Analysis and Design of Telecommunications Systems: prerequisite

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Abstract

This document is the prerequisite for Analysis and Design of Telecommunications Systems taught at the University of Liège. It is intended to be a reminder of some basic knowledge.

For a textbook, I recommend: *Digital and Analog Communication Systems*, by L. Couch, Prentice Hall [1].

Contents

1 Fourier transform and spectrum analysis

Definition 1. [FOURIER transform] Let $g(t)$ be a deterministic signal (also called a *waveform*), the FOURIER transform of $g(t)$, denoted $\mathcal{G}$, is

$$\mathcal{G}(f) = \int_{-\infty}^{+\infty} g(t)e^{-2\pi jft}dt \tag{1.1}$$

where f is the frequency parameter with units in *Hertz*, Hz (that is $\frac{1}{s}$). It is also related to angular pulsation ω by $\omega = 2\pi f$.

Note that f is the parameter of the FOURIER transform $\mathcal{G}(f)$, that is also called *(two-sided) spectrum* of $g(t)$, because it contains both positive and negative components.

Definition 2. [Inverse FOURIER transform] The time waveform $g(t)$ is obtained by taking the inverse FOURIER transform of $\mathcal{G}(f)$, defined as follows

$$g(t) = \int_{-\infty}^{+\infty} \mathcal{G}(f)e^{2\pi jft}dt \tag{1.2}$$

The functions $g(t)$ and $\mathcal{G}(f)$ are said to constitute a FOURIER transform pair, because they are two representations of a same signal.

1.1 Properties

Proposition 3. [Spectral symmetry of real signals] *If $g(t)$ is real, then*

$$\mathcal{G}(-f) = \mathcal{G}^*(f) \tag{1.3}$$

where $()^*$ denotes the conjugate operation.

A consequence is that the negative components can be obtained from the positives ones. Therefore, the positive components are sufficient to describe the original waveform $g(t)$.

Proposition 4. [RAYLEIGH] *The total energy in the time domain and the frequency domain are equal:*

$$E = \int_{-\infty}^{+\infty} |g(t)|^2 \, dt = \int_{-\infty}^{+\infty} \|\mathcal{G}(f)\|^2 \, df \tag{1.4}$$

After integration, $\|\mathcal{G}(f)\|^2$ provides the total energy of the signal in Joule, J. Therefore, it is sometimes named *energy spectral density.*

Proposition 5. [Linearity] *If $g(t) = c_1g_1(t) + c_2g_2(t)$ then $\mathcal{G}(f) = c_1\mathcal{G}_1(f) + c_2\mathcal{G}_2(f)$.*

This property of linearity is essential in telecommunications, because most systems (channels, filters, etc) are linear. The consequence is that any linear system can only modify the spectral content a signal, but is incapable to add new frequencies.

This property also highlights that a spectrum analysis is adequate for dealing with linear systems. To the contrary, it is more difficult to analyze a non-linear system (for example a squaring operator, $g^2(t)$) in terms of frequencies.

Operation	Function	FOURIER transform
Conjugate symmetry	$g(t)$ is real	$\mathcal{G}(f) = \mathcal{G}^*(-f)$
Linearity	$c_1 g_1(t) + c_2 g_2(t)$	$c_1 \mathcal{G}_1(f) + c_2 \mathcal{G}_2(f)$
Time scaling	$g(at)$	$\frac{1}{\|a\|} \mathcal{G}\left(\frac{f}{a}\right)$
Time shift (delay)	$g(t - t_0)$	$\mathcal{G}(f)\, e^{-2\pi j f t_0}$
Convolution	$g(t) \otimes h(t) = \int_{-\infty}^{+\infty} g(\tau) h(t-\tau)\, d\tau$	$\mathcal{G}(f)\mathcal{H}(f)$
Frequency shift	$g(t)\, e^{2\pi j f_c t}$	$\mathcal{G}(f - f_c)$
Modulation	$g(t) \cos(2\pi f_c t)$	$\frac{\mathcal{G}(f-f_c)+\mathcal{G}(f+f_c)}{2}$
Temporal derivation	$\frac{d}{dt} g(t)$	$2\pi j f\, \mathcal{G}(f)$
Temporal integration	$\int_{-\infty}^{t} g(\tau)\, d\tau$	$\frac{1}{2\pi j f}\, \mathcal{G}(f)$

Table 1: Main properties of the FOURIER transform.

1.2 Specific signals

Some waveforms play a specific role in communications.

1. The DIRAC *delta function*, denoted $\delta(t)$, is defined by

$$\delta(t) = \begin{cases} \infty, & x = 0 \\ 0 & x \neq 0 \end{cases} \tag{1.5}$$

and

$$\int_{-\infty}^{+\infty} \delta(t)\, dt = 1 \tag{1.6}$$

The DIRAC delta function is not a true function, so it is said to be a singular function. However, it is considered as a function in the more general framework of the theory of distributions. Note that

$$g(t) \otimes \delta(t) = g(t) \tag{1.7}$$

According to equation 1.6,

$$\int_{-\infty}^{+\infty} \delta(t) e^{-2\pi j f t} = e^0 = 1 \tag{1.8}$$

As a consequence, 1 is the FOURIER transform of $\delta(t)$. Likewise, $\delta(f)$ is the FOURIER transform of 1. This explains the central role of the DIRAC delta function in signal processing and communications.
The DIRAC delta function is also called the *unit impulse function*.

2. *Rectangular pulse*

$$\mathrm{rect}(t) = \begin{cases} 1 & -\frac{1}{2} < t < \frac{1}{2} \\ 0 & |t| > \frac{1}{2} \end{cases} \tag{1.9}$$

The rectangular pulse is the most common shape form for the representation of digital signals.

3. *Step function* (HEAVISIDE function)

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases} \tag{1.10}$$

4. *Sign function*

$$\text{sign}(t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases} \tag{1.11}$$

5. *Sinc function*

$$sinc(t) = \frac{\sin(\pi t)}{\pi t} \tag{1.12}$$

1.3 FOURIER transform pairs

Temporal waveform	FOURIER transform
$\delta(t)$	1
1	$\delta(f)$
$\cos(2\pi f_c t)$	$\frac{1}{2}[\delta(f-f_c)+\delta(f+f_c)]$
$\sin(2\pi f_c t)$	$\frac{1}{2j}[\delta(f-f_c)-\delta(f+f_c)]$
$\text{rect}\left(\frac{t}{T}\right)$	$T\,sinc(fT)$
$sinc(2Wt)$	$\frac{1}{2W}\,\text{rect}\left(\frac{f}{2W}\right)$
$\text{sign}(t)$	$\frac{1}{\pi j f}$
$u(t)$	$\frac{1}{2}\delta(f)+\frac{1}{2\pi j f}$
$\frac{1}{\pi t}$	$-j\,\text{sign}(f)$
$\sum_{i=-\infty}^{+\infty}\delta\left(t-iT_0\right)$	$\frac{1}{T_0}\sum_{n=-\infty}^{+\infty}\delta\left(f-\frac{n}{T_0}\right)$
$e^{-\pi t^2}$	$e^{-\pi f^2}$
$e^{-at}u(t),\ a>0$	$\frac{1}{a+2\pi j f}$
$e^{-a\|t\|},\ a>0$	$\frac{2a}{a^2+(2\pi f)^2}$

Table 2: Some common FOURIER transform pairs.

2 Signals

2.1 Continuous-time and discrete-time signals

A *continuous-time* signal $g(t)$ is a signal of the real variable t (time). For example, the signal $\cos(2\pi f_c t)$ is a function of time.

A *discrete-time* signal $g[n]$ is a sequence where the values of the index n are integers. Such a signal carries digital information.

2.2 Analog and digital signals

An *analog* signal is a signal with a continuous range of amplitudes.

A *digital* signal is a member of a set of M unique signals, defined over the T period, that represent M data symbols.

Figure 2.1 represents an analog and a digital signal (top row). In practice, digital signals are "materialized" by a continuous-time signal, named a *representation*. Such representations are illustrated in Figure 2.1 (bottom row).

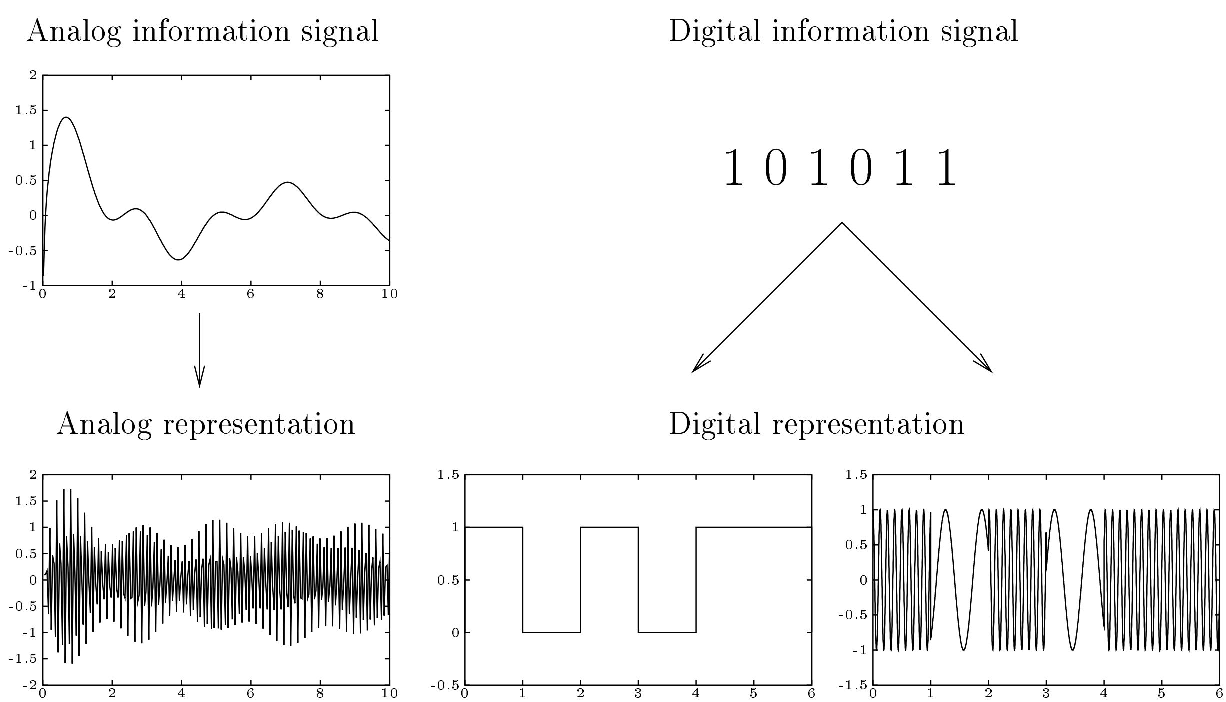

Figure 2.1: Examples of representations of an continuous-time signal (left) and a discrete-time signal (right).

2.3 Energy and power

The *power* in watts $[W]$ delivered by a voltage signal $v(t)$ in volts to a resistive load R in ohms $[\Omega]$ is given by

$$P = v(t)\, i(t) = \frac{v^2(t)}{R} = R\, i^2(t) \tag{2.1}$$

where $i(t)$ is the current.

In communications, the value of the resistive load is normalized to $1\,[\Omega]$. Therefore, the *instantaneous normalized power* is given by

$$p(t) = v^2(t) = i^2(t) \tag{2.2}$$

The *averaged normalized power* is given by

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} v^2(t)\, dt \tag{2.3}$$

A signal $v(t)$ is called a *power* signal if and only if its averaged power is non-zero and finite, that is $0 < P < \infty$.

Example 6. *Power of a sinusoid* $A\cos(2\pi f_c t)$. The averaged normalized power delivered to a $1\,[\Omega]$ load is

$$\begin{aligned} P &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} A^2 \cos^2(2\pi f_c t)\; dt & (2.4)\\ &= \lim_{T\to\infty} \frac{A^2}{4T} \int_{-T}^{T} (1+\cos(4\pi f_c t)\; dt & (2.5)\\ &= \frac{A^2}{2} + \lim_{T\to\infty} \frac{A^2}{4T} \left(\frac{\sin(4\pi f_c T)}{4\pi f_c} - \frac{\sin(-4\pi f_c T)}{4\pi f_c} \right) & (2.6)\\ &= \frac{A^2}{2} + \frac{A^2}{2} \lim_{T\to\infty} \frac{\sin(4\pi f_c T)}{4\pi f_c T} = \frac{A^2}{2} & (2.7) \end{aligned}$$

Therefore, $\frac{A^2}{2}$ is the power of a sinusoid $A\cos(2\pi f_c t)$. The sinusoid is a power signal.

The *energy* in Joules of a voltage signal $v(t)$ in volts is given by

$$E = \int_{-\infty}^{+\infty} v^2(t)\, dt \tag{2.8}$$

A signal $v(t)$ is called an *energy* signal if and only if its energy is non-zero and finite, that is $0 < E < \infty$.

2.4 Deterministic vs stochastic tools

As shown in Figure 2.2, the deterministic or stochastic nature of signals depends on the signal and the location in the communication chain.

	transmitter	receiver
User's signal	*deterministic*	*stochastic*
Noise and interference	*stochastic*	*stochastic*

Figure 2.2: Deterministic or stochastic nature of signals.

In fact, only the user's signal at the transmitter is fully known; it would make no sense to send a signal that would be known by the receiver, at least from a communication engineer's point of view.

Therefore, we need to adapt the tools for describing signals to their intrinsic nature. More precisely, it appears that stochastic signals can only be described in terms of statistics (mean, average, autocorrelation function, etc).

Figure 2.3 presents the tools used for describing the power of a signal according to its intrinsic nature.

	deterministic	*stochastic*
signal to consider	voltage / current	power
power analysis	instantaneous power $p(t) = \frac{\lvert v(t)\rvert^2}{R} = R\,\lvert i(t)\rvert^2$	Power Spectral Density (PSD) $E\left\{X^2(t)\right\} = \int_{-\infty}^{+\infty} \gamma_X(f)df$

Figure 2.3: Description of power adapted to the intrinsic nature of signals.

2.5 Decibel

The decibel is a base 10 logarithm measure, used mainly for powers:

$$x \leftrightarrow 10\log_{10}(x)\,[dB] \tag{2.9}$$

When describing powers, decibels should be expressed in dB of watts: dBW. Note that dB is often a shortcut of dBW.

Typical values are given in the following table:

$x\,[W]$	$10\log_{10}(x)\,[dBW]$
$1\,[W]$	$0\,[dBW]$
$2\,[W]$	$3\,[dBW]$
$0,5\,[W]$	$-3\,[dBW]$
$5\,[W]$	$7\,[dBW]$
$10^n\,[W]$	$10\times n\,[dBW]$

Example 7. Power conversion in $[dB]$. Assume $P = 25\,[W]$. Because $25 = 100/2/2$, we have that

$$10\log_{10}(25) = 10\log_{10}(100) - 10\log_{10}(2) - 10\log_{10}(2) \tag{2.10}$$
$$= 20 - 3 - 3 = 14\,[dBW] \tag{2.11}$$

2.6 Digitization of analog signals

A waveform $g(t)$ is said to be *band-limited* to B hertz if

$$\mathcal{G}(f) = 0, \text{ for } |f| \geq B \tag{2.12}$$

2.6.1 Sampling theorem

Theorem 8. [SHANNON] *Any physical waveform $w(t)$, band-limited to B hertz, can be entirely represented by the following samples series*

$$g\left[\frac{n}{f_s}\right] \tag{2.13}$$

where n is an integer and f_s is the sampling, if

$$f_s \geq 2B \tag{2.14}$$

The condition $f_s \geq 2B$ is named the NYQUIST criterion.

2.6.2 Impulse sampling

The *impulse-sampled* series of a waveform is obtained by multiplying it with a train of unit-weight impulses:

$$g_s(t) = g(t) \sum_{n=-\infty}^{+\infty} \delta\left(t - nT_s\right) \tag{2.15}$$

The FOURIER transform of $g_s(t)$ is then

$$\mathcal{G}_s(f) = \mathcal{G}(f) \otimes \frac{1}{T_s} \sum_{n=-\infty}^{+\infty} \delta\left(f - \frac{n}{T_s}\right) \tag{2.16}$$

$$= f_s \sum_{n=-\infty}^{+\infty} \mathcal{G}\left(f - nf_s\right) \tag{2.17}$$

The spectrum of the impulse sampled signal is the spectrum of the unsampled signal that is *repeated every* f_s Hz, where f_s is the sampling frequency. This is shown in Figure 2.4.

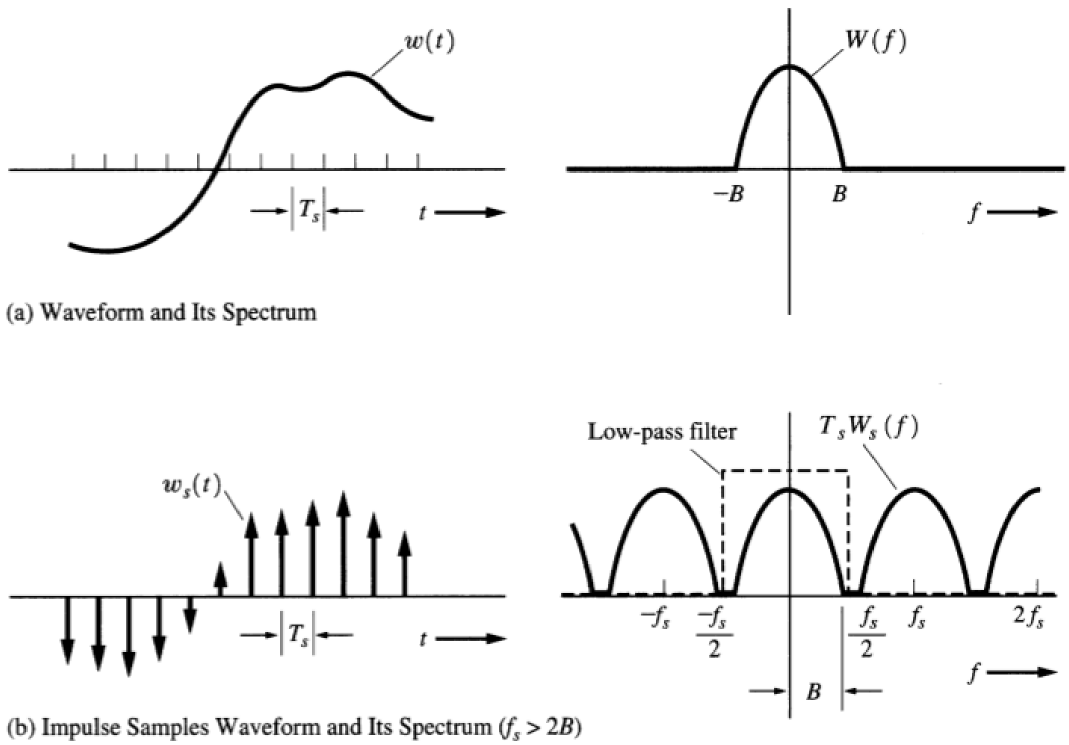

Figure 2.4: Effects of impulse sampling on a waveform $w(t)$ [1].

3 Linear systems

3.1 Linear time-invariant systems

An electronic filter or system $\psi(t)$ is *linear* when the principle of superposition holds. That is when the output $y(t)$ to a combination of inputs follows

$$y(t) = \psi\left(ag_1(t) + bg_2(t)\right) = a\psi\left(g_1(t)\right) + b\psi\left(g_2(t)\right) \tag{3.1}$$

The system $\psi(t)$ is *time-invariant* if, for any delayed input $g(t-\tau)$, the output is also delayed by the same amount $y(t-\tau)$.

3.2 Impulse response and transfer function

The *impulse response* to a filter is the response $h(t)$ when the input is a forcing DIRAC delta function.

The *transfer function* or *frequency response* is the FOURIER transform of the impulse response, $\mathcal{H}$.

Using the convolution theorem, we get that if $y(t)$ is the output of a filter expressed by its impulse response $h(t)$ to an input $g(t)$, then

$$\mathcal{Y}(f) = \mathcal{H}(f)\mathcal{G}(f) \tag{3.2}$$

3.3 Distortionless transmission

In communication, a *distortionless channel* or *ideal channel* is a channel whose output is a proportion of the delayed version of the input

$$y(t) = Ag(t-\tau) \tag{3.3}$$

The corresponding frequency response of an ideal channel is then

$$\frac{\mathcal{Y}(f)}{\mathcal{G}(f)} = Ae^{-2\pi jf\tau} \tag{3.4}$$

4 Random variables and stochastic processes

4.1 Gaussian random variable

The *Gaussian distribution*, also known as the *normal distribution*, is one of the most (if not the most) important distribution.

Definition 9. The probability density function (pdf) of a *Gaussian* distribution is given by

$$\mathrm{pdf}_X = f_X(x) = \frac{1}{\sigma_X\sqrt{2\pi}} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}} \tag{4.1}$$

where μ_X and σ_X^2 are the mean and variance respectively.

From this expression, we can see that the mean and variance of X suffice to characterize a Gaussian random variable completely. The Gaussian is also important because of the statistical of law of large numbers. Basically, this law states that the average of independent random variables of equal mean tends to a Gaussian distribution. Therefore it is a good approximation for the sum of a number of independent random variables with arbitrary one-dimensional probability density functions.

The Gaussian character of a distribution is preserved by linear operations, as stated hereafter.

Proposition 10. *If the* input *of a linear system is Gaussian, then the* output *is also a Gaussian.*

4.2 Stochastic processes

A real *random process* (or *stochastic process*) is an indexed set of real functions of some parameter (usually time) that has certain statistical properties.

As shown in Figure 4.1, each trajectory represents a possible path, named *observation* or *realization*. Because there are many trajectories, several values are possible for each time t. The common choice is to concentrate the information of possible values at a given time t_1, by a random variable $X(t_1)$. When considering all these random variables, we get the stochastic process $X(t)$.

4.2.1 Stationarity

Let $X(t)$ be a stochastic process (note that we use a "capital" letter X for stochastic processes). Because stochastic processes $X(t)$ are a collection of random variables, it is interesting to analyze how they compare over time. This leads to the concept of autocorrelation and stationarity.

Definition 11. [Autocorrelation function] The autocorrelation of a real stochastic process is defined as

$$\Gamma_{XX}(t_1, t_2) = E\{X(t_1)X(t_2)\} \tag{4.2}$$

The autocorrelation makes a first link between two random variables of $X(t)$ taken at different times.

Definition 12. A random process is said to be *Wide-Sense Stationary [WSS]* if

1. $\mu_X(t) =$ constant; the mean of the process does not depend on time.
2. $\Gamma_{XX}(t_1, t_2) = \Gamma_{XX}(\tau)$, where $\tau = t_1 - t_2$.

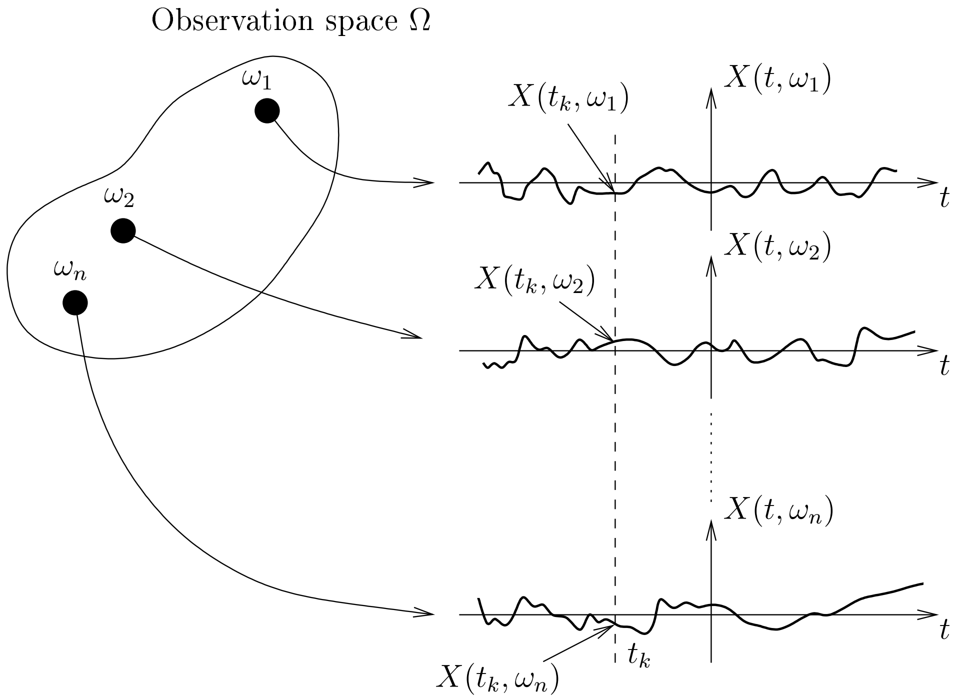

Figure 4.1: Possible trajectories of a stochastic process.

The autocorrelation function of stationary stochastic processes is an essential tool because, for $\tau = 0$, it expresses the average power:

$$\Gamma_{XX}(\tau = 0) = E\left\{X^2(t)\right\} \tag{4.3}$$

In practice, we thus consider that the power P_X of a stochastic process is given by:

$$P_X = E\left\{X^2(t)\right\} \tag{4.4}$$

Consequently, the FOURIER transform of the autocorrelation provides the power distribution in the frequency domain. This leads to the notion of power spectral density of wide-sense stationary stochastic processes as defined hereafter.

Definition 13. [Power spectrum or power spectral density of a stationary process]

$$\gamma_X(f) = \int_{-\infty}^{+\infty} \Gamma_{XX}(\tau)\, e^{-2\pi j f \tau} d\tau \tag{4.5}$$

In practice,

$$P_X = E\left\{X^2(t)\right\} = \int_{-\infty}^{+\infty} \gamma_X(f) df \tag{4.6}$$

Therefore, $\gamma_X(f)$ expresses the contribution of each frequency to the total power.

Example 14. Let us consider a signal with a random phase $\theta \in [0,\, 2\pi]$ or $[-\pi, +\pi]$

$$X(t) = A_c \cos(2\pi f_c t + \Theta) \tag{4.7}$$

This is typical for the carrier signal of a modulated signal. The mean of $X(t)$ is computed as

$$\mu_X(t) = E\left\{X(t)\right\} = \int_{-\pi}^{+\pi} A_c \cos\left(2\pi f_c t + \theta\right) \frac{1}{2\pi}\, d\theta = 0 \tag{4.8}$$

The autocorrelation is obtained as follows:

$$\begin{aligned}
\Gamma_{XX}\left(t_1, t_2\right) &= E\left\{X(t_1)X(t_2)\right\} && (4.9)\\
&= \int_{-\pi}^{+\pi} A_c \cos\left(2\pi f_c t_1 + \theta\right) A_c \cos\left(2\pi f_c t_2 + \theta\right) \frac{1}{2\pi}\, d\theta && (4.10)\\
&= \frac{A_c^2}{2} \cos\left[2\pi f_c (t_2 - t_1)\right] = \frac{A_c^2}{2} \cos\left[2\pi f_c \tau\right] && (4.11)
\end{aligned}$$

We then conclude that the signal is wide-sense stationary and compute its power spectral density:

$$\gamma_X(f) = \frac{A_c^2}{4}\left[\delta(f - f_c) + \delta(f + f_c)\right] \tag{4.12}$$

4.2.2 Power spectral density and linear systems (= filtering)

Consider a wide-sense stationary process $X(t)$, a linear system whose transfer function is given by $\mathcal{H}(f)$, and $Y(t)$ the output process.

Theorem 15. *The mean of $Y(t)$ is given by:*

$$\mu_Y = \mu_X \mathcal{H}(0) \tag{4.13}$$

Definition 16. [Wiener-Kintchine] The power spectral density $Y(t)$ is given by:

$$\gamma_Y(f) = \left\|\mathcal{H}(f)\right\|^2 \gamma_X(f) \tag{4.14}$$

In addition, if the stochastic process $X(t)$ is Gaussian, then the filtered output $Y(t)$ is also Gaussian. Remember that an integral is a linear process, so that the integration of a Gaussian process also results in a Gaussian process.

Proposition 17. *[Sum of (stationary) stochastic processes]. Consider the sum*

$$Y(t) = X(t) + N(t) \tag{4.15}$$

If both signals are uncorrelated (which they are if they are independent), then

$$\gamma_{YY}(f) = \gamma_{XX}(f) + \gamma_{NN}(f) \tag{4.16}$$

4.2.3 Noise and white noise

Definition 18. [*White noise*] A white noise is defined as a stochastic process whose power spectral density is constant for each frequency

$$\gamma_N(f) = \frac{N_0}{2}\left[\frac{W}{Hz}\right] \tag{4.17}$$

In practice, there is no “pure” white noise, but it is not critical as long as its power spectral density is constant inside the useful bandwidth.

A common signal in telecommunications is a *wide-sense stationary zero-mean white Gaussian noise*. This signal is characterized by the following properties:

- the probability density function of the voltage of the noise is a Gaussian.
- the observed mean voltage has a zero mean.
- its power spectrum is constant for each frequency.

The power of a white noise (for a B large bandwidth) is

$$P_N = N = \int_{-\infty}^{+\infty} \gamma_N(f)\, df = 2 \int_{f_c-\frac{B}{2}}^{f_c+\frac{B}{2}} \frac{N_0}{2}\, df = 2 \times B \times \frac{N_0}{2} = B\, N_0 \tag{4.18}$$

5 Line coding and spectra

5.1 Line coding

Line coding consists to transform a series of bits into a continuous signal $X(t)$. In this signal, each time period T_b is dedicated to one bit (or several) of the bit stream. In other words, we take a pulse waveform $p(t)$, limited to the $[-\frac{T_b}{2}, \frac{T_b}{2}]$ interval ($p(t)$ is zero outside of that interval), and build the signal

$$X(t) = \sum_{k=-\infty}^{+\infty} A_k p(t - kT) \tag{5.1}$$

where A_k is a random variable that encodes the digital information. For example, it is common to take $A_k = \pm A$ and a rectangular unit pulse for $p(t)$.

We can distinguish, among all the possibilities, the following popular signaling format (see Figure 5.1):

1. *Nonreturn-to-zero (NRZ)* techniques. This format is obtained for a rectangular pulse shape $g(t)$. There are two variants: unipolar and polar signaling. For polar signaling, one A_k is equal to 0. For polar signaling, we have $A_k = \pm A$.
2. *Return-to-zero (RZ)* techniques. After a period of of αT_b, with $\alpha < 1$, the signal returns to zero.
3. MANCHESTER signaling, also known as Phase Encoding. It is a line coding in which the encoding of each data bit has at least one transition and occupies the same time. It therefore has no DC component, and is self-clocking.
4. *Multi-level* signaling. Multiple successive bits are encoded over one T_b period. Therefore, the signaling mechanism needs more than two levels to represent a symbol.

5.2 General formula for the power spectral density of baseband digital signals

The following expression provides the general formula for the power spectral density of baseband digital signals, when there is no correlation between the successive bits and the $\mathcal{P}(f)$ is the FOURIER transform of the pulse shape:

$$\gamma_X(f) = \|\mathcal{P}(f)\|^2 \frac{1}{T_b} \left[\sigma_A^2 + \mu_A^2 \sum_{m=-\infty}^{+\infty} \frac{1}{T_b} \delta\left(f - \frac{m}{T_b}\right)\right] \tag{5.2}$$

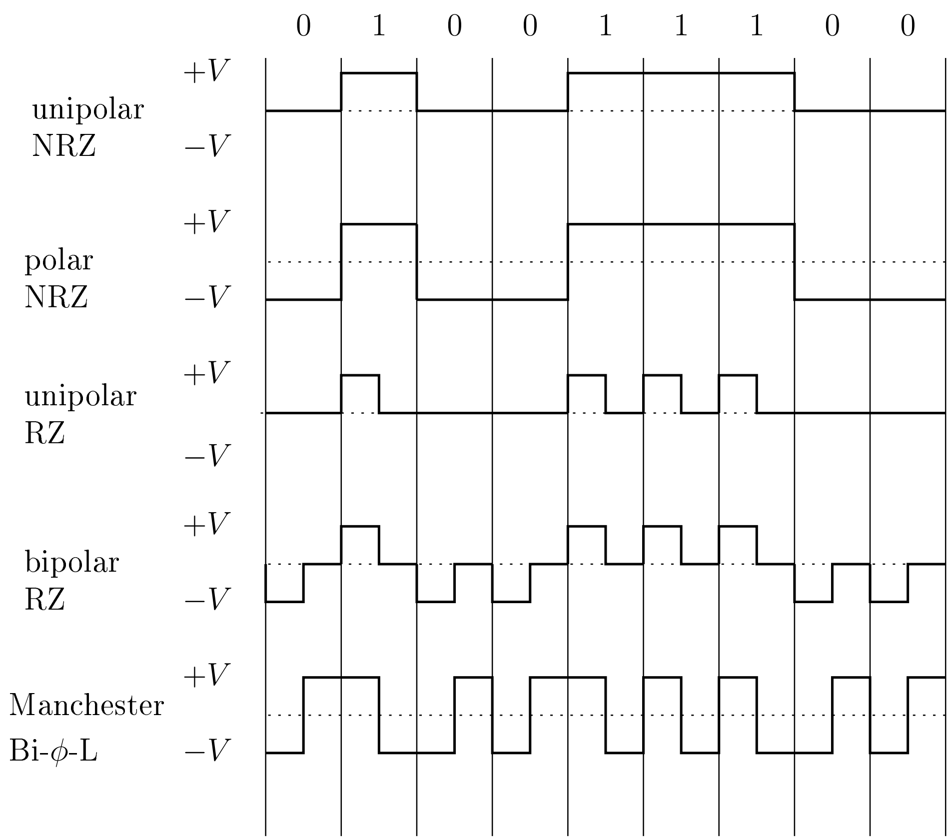

Figure 5.1: Some binary signaling formats.

6 Power budget link for radio transmissions

Consider two antennas in free space that are separated by a distance d. One antenna is transmitting a total power of P_T watts of power and the other is receiving P_R watts of power in its terminal impedance. In the direction of transmission, the transmitting antenna has a gain G_T, and the receiving antenna has a gain G_R. This situation is depicted in Figure 6.1.

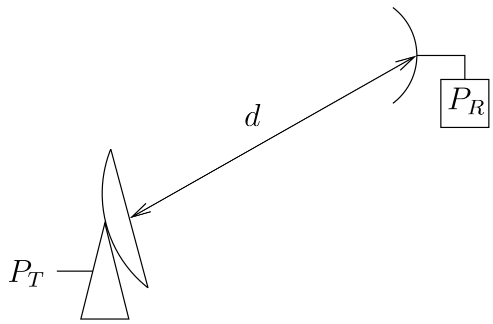

Figure 6.1: Link between two antennas in free space (no ground or obstacle).

Theorem 19. [Friis formula] *The ratio between the transmitting power and the receiving power, called free space loss L_{FS}, is given by*

$$L_{FS} = \frac{P_T}{P_R} = \left(\frac{4\pi d}{\lambda}\right)^2 \frac{1}{G_T G_R} \tag{6.1}$$

In decibels, the Friis transmission equation becomes

$$10 \log \frac{P_T}{P_R} = 32,5 + 20 \log f_{[MHz]} + 20 \log d_{[km]} - G_{T\,[dB]} - G_{R\,[dB]} \tag{6.2}$$

where f is given in MHz and d is measured in km, for convenience.

Example 20. Consider two identical antennas separated by a distance of $100\,[m]$. Both antennas have a directive gain of $15\,[dB]$ in the direction of transmission. If the transmitting antenna sends a power of $5\,[W]$ at a frequency of $3\,[GHz]$, then the received power is

$$P_R = P_T G_T G_R \left(\frac{\lambda}{4\pi d}\right)^2 \tag{6.3}$$

$$= 5 \times 31.62 \times 31.62 \times \left(\frac{0.1}{4\pi \times 100}\right)^2 \tag{6.4}$$

$$= 31.7\,[\mu W] \tag{6.5}$$

7 Information theory

7.1 Channel capacity

One of the central notion in communication is that of *channel capacity.*

Definition 21. The *channel capacity* is the tightest upper bound on the rate of information that can be reliably transmitted over a communications channel.

Theorem 22. [Shannon-Hartley] *The channel capacity C (conditions for the error rate $P_e \to 0$) is given by*

$$C\,[b/s] = B\,\log_2\left(1 + \frac{S}{N}\right) \tag{7.1}$$

where

- B is the channel bandwidth in Hz
- $\frac{S}{N}$ the signal-to-noise ratio (in watts/watts, not in dB).

7.2 On the importance of the $\frac{E_b}{N_0}$ ratio for digital transmissions

Assume an infinite bandwidth and a Gaussian white channel, then

$$C = \lim_{B\to\,\infty}\left\{B\,\log_2\left(1 + \frac{S}{N}\right)\right\} \tag{7.2}$$

As

- $S = E_b R_b$ (E_b is the energy of one bit and $R_b = \frac{1}{T_b}$ is the bitrate)
- $N = B\,N_0$

we have

$$\begin{aligned} C &= \lim_{B\to\,\infty}\left\{B\,\log_2\left(1 + \frac{E_b R_b}{B\,N_0}\right)\right\} = \lim_{x\to 0}\left\{\frac{\log_2\left(1 + x\frac{E_b R_b}{N_0}\right)}{x}\right\} \\ &= \log_2 e\,\lim_{x\to 0}\left\{\frac{1}{1 + x\frac{E_b R_b}{N_0}}\frac{E_b R_b}{N_0}\right\} = \frac{1}{\ln 2}\frac{E_b R_b}{N_0} \end{aligned} \tag{7.3}$$

At maximum capacity: $C = R_b$, so that $\frac{E_b}{N_0} = \ln 2 \equiv -1.59\,[dB]$ is the absolute minimum.

List of symbols

$\|.\|$	norm
X^*	conjugate of X
B	bandwidth
d	distance
dB	decibel
E_b	bit energy
f_c	carrier frequency
f_s	sampling frequency
f	frequency
G_R	receiver antenna gain
$g(t)$	waveform
$\mathcal{H}(f)$	transfer function or frequency response
$h(t)$	impulse response
$N_0/2$	power spectral density of noise
P	power
P_e	bit error probability
$p(A)$	probability of A
$p(t)$	instantaneous normalized power
P_T	transmit power
R_b	bit rate
$\Gamma_{XX}(\tau)$	autocorrelation of a WSS random process $X(t)$
$\gamma_X(f)$	power spectral density of WSS random process $X(t)$
$\text{sign}(t)$	sign function
T	time internal, period, sampling period
T_b	bit time
t	time
σ_X^2	variance of the random variable X
W	bandwidth
$\mathcal{G}(f)$	FOURIER transform of $g(t)$
$x[n]$	discrete-time signal
$\otimes$	convolution
$\delta(t)$	unit impulse function, DIRAC
τ	time delay

References

[1] L. Couch, *Digital and analog communication systems.* Prentice Hall, sixth ed., 2001.