and the electric charge on Γ_c. For iteration $i+1$, a new source $v_{s,2i+2}$ for the initial configuration has to be then calculated. This is done by projecting v_{2i+1} from its support mesh to that of Ω. This projection can be limited to the layer of FEs touching Γ_c in $\Omega \backslash \Omega_c$, denoted Ω_l, i.e.

$$(\operatorname{grad} v_{s,2i+2}, \operatorname{grad} v')_{\Omega_l} - (\operatorname{grad} v_{2i+1}, \operatorname{grad} v')_{\Omega_l} = 0, \forall v' \in F(\Omega_l). \tag{10}$$

A perturbation electric scalar potential problem is defined in Ω as

$$(-\varepsilon \operatorname{grad} v_{2i+2}, \operatorname{grad} v')_{\Omega} - \langle \boldsymbol{n} \cdot \boldsymbol{d}_{2i+2}, v' \rangle_{\Gamma_d} = 0, \forall v' \in F(\Omega), \tag{11}$$

with non-homogeneous Dirichlet BC $v_{2i+2} = -v_{s,2i+2} \mid_{\Gamma_c}$ and Neumann BC $\boldsymbol{n} \cdot \boldsymbol{d}_{2i+2} \mid_{\Gamma_d} = -\boldsymbol{n} \cdot \boldsymbol{d}_{2i+1} \mid_{\Gamma_d}$. The latter is not known in a strong sense. The associated surface integral term in (11) can be evaluated rather via the weak formulation of problem $2i+1$ now applied to Ω_l, as

$$\begin{aligned} \langle \boldsymbol{n} \cdot \boldsymbol{d}_{2i+2}, v' \rangle_{\Gamma_d} &= -\langle \boldsymbol{n} \cdot \boldsymbol{d}_{2i+1}, v' \rangle_{\Gamma_d} \\ &= -(-\varepsilon \operatorname{grad} v_{s,2i+2}, \operatorname{grad} v')_{\Omega_l} \forall v' \in F(\Omega_l), \end{aligned} \tag{12}$$

benefiting from the projection $v_{s,2i+2}$ of v_{2i+1} already done by (10).

This iterative process is repeated until convergence for a given tolerance.

D. Electrostatic charges

A suitable treatment of the surface integral term in (9) consists in naturally defining a global electric charge in the weak sense [5]. A test function v' is chosen as equal to one on $\Gamma_{c,p}$ and as continuously varying towards 0 in the layer $\Omega_{l,p}$ of FEs touching $\Gamma_{c,p}$ in $\Omega_p \backslash \Omega_{c,p}$. The electric charge can be naturally calculated at the post-processing stage as the volume integral in (9) limited to $\Omega_{l,p}$ as

$$Q_p = Q_u + Q, \tag{13}$$

with $Q_u = -(-\varepsilon \operatorname{grad} v_s, \operatorname{grad} v')_{\Omega_{l,p}}$,

and $Q = -(-\varepsilon \operatorname{grad} v, \operatorname{grad} v')_{\Omega_{l,p}}$.

The calculation of Q_u needs to use the unperturbed potential v_u expressed in the mesh of $\Omega_{l,p}$, which is actually the projected potential v_s. This justifies the interest of projecting v_u in $\Omega_{l,p}$.

In case of iterative sequence of perturbation problems, the total charge on $\Gamma_{c,p}$ is given by the summation of elementary charges obtained at each iteration i.

E. Electrostatic forces

As previously mentioned, the perturbed electric field $\boldsymbol{e}_p$ can be computed in the layer $\Omega_{l,p}$ of FEs touching $\Gamma_{c,p}$ in $\Omega_p \backslash \Omega_{c,p}$. The electric force distribution is calculated thus by locally applying the virtual work principle [7] in this transition layer. At the discrete level, the force at each node of $\Gamma_{c,p}$ is obtained by deriving the electric energy in the considered layer of FEs with respect to a virtual displacement. The contribution of a reference element Δ to the force in a given direction is

$$F_r = \int_{\Delta} \frac{\varepsilon}{2} (-2\, \boldsymbol{e}_p\, J^{-1} \frac{\partial J}{\partial u}\, \boldsymbol{e}_p + \boldsymbol{e}_p\, \boldsymbol{e}_p \frac{\partial |J|}{\partial u})\, d\Delta, \tag{14}$$

for a virtual displacement r in this direction. J is the geometrical Jacobian matrix with determinant $|J|$.

Given the non-linearity of the force, a direct summation of the forces at each iteration is not possible. The total electric field $\boldsymbol{e}_p$ has to be updated at each iteration before computing the total force by (14).

Published in: ***IEEE TRANSACTIONS ON MARGNETICS, VOL. 44, NO. 4, APRIL 2008.***
Status: Postprint (Author´s version)
Digital Object Identifier: 10.1109/TMAG.2007.916146

4. Application

In order to illustrate and validate the iterative perturbation method for electrostatic field distortions, we consider a parallel-plate capacitor (plate length and plate separation 200 μm). The conductive parts Ω_c of the capacitor are two electrodes between which the electric potential difference is $\Delta \mathrm{V} = 1$ V (upper electrode fixed to 1 V). A square conductor ($\mathrm{side} = 20$ μm, at fixed potential 0.7 V) is considered as a perturbing region $\Omega_{c,p}$ inside the capacitor. The geometry of this conductor accounts for fringing field effects. Its distance from the upper electrode is denoted d.

Fig. 1 shows examples of meshes for the unperturbed and perturbation problems. An adapted mesh, specially fine in the vicinity of the corners of the perturbing conductor, is used. Note that any intersection of perturbation problem boundaries with the unperturbed problem material regions is allowed.

Fig. 2 illustrates the sequence of associated solutions to be considered with the developed perturbation method, i.e. the unperturbed fields v_u and $\boldsymbol{e}_u$, the perturbation fields v and $\boldsymbol{e}$, and the perturbed fields v_p and $\boldsymbol{e}_p$.

For the position $d = 20$ μm, a sequence of perturbation problems has been carried out. The relative error of v_p and the y-component of $\boldsymbol{e}_p$ computed near the conductor at some iterations is depicted in Fig. 3.

The relative error for v_p and $\boldsymbol{e}_p$ is below 1% at iteration 15 and 23, respectively. Note that v_p converges faster than $\boldsymbol{e}_p$ (Fig. 3). Applying the Aitken acceleration to the iterative procedure speeds up the convergence. In that case, only 5 iterations are required for both v_p and $\boldsymbol{e}_p$ (Fig. 4).

At odd iterations, the electric charge Q_p and the electric forces on the outer surface of the conductor are computed. In order to compare with the FE solution, we consider the summation F_p of the values of the y-component of the electric forces at each node of half the sides of the conductor from one corner facing electrode at 1 V. Fig. 5 shows the convergence of Q_p and F_p as a function of iteration i.

The relative error for Q_p and F_p is below 1% at iteration 25 and 27, respectively. Note that Q_p and F_p have converged nearly at the same time as $\boldsymbol{e}_p$. When the Aitken acceleration is used, the number of iterations is reduced to 5 and 7, respectively.

Several relative positions d of the conductor are listed in Table I. For each of them, the perturbation problem is solved and an iterative process is carried out till convergence (i.e. the relative error of v_p, $\boldsymbol{e}_p$, Q_p and F_p below 1%). The number of iterations to achieve convergence without and with Aitken acceleration is shown as well.

As expected, more iterations are needed when the conductor is close to an electrode. It can be seen that $\boldsymbol{e}_p$, Q_p and F_p need nearly the same number of iterations to achieve convergence at each considered position d. Furthermore, Aitken acceleration has proven to be very efficient.

TABLE I

NUMBER OF ITERATIONS TO ACHIEVE CONVERGENCE WITHOUT AND WITH AITKEN ACCELERATION (BETWEEN PARENTHESES) FOR SEVERAL DISTANCES (CONDUCTOR AT $0.7\ V$ AND ELECTRODE AT $1\ V$).

$d(\mu\mathrm{m})$	$v_p(\mathrm{V})$	$\boldsymbol{e}_p(\mathrm{V/m})$	$F_p(\mathrm{N})$	$Q_p(\mathrm{C})$
5	39(7)	61(7)	69(9)	61(9)
10	25(5)	37(7)	43(9)	39(9)
20	15(3)	23(5)	27(7)	25(7)
50	1(-)	5(5)	7(7)	19(7)

Published in: ***IEEE TRANSACTIONS ON MARGNETICS, VOL. 44, NO. 4, APRIL 2008.***
Status: Postprint (Author´s version)
Digital Object Identifier: 10.1109/TMAG.2007.916146

5. Conclusion

An iterative finite element perturbation method has been developed for efficiently computing electrostatic field distortions and forces on moving conductors. First, an unperturbed problem (in the absence of some perturbing regions) is solved with the conventional FE method in the complete domain. Second, a perturbation problem is solved in a reduced region with an additional conductor using the solution of the unperturbed problem as a source. Benefits of the projections in reduced supports around some boundaries are pointed out. This way, charges and forces are computed. This approach uses independent meshes which are adapted for each associated problem. A projection from one mesh to another is used to feed a given problem with its sources. For close relative positions where the coupling between the source and perturbing regions is significant, an accurate solution can be obtained via an iterative procedure. The application of Aitken acceleration to the iterative process has been proven to be very efficient.

Acknowledgment

This work is supported by the Belgian French Community (ARC 03/08-298) and the Belgian Science Policy (IAPP 6/21).

References

1. Z. Badics et al., "An effective 3-D finite element scheme for computing electromagnetic field distortions due to defects in eddy-current nondestructive evaluation," *IEEE Trans. Magn.*, vol. 33, no. 2, pp. 1012−1020, 1997.
2. P. Dular and R.V. Sabariego, "A perturbation method for computing field distorsions due to conductive regions with h-conform magnetodynamic finite element formulations," *IEEE Trans. Magn.*, vol. 43, no. 4, pp. 1293−1296, 2007.
3. M. Boutaayamou, R.V. Sabariego, P. Dular, "A Perturbation Method for the 3D Finite Element Modeling of Electrostatically Driven MEMS," *8th Proc. EuroSimE 2007*, London, pp. 50−54, 2007.
4. I. Sebestyen, "Electric Field Calculation for HV Insulators Using Domain Decomposition Method," *IEEE Trans. Magn.*, vol. 38, no. 2, pp. 1213−1216, 2002.
5. P. Dular, W. Legros, A. Nicolet, "Coupling of local and global quantities in various finite element formulations and its application to electrostatics, magnetostatics and magnetodynamics," *IEEE Trans. Magn.*, vol. 34, no. 5 , pp. 3078−3081, 1998.
6. C. Geuzaine, B. Meys, F. Henrotte, P. Dular, W. Legros, "A Galerkin projection method for mixed finite elements," *IEEE Trans. Magn.*, vol. 35, no. 3, pp. 1438−1441, 1999.
7. J. L. Coulomb and G. Meunier, "Finite element implementation of virtual work principle for magnetic or electric force and torque computation," *IEEE Trans. Magn.*, vol. 20, no. 5 , pp. 1894−1896, 1984.

Published in: ***IEEE TRANSACTIONS ON MARGNETICS, VOL. 44, NO. 4, APRIL 2008.***
Status: Postprint (Author´s version)
Digital Object Identifier: 10.1109/TMAG.2007.916146

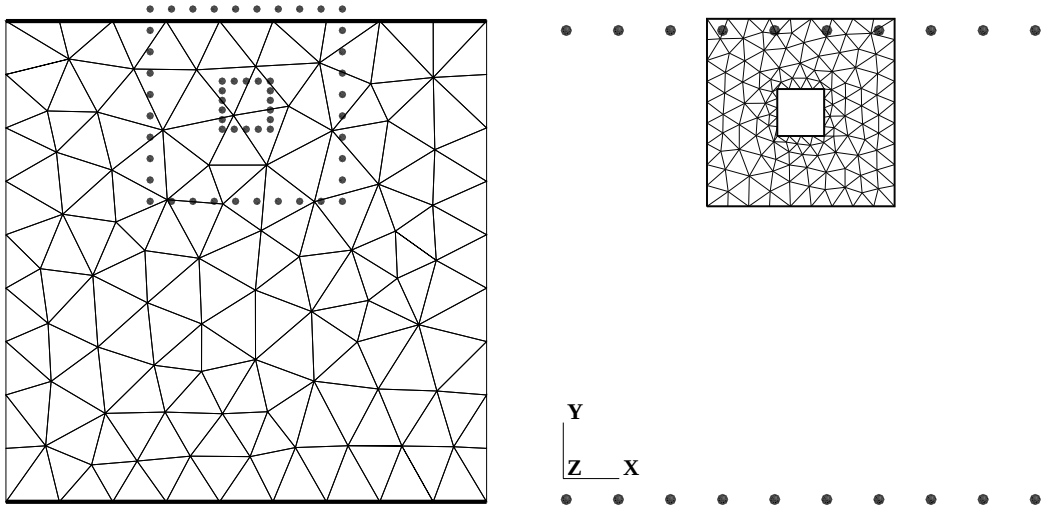

Fig. 1. Mesh of Ω (*left*) and adapted mesh of Ω_p (*right*).

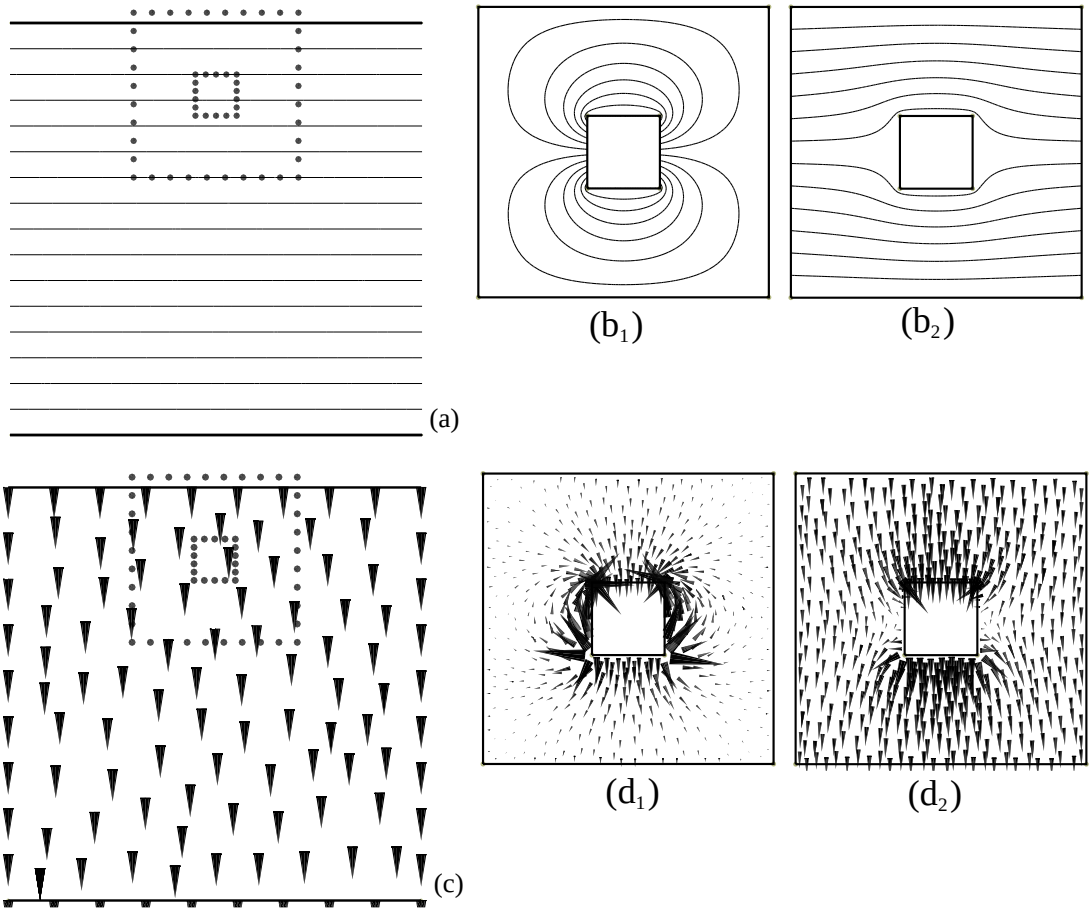

Fig. 2. The unperturbed fields v_u (a) and $\boldsymbol{e}_u$ (c); the perturbation fields v (b_1) and $\boldsymbol{e}$ (d_1); the perturbed fields v_p (b_2) and $\boldsymbol{e}_p$ (d_2).

Published in: ***IEEE TRANSACTIONS ON MARGNETICS, VOL. 44, NO. 4, APRIL 2008.***
Status: Postprint (Author´s version)
Digital Object Identifier: 10.1109/TMAG.2007.916146

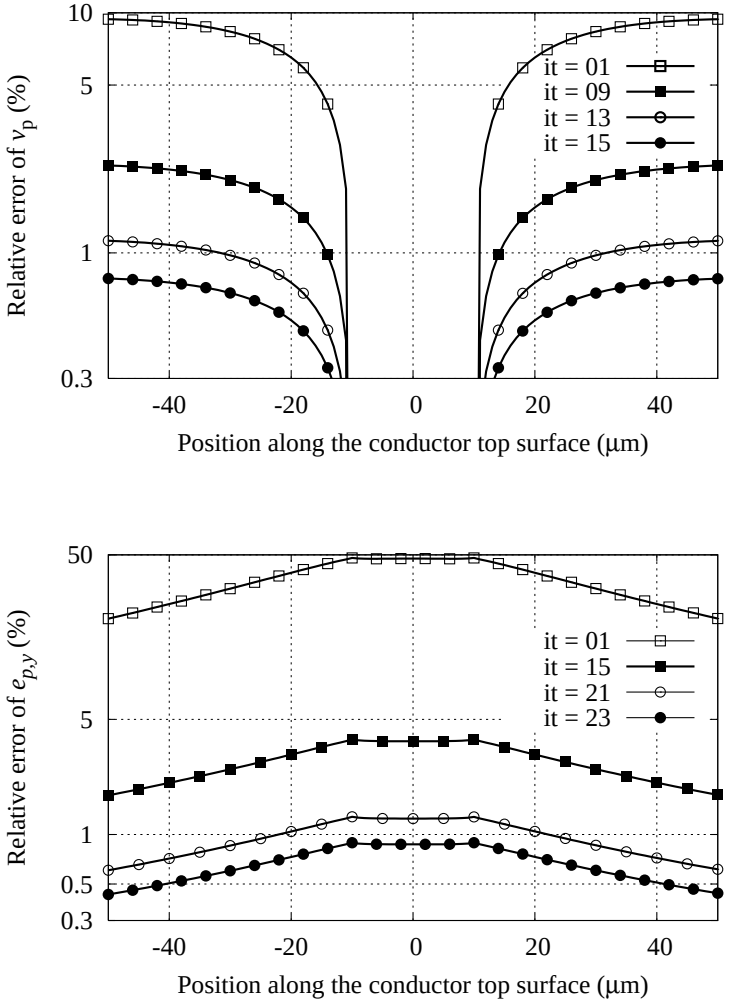

Fig. 3. Relative error of v_p (*top*) and $\boldsymbol{e}_p$ (y-component) (*bottom*) computed along the conductor top surface for some iterations.

Published in: ***IEEE TRANSACTIONS ON MARGNETICS, VOL. 44, NO. 4, APRIL 2008.***
Status: Postprint (Author´s version)
Digital Object Identifier: 10.1109/TMAG.2007.916146

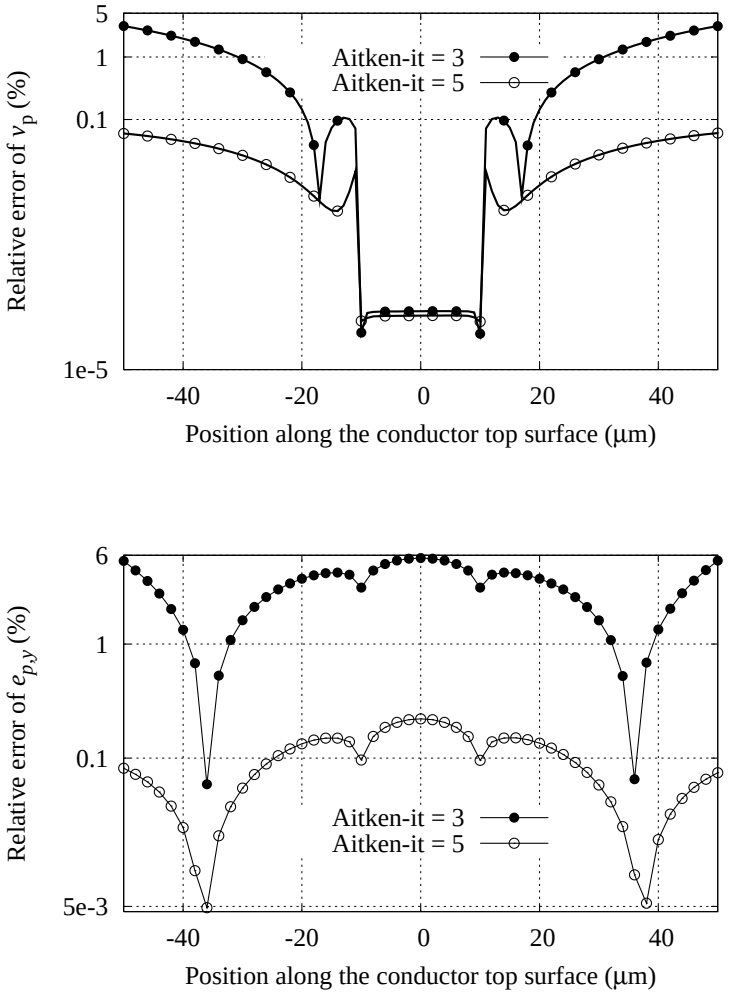

Fig. 4. Relative error of v_p (*top*) and $\boldsymbol{e}_p$ (y-component) (*bottom*) computed along the conductor top surface for some iterations with Aitken acceleration.

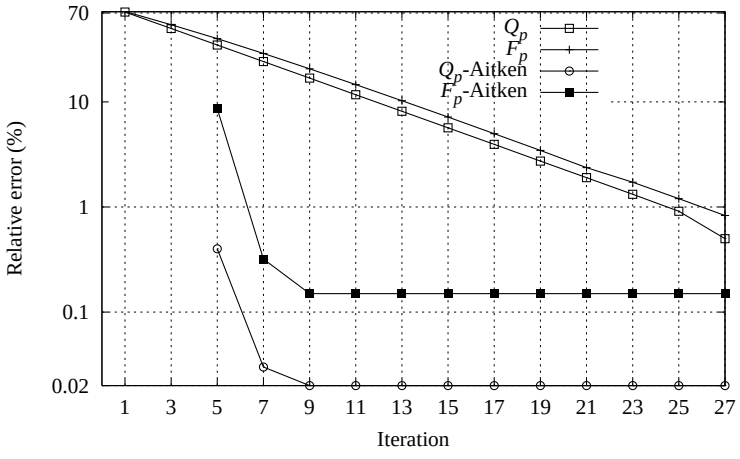

Fig. 5. Relative error of Q_p and F_p as a function of iteration number i.