Analysis and Design of Telecommunications Systems

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Practical information

- Instructor: Marc Van Droogenbroeck
- Assistant: Renaud Vandeghen
- Evaluation
  - oral(!): theory (1/2 of the final note) + exercise (1/2 of the final note)
- Slides: `https://orbi.uliege.be/handle/2268/170889`
- *New from September 2019*: manual of exercises: `https://orbi.uliege.be/handle/2268/239453`

General overview

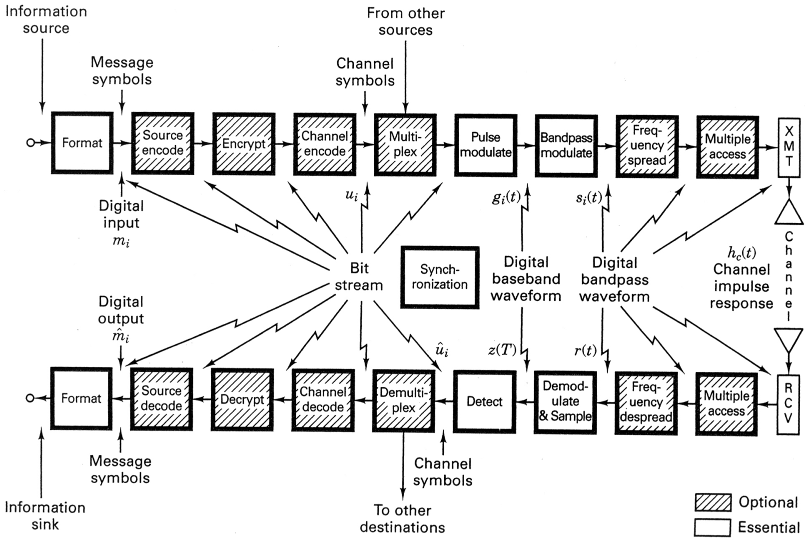
In this course:

1. Theory on signals and noise
2. Study of specific aspects of telecommunications systems
3. Towards engineering rules

Our main driving question: how?

$\longrightarrow$ understand, model, build systems, and find rules

Digital communication system (block diagram)



Components of a telecommunications system I

Main components:

1. signals: useful signal ($\equiv$ payload), noise
2. electronics (transmitter, receiver, connectors, repeaters, etc.)
3. channel: cable (+ adapters), wireless
4. propagation issues: fading (statistical effects)

Components of a telecommunications system II

Main concerns related to signals:

- Signal source handling (preparation of the signal, at the source, in the transmitter):
  - filtering (remove what is useless for communications)
  - analog $\leftrightarrow$ digital (digitization)
  - remove the redundancy in the signal: compression
- Signal over the channel
  - signal shaping to make it suitable for transmission (*coding*, *modulation*, *multiplexing*, etc.)
  - signal power versus the noise power
- Signals in the equipment
  - limit noise (by adding a *filter* and an *amplifier* at proper places!)

Components of a telecommunications system III

Main constraints:

1. *shared* channel
   - bandwidth
2. perturbations due to *noise*.
   Goals of a good design:
   - increase the Signal to Noise ratio $\left(\frac{S}{N}\right)$ to reach the maximal channel capacity (typically in the presence of white Gaussian noise)
   - enable a communication with multiple/many users (seen as interference for the main link)
3. *power* consumption

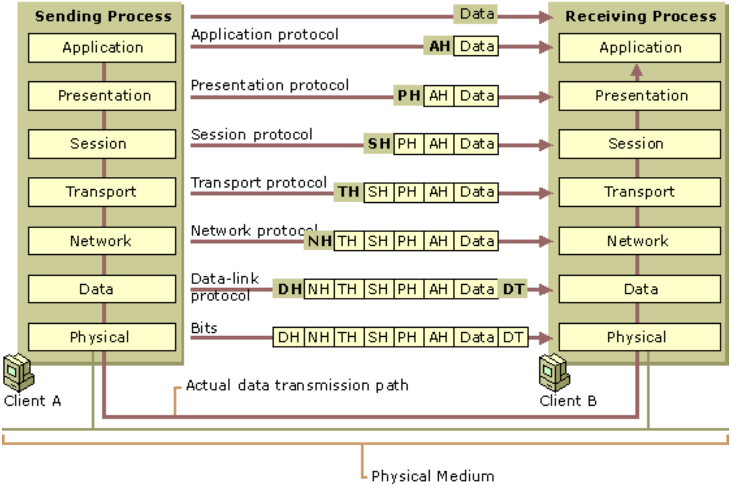
Outline

Outline

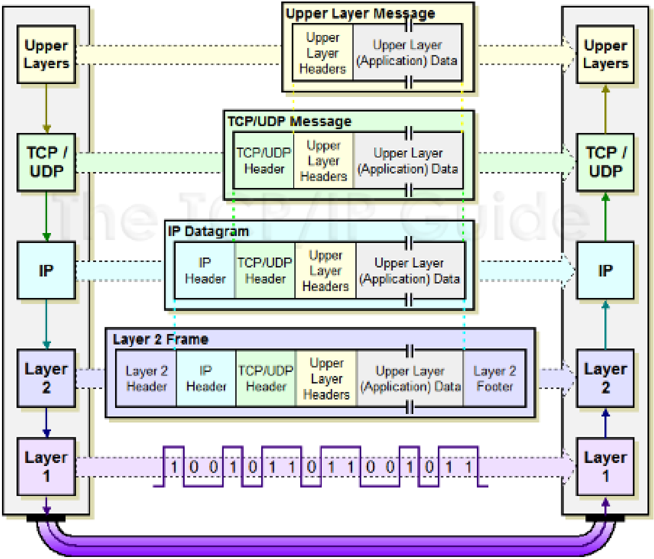
Reminder

- Physical layer (not all the transmission protocols)
- Deterministic - stochastic processes
- A tool for characterizing stochastic processes (including noise) and linear systems: the power spectrum or power spectral density
- Properties of the power spectrum
- Gaussian process
- Noise
- Modulation
- Digital communications

Network ⇒ protocols I



Network ⇒ protocols II



Deterministic vs stochastic tools

	transmitter	receiver
User's signal	*deterministic*	*stochastic*
Noise and interferences	*stochastic*	*stochastic*

	deterministic	*stochastic*
signal to consider	voltage / current	"average" power
power analysis	instantaneous power $p(t) = \frac{\lvert v(t)\rvert^2}{R} = R\,\lvert i(t)\rvert^2$	Power Spectral Density (PSD) $E\left\{X^2(t)\right\} = \int_{-\infty}^{+\infty} \gamma_X(f)df$

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Central notion for stochastic processes: stationarity

Wide-sense stationary stochastic process

- Definition: $X(t)$ is a *wide-sense stationary* stochastic process if
  1. its mean does not depend on time, that is
  $$\mu_X(t) = E\{X(t)\} = \mu_X = \text{constant} \tag{1}$$
  2. its auto-correlation function does depend on the time difference only, that is
  $$\Gamma_{XX}(t_1, t_2) = \Gamma_{XX}(\tau) \tag{2}$$
  with $\tau = t_2 - t_1$.
- It's a reasonable assumption for the development of models and experiments in most of the cases, but it is not perfect.

What if a process is not stationary?

If a process $X(t)$ is not stationary, then

- you cannot define a PSD for a non-stationary process.
- But, you can sometimes make it stationary.
  2 classical ways worth trying:
  1. inject a random (independent) phase, suited for modulated signals: $S(t) = X(t)\cos(2\pi f_c t + \Theta)$ with $\mathrm{pdf}_{\Theta}(\theta) = \frac{1}{2\pi}$ for $\theta \in [0, 2\pi]$.
  2. inject a random (independent) time shift ($\equiv$jitter), suited for digital signals: $X(t + T_0)$ with $\mathrm{pdf}_{T_0}(t_0) = \frac{1}{T}$ for $t_0 \in [0, T]$.

Let $X(t)$ be a (wide-sense) *stationary* stochastic process (note that we use a “capital” letter X for stochastic processes).

Definition (Auto-correlation function)

$$\Gamma_{XX}(\tau) = E\{X(t+\tau)X(t)\} \qquad \forall t \tag{3}$$

Very useful because it expresses the average power (when $\tau = 0$):

$$\Gamma_{XX}(\tau = 0) = E\left\{X^2(t)\right\} \tag{4}$$

In practice, we have that the power P_X of a stochastic process is given by:

$$\boxed{P_X = E\left\{X^2(t)\right\}} \tag{5}$$

Definition (Power spectrum or power spectral density of a stationary process PSD)

$$\gamma_X(f) = \int_{-\infty}^{+\infty} \Gamma_{XX}(\tau)\, e^{-2\pi jf\tau} d\tau \tag{6}$$

Summary:

$$\begin{aligned} P_X &= E\left\{X^2(t)\right\} && (7)\\ &= E\left\{X(t+0)X(t)\right\} && (8)\\ &= \Gamma_{XX}(\tau = 0) = \int_{-\infty}^{+\infty} \gamma_X(f) e^{2\pi jf0} df && (9)\\ &= \int_{-\infty}^{+\infty} \gamma_X(f) df && (10) \end{aligned}$$

Therefore, $\gamma_X(f)$ expresses the distribution of power for all the frequencies.

Example of a PSD

Let us consider a signal with a random phase θ, uniformly distributed over $[-\pi, +\pi]$ (or, likewise, $[0, 2\pi]$): $\text{pdf}_\Theta(\theta) = \frac{1}{2\pi}$

$$X(t) = A_c \cos(2\pi f_c t + \Theta) \tag{11}$$

1. *Mean of $X(t)$?*

$$\mu_X(t) = E\{X(t)\} = \int_{-\pi}^{+\pi} X(t)\, \text{pdf}_\Theta(\theta)\, d\theta = \int_{-\pi}^{+\pi} A_c \cos(2\pi f_c t + \theta) \frac{1}{2\pi} d\theta = 0 \tag{12}$$

2. *Auto-correlation?*

$$\begin{aligned}
\Gamma_{XX}(t_1, t_2) &= E\{X(t_1)X(t_2)\} && (13)\\
&= \int_{-\pi}^{+\pi} A_c \cos(2\pi f_c t_1 + \theta)\, A_c \cos(2\pi f_c t_2 + \theta) \frac{1}{2\pi} d\theta \\
&= \frac{A_c^2}{2} \cos[2\pi f_c(t_2 - t_1)] = \frac{A_c^2}{2} \cos[2\pi f_c \tau] && (14)
\end{aligned}$$

The signal is stationary. So,

- Power spectral density?

$$\gamma_X(f) = \frac{A_c^2}{4}[\delta(f - f_c) + \delta(f + f_c)] \tag{15}$$

Practical link between $x(t)$ and $X(t)$? I

Let:

- $X(t)$ be the stochastic (unknown) process behind the scenes,
- $x(t)$ be the observation. This is what you measure/observe.

One can *estimate* the PSD of $X(t)$ by taking the Fourier transform of the observation $x(t)$ (to the square).

Proof.
We want to show that $\gamma_x(f)$ can be estimated by $\|\mathcal{X}(f)\|^2$.
[The theory of estimation provides other ways to estimate $\gamma_x(f)$ though]

Practical link between $x(t)$ and $X(t)$? II

Assume $x(t)$ is deterministic and has finite energy, that is

$$\int_{-\infty}^{+\infty} |x(t)|^2 \, dt \tag{16}$$

Let us define:

- a *"pseudo" auto-correlation* function by

$$\Gamma_{xx}(\tau) = \int_{-\infty}^{+\infty} x(t)x(t+\tau)dt \tag{17}$$

- and a "pseudo" PSD, which is an estimate of the true PSD, as

$$\gamma_x(f) = \int_{-\infty}^{+\infty} \Gamma_{xx}(\tau)\, e^{-2\pi j f \tau} \, d\tau \tag{18}$$

Some calculations lead to:

$$
\begin{aligned}
\gamma_x(f) &= \int_{-\infty}^{+\infty} \Gamma_{xx}(\tau) e^{-2\pi j f \tau} d\tau && (19)\\
&= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} x(t) x(t+\tau) dt \right) e^{-2\pi j f \tau} d\tau && (20)\\
&= \int_{-\infty}^{+\infty} x(t) \left(\int_{-\infty}^{+\infty} x(t+\tau) e^{-2\pi j f \tau} d\tau \right) dt && (21)\\
&= \int_{-\infty}^{+\infty} x(t) \left(\mathcal{X}(f) e^{2\pi j f t} \right) dt && (22)\\
&= \mathcal{X}(f) \int_{-\infty}^{+\infty} x(t) e^{2\pi j f t} dt && (23)\\
&= \mathcal{X}(f) \mathcal{X}^*(f) && (24)\\
&= \|\mathcal{X}(f)\|^2 && (25)
\end{aligned}
$$

Practical link between $x(t)$ and $X(t)$? IV

In decibels (remember that $v \leftrightarrow 10\log_{10}(v)\,[dB]$),

$$\gamma_x(f)\,[dB] = 10\,\log_{10}\|\mathcal{X}(f)\|^2 = 20\,\log_{10}\|\mathcal{X}(f)\| \qquad (26)$$

So, $\gamma_X(f)\,[dB]$ can be *estimated* with the help of $\|\mathcal{X}(f)\|$.

However,

- $\gamma_X(f)$ is not based on a single observation and therefore $\gamma_X(f)$ is *not equal to* $\gamma_x(f)$.

$x\,[W]$	$10\log_{10}(x)\,[dBW]$
$1\,[W]$	$0\,[dBW]$
$2\,[W]$	$3\,[dBW]$
$0,5\,[W]$	$-3\,[dBW]$
$5\,[W]$	$7\,[dBW]$
$10^n\,[W]$	$10\times n\,[dBW]$

Power spectral density and linear systems (= filtering)

Consider a stationary process $X(t)$, a linear system whose transfer function is given by $\mathcal{H}(f)$, and $Y(t)$ the output process.

Theorem (Mean of a filtered stochastic process)

Mean of $Y(t)$:

$$\mu_Y = \mu_X \mathcal{H}(0) \tag{27}$$

Theorem (Wiener-Kintchine)

Power spectrum of a filtered stochastic process $Y(t)$:

$$\gamma_Y(f) = \|\mathcal{H}(f)\|^2 \gamma_X(f) \tag{28}$$

Useful considerations about the PSDs

- Sum of (stationary) stochastic processes:

$$Y(t) = K(t) + N(t) \tag{29}$$

If both signals are uncorrelated (which they are if they are independent), then

$$\gamma_{YY}(f) = \gamma_{KK}(f) + \gamma_{NN}(f) \tag{30}$$

What does it mean that a process $X(t)$ is "Gaussian"?

- the probability density function (pdf) of its voltage/current is Gaussian distributed:

$$\mathrm{pdf}_X = f_X(x) = \frac{1}{\sigma_X \sqrt{2\pi}} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}} \tag{31}$$

- the mean and variance of X suffice to characterize it.
- it is a good approximation for the sum of a number of independent random variables with arbitrary one-dimensional pdfs.

Useful properties:

- If the *input* of a linear system is a Gaussian stochastic process, then the *output* is also a Gaussian process.

Definition (White noise)

A white noise is defined as a stochastic process whose power spectral density is constant for each frequency

$$\gamma_N(f) = \frac{N_0}{2} \left[\frac{W}{Hz}\right] \tag{32}$$

In practice, there is no “pure” white noise, but it does not matter as long as the PSD is constant inside the useful bandwidth.

A common signal in telecommunications is a *wide-sense stationary zero-mean white Gaussian noise*:

- the probability density function of the voltage of the noise is a Gaussian.
- the observed mean voltage has a zero mean.
- its power spectrum is constant for each frequency.

Fact

Power of a white noise (for a B large bandwidth)

$$P_N = N = \int_{-\infty}^{+\infty} \gamma_N(f)\, df = 2 \int_{f_c-\frac{B}{2}}^{f_c+\frac{B}{2}} \frac{N_0}{2}\, df = 2 \times B \times \frac{N_0}{2} = B\, N_0 \tag{33}$$

Principle: modulation is all about using of a carrier f_c for transmitting information

$$\underbrace{\overbrace{s(t) = A(t)\cos\left(2\pi f(t)t + \phi(t)\right)}^{\text{Frequency modulation [FM]}}}_{\text{Amplitude} \qquad \text{Phase}}$$

Amplitude modulation [AM] Phase modulation [PM]

Consequences of modulation

- On analog signals:
  - frequency band is shifted towards the carrier frequency ($\Rightarrow f_c$)
  - bandwidth modification
- On digital signals:
  - power spectral density is shifted

$$\gamma_S(f) = \frac{\gamma_M(f - f_c) + \gamma_M(f + f_c)}{4} \tag{34}$$

    (note the presence of 4!)
  - shape of the power spectral density may be tailored

Rice's decomposition

An expression such as

$$s(t) = A(t)\cos(2\pi f_c t + \phi(t)) \tag{35}$$

may also be written as

$$\begin{aligned} s(t) &= A(t)\cos(2\pi f_c t + \phi(t)) \\ &= s_I(t)\cos(2\pi f_c t) - s_Q(t)\sin(2\pi f_c t) \end{aligned} \tag{36}$$

with

$$s_I(t) = A(t)\cos\phi(t) \tag{37}$$

$$s_Q(t) = A(t)\sin\phi(t) \tag{38}$$

Rice's decomposition is essential as it says that **any** modulated signal can be decomposed into two amplitude modulated signals (proof will follow).
Many receivers in telecommunications use this principle!

Information theory and channel capacity: there is *maximum* bit rate! I

Theorem (Shannon-Hartley)

Channel capacity C *(conditions for the error rate* $P_e \rightarrow 0$*)*

$$C\,[b/s] = B\,\log_2\left(1 + \frac{S}{N}\right) \tag{39}$$

where

- B is the channel bandwidth in Hz.
- $\frac{S}{N}$ the signal-to-noise ratio (in watts/watts, not in dB).

Assume infinite bandwidth and a white Gaussian channel,

$$C = \lim_{B \to \infty} \left\{ B \log_2 \left(1 + \frac{S}{N} \right) \right\} \tag{40}$$

As

- $S = E_b R_b$ (E_b is the energy of one bit and $R_b = \frac{1}{T_b}$ is the bitrate)
- $N = B\, N_0$

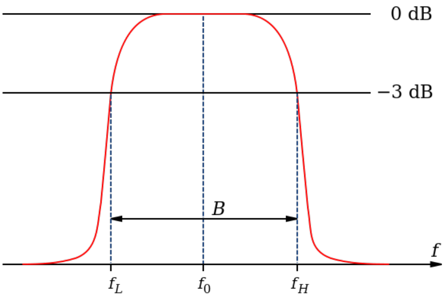
Therefore,

$$\begin{aligned} C &= \lim_{B \to \infty} \left\{ B \log_2 \left(1 + \frac{E_b R_b}{B\, N_0} \right) \right\} = \lim_{x \to 0} \left\{ \frac{\log_2 \left(1 + x \frac{E_b R_b}{N_0} \right)}{x} \right\} \\ &\stackrel{\text{H}}{=} \log_2 e \lim_{x \to 0} \left\{ \frac{1}{1 + x \frac{E_b R_b}{N_0}} \frac{E_b R_b}{N_0} \right\} = \frac{1}{\ln 2} \frac{E_b R_b}{N_0} \end{aligned} \tag{41}$$

At maximum capacity: $C = R_b$, so that $\frac{E_b}{N_0} = \ln 2 \equiv -1.59\,[dB]$ is the absolute minimum.

Outline

Towards a dedicated representation for bandpass signals



Why do we use "frequency-based" representations?

[Systems] it is convenient for *linear systems*, and most communication systems are linear (channel, filter, etc.).

[Sharing] channels can be shared if signals are "frequency"-friendly (multiplexing).

What type of *frequency signals* do we have?

1. in its original form, a signal is in its *baseband* (voice in the telephone network is in the [300 Hz, 3400 Hz] band).
2. *bandwidth* $\equiv$ band occupied by the signal.
3. for digital signals, sampling $\longrightarrow$ *sampling frequency/rate* (driven by Nyquist's criterion and relating to the highest frequency, thus the frequency content).
4. spectrum
    1. $\mathcal{V}(f)$ for deterministic signals
    2. $\gamma_V(f)$ for stochastic processes.

Purposes of dealing with bandpass signals specifically

1. Helps to reduce the sampling rate (not twice the highest frequency!)
2. There are many more convenient *representations* of such a signal (*analytic signal, complex envelope, baseband equivalent, quadrature components*, etc.):
    1. They are equivalent, but not equal.
    2. Which one is the most appropriate depends on the context (hard to foresee).
    3. See these representations as *tools*!
    4. The many representations are convenient theoretical and practical ways to process signals.
    5. Almost every receiver uses the underlying theory of bandpass signals.

Representation of bandpass signals: outline

Three "components" related to the frequency analysis:

1. Representation of *deterministic* band-limited (bandpass) signals:
   - *rethinking the sampling frequency* of band-limited signals
   - towards alternative representations
     - bandpass equivalent
     - analytic signal
     - complex envelope
     - and more !
2. Representation of bandpass *systems*
3. Representation of bandpass *stochastic* processes

Sampling of bandpass signals

Theorem ("Revised" sampling theorem)

Assume that $v(t)$ is a deterministic, energy limited signal, with a W large bandwidth, whose Fourier *transform $\mathcal{V}(f)$ is upper bounded by f_u (that is $\mathcal{V}(f) = 0$ for $f > f_u$).*
Then, it is possible to characterize this signal with samples $v[nT_s]$, $n \in \{-\infty, +\infty\}$, taken at a f_s sampling frequency if this frequency is equal to $f_s = \frac{2f_u}{k}$, where k is the largest integer strictly smaller than $\frac{f_u}{W}$.

It should be noted that not all sampling frequencies are valid (in order to reconstruct $v(t)$ perfectly), except for all those that are strictly larger than $2f_u$.

Important practical consequences:

1. sampling frequency: from $f_s = 2f_u$ to $f_s = \frac{2f_u}{k}$. k times less!
2. there are other ways to look at bandpass signals: new representations!

Representations of deterministic bandpass signals I

Deterministic signal	Stochastic signal	
$v(t)$	$X(t)$	$\Gamma_{XX}(\tau)$ or $\gamma_X(f)$

Definition (Bandpass)

A *bandpass* signal $v(t)$ is a signal for which there exist two values, B and f_0, such that $B \ll f_0$, and

$$\forall f \notin \left[f_0 - \frac{B}{2},\, f_0 + \frac{B}{2}\right],\ \mathcal{V}(f) = 0 \tag{42}$$

Representations of deterministic bandpass signals II

Definition (Equivalent baseband)

Assume a deterministic bandpass signal $v(t)$.
$\overline{v}(t)$ is an *equivalent baseband* of $v(t)$ if there exists a frequency f_0, comprised inside the frequency band of $v(t)$, such that

$$v(t) = \mathrm{Re}\left(\overline{v}(t)e^{2\pi j f_0 t + j\varphi_0}\right) \tag{43}$$

Note: in fact, $\overline{v}$ covers a *family* of equivalent baseband signals, since

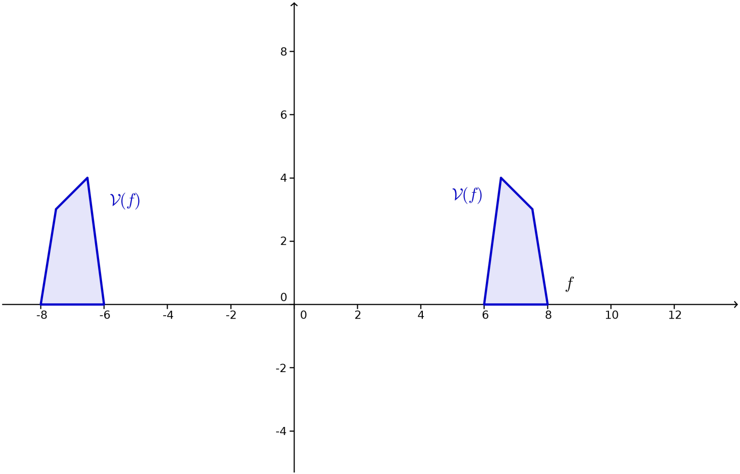
$$\overline{v}(t) = (v(t) + jz(t))e^{-2\pi j f_0 t - j\varphi_0} \tag{44}$$

are all valid candidates.

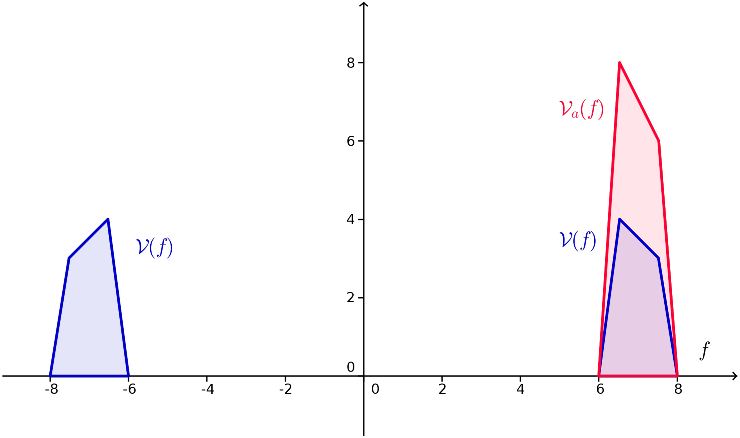
Summary of some representations of a bandpass signal

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
{\color{blue} v(t)} \arrow[rr, leftrightarrow, "v_I(t)\cos(.) - v_Q(t)\sin(.)"] \arrow[d, shift left, "\otimes\left(\delta(t) + \frac{j}{\pi t}\right)"] & & v_I(t) + j v_Q(t) \arrow[d, leftrightarrow, "\equiv"] & \\
v_a(t) \arrow[u, shift left, "\mathrm{Re}(.)"] \arrow[rr, shift left, "\times e^{-2\pi j f_0 t}"] & & e_v(t) \arrow[ll, shift left, "\times e^{+2\pi j f_0 t}"] \arrow[r, leftrightarrow, "\equiv"] & a_v(t) e^{j\phi_v(t)}
\end{tikzcd}
$$

Working on the spectrum I

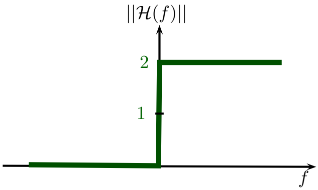


Working on the spectrum II



Why do we double the height of $\mathcal{V}(f)$ to define $\mathcal{V}_a(f)$?

Analytic signal I



Let us consider the following filter that removes the negative frequency components of a signal with the (Heaviside) step function:

$$\mathcal{H}(f) = \begin{cases} 0 & \text{if } f < 0 \\ 2 & \text{if } f \geq 0 \end{cases} = 1 + \mathrm{sign}(f) \tag{45}$$

whose *impulse response* is

$$h(t) = \delta(t) + \frac{j}{\pi t} \tag{46}$$

Definition (Analytic signal)

A signal which has no negative-frequency components is called an *analytic signal*. In the time domain, it is obtained as

$$v_a(t) = v(t) \otimes \left(\delta(t) + \frac{j}{\pi t} \right) \quad (47)$$

$$= v(t) + jv(t) \otimes \frac{1}{\pi t} \quad (48)$$

The norm of the analytic signal is named "*envelope*".

Representation map: analytic signal

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
v(t) \arrow[rr, leftrightarrow, "v_I(t)\cos(.) - v_Q(t)\sin(.)"] \arrow[d, shift left, "\otimes \left(\delta(t) + \frac{j}{\pi t}\right)"] & & v_I(t) + j v_Q(t) \arrow[d, leftrightarrow, "\equiv"] & \\
v_a(t) \arrow[u, shift left, "\mathrm{Re}(.)"] \arrow[rr, shift left, "\times e^{-2\pi j f_0 t}"] & & e_v(t) \arrow[ll, shift left, "\times e^{+2\pi j f_0 t}"] \arrow[r, leftrightarrow, "\equiv"] & a_v(t) e^{j\phi_v(t)}
\end{tikzcd}
$$

Hilbert transform

Definition (Hilbert transform)

The Hilbert transform of a signal $v(t)$, denoted as $\widetilde{v}(t)$, is defined by

$$\widetilde{v}(t) = v(t) \otimes \frac{1}{\pi t} \tag{49}$$

With this definition,

$$v_a(t) = v(t) + j\widetilde{v}(t) \tag{50}$$

Properties of the Hilbert transform

- The energy (or power) of a signal and that of its Hilbert transform are equal.
- [Hilbert *transform of a modulated signal*] Assume that $v(t)$ is a baseband signal, then

$$v(t)\widetilde{\cos(2\pi f_c t)} = v(t)\sin(2\pi f_c t) \tag{51}$$

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So,

$$v(t) = \mathrm{Re}\left(v_a(t)\right) \tag{53}$$

$$\begin{tikzcd}
v(t) \arrow[r, leftrightarrow, "v_I(t)\cos(.) - v_Q(t)\sin(.)"] \arrow[d, shift left, "\otimes\left(\delta(t) + \frac{j}{\pi t}\right)"] & v_I(t) + jv_Q(t) \arrow[d, leftrightarrow, "\equiv"] & \\
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\end{tikzcd}$$

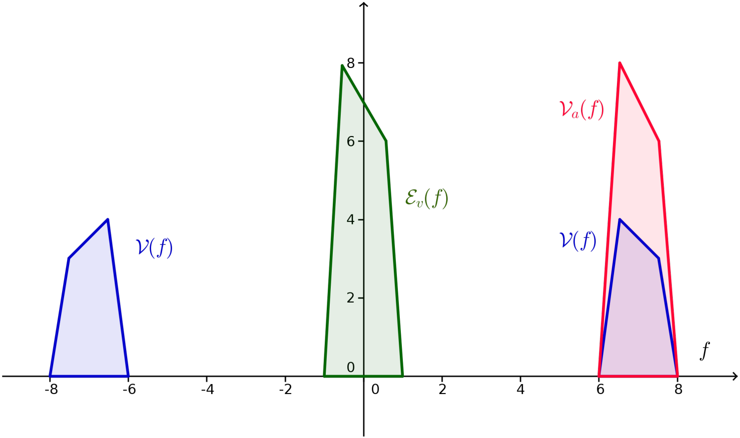
Going backwards II

Properties of the analytic signal

- it has no negative frequency
- it carries the same power as the original signal
- $v(t)$ can be reconstructed from $v_a(t)$
- $v_a(t)$ is not a real signal. Is that a problem?

Why would we want to use $v_a(t)$?

$$\mathcal{E}_v(f) = \mathcal{V}_a(f + f_0) \tag{54}$$



Definition (Complex envelope of a signal)

The signal that results from a right-to-left shift of the analytic signal in the frequency domain is named the *complex envelope* of the signal. It is denoted as $e_v(t)$.

Mathematically, the complex envelope and its spectrum are related to the analytic signal as follows:

$$\mathcal{E}_v(f) = \mathcal{V}_a(f + f_0) \tag{55}$$

$$e_v(t) = v_a(t)e^{-2\pi j f_0 t} \tag{56}$$

It remains to find a good practical method to determine the complex envelope!

Representation map: complex envelope

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
{\color{blue} v(t)} \arrow[r, leftrightarrow, "v_I(t)\cos(.) - v_Q(t)\sin(.)"] \arrow[d, shift left, "\otimes\left(\delta(t) + \frac{j}{\pi t}\right)"] & v_I(t) + jv_Q(t) \arrow[d, leftrightarrow, "\equiv"] & \\
{\color{red} v_a(t)} \arrow[u, shift left, "\mathrm{Re}(.)"] \arrow[r, shift left, "\times e^{-2\pi j f_0 t}"] & {\color{red} e_v(t)} \arrow[l, shift left, "\times e^{+2\pi j f_0 t}"] \arrow[r, leftrightarrow, "\equiv"] & a_v(t)e^{j\phi_v(t)}
\end{tikzcd}
$$

Question

What sampling frequency to choose for the complex envelope?

$$e_v(t) = v_I(t) + jv_Q(t) \tag{57}$$

$$\begin{tikzcd}
v(t) \arrow[rr, leftrightarrow, "v_I(t)\cos(.) - v_Q(t)\sin(.)"] \arrow[d, shift left, "\otimes\left(\delta(t) + \frac{j}{\pi t}\right)"] & & v_I(t) + jv_Q(t) \arrow[d, leftrightarrow, "\equiv"] & \\
v_a(t) \arrow[u, shift left, "\mathrm{Re}(.)"] \arrow[rr, shift left, "\times e^{-2\pi j f_0 t}"] & & e_v(t) \arrow[ll, shift left, "\times e^{+2\pi j f_0 t}"] \arrow[r, leftrightarrow, "\equiv"] & a_v(t)e^{j\phi_v(t)}
\end{tikzcd}$$

In-phase component: $v_I(t)$

$$\begin{aligned}
v_I(t) &= \mathrm{Re}\,(e_v(t)) && (58)\\
&= \mathrm{Re}\left(v_a(t)e^{-2\pi j f_0 t}\right) && (59)\\
&= \mathrm{Re}\left((v(t)+j\widetilde{v}(t))\,e^{-2\pi j f_0 t}\right) && (60)\\
&= v(t)\cos(2\pi f_0 t)+\widetilde{v}(t)\sin(2\pi f_0 t) && (61)
\end{aligned}$$

Quadrature component: $v_Q(t)$

$$\begin{aligned}
v_Q(t) &= \mathrm{Im}\,(e_v(t)) && (62)\\
&= \mathrm{Im}\left(v_a(t)e^{-2\pi j f_0 t}\right) && (63)\\
&= -v(t)\sin(2\pi f_0 t)+\widetilde{v}(t)\cos(2\pi f_0 t) && (64)
\end{aligned}$$

Theoretical determination of Rice components II

Therefore,

$$
\begin{aligned}
v(t) &= \mathrm{Re}\left(v_a(t)\right) && (65)\\
&= \mathrm{Re}\left(e_v(t)e^{2\pi j f_0 t}\right) && (66)\\
&= \mathrm{Re}\left(\left(v_I(t)+jv_Q(t)\right)e^{2\pi j f_0 t}\right) && (67)\\
&= v_I(t)\cos(2\pi f_0 t) - v_Q(t)\sin(2\pi f_0 t) && (68)
\end{aligned}
$$

$$
\begin{tikzcd}
v(t) \arrow[r, leftrightarrow, "v_I(t)\cos(.) - v_Q(t)\sin(.)"] \arrow[d, shift left, "\otimes\left(\delta(t)+\frac{j}{\pi t}\right)"] & v_I(t)+jv_Q(t) \arrow[d, leftrightarrow, "\equiv"] & \\
v_a(t) \arrow[u, shift left, "\mathrm{Re}(.)"] \arrow[r, shift left, "\times e^{-2\pi j f_0 t}"] & e_v(t) \arrow[l, shift left, "\times e^{+2\pi j f_0 t}"] \arrow[r, leftrightarrow, "\equiv"] & a_v(t)e^{j\phi_v(t)}
\end{tikzcd}
$$

$$v(t) = v_I(t)\cos(2\pi f_0 t) - v_Q(t)\sin(2\pi f_0 t) \qquad (69)$$

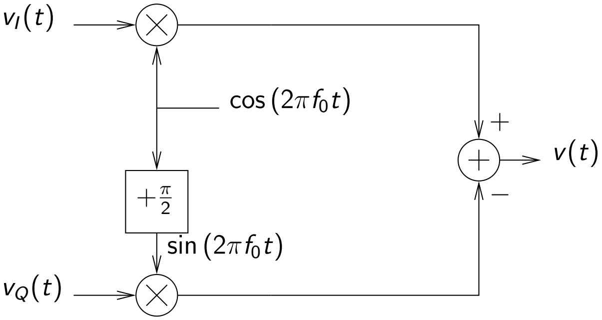
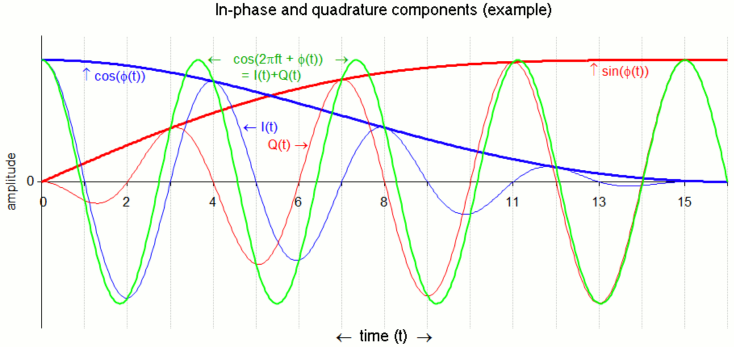


Figure: Recover $v(t)$ starting from the Rice components.

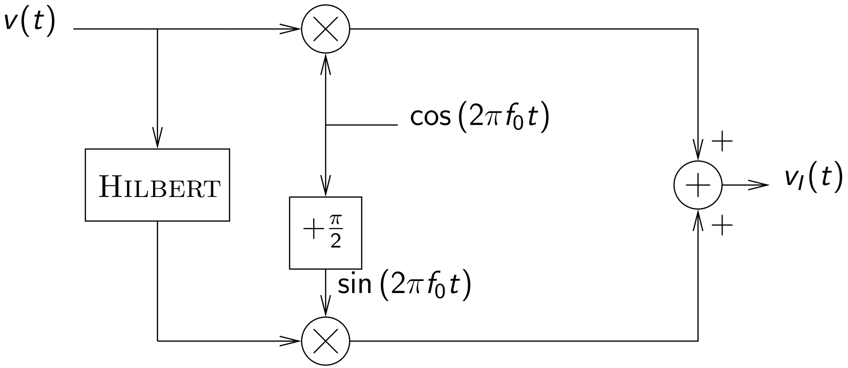
$$\cos\left(2\pi ft + \phi(t)\right) = \cos\phi(t)\cos\left(2\pi ft\right) - \sin\phi(t)\sin\left(2\pi ft\right)$$

Any bandpass signal can be seen as the sum of two modulated signals by lowpass signals.



A practical method to compute Rice components I

Theoretically,



A practical method to compute Rice components II

$$
\begin{aligned}
v(t) \times 2\cos(2\pi f_0 t) &= 2\left[v_I(t)\cos(2\pi f_0 t) - v_Q(t)\sin(2\pi f_0 t)\right]\cos(2\pi f_0 t) \\
&= 2\left[v_I(t)\cos^2(2\pi f_0 t) - v_Q(t)\sin(2\pi f_0 t)\cos(2\pi f_0 t)\right] \\
&= v_I(t) + v_I(t)\cos(4\pi f_0 t) - v_Q(t)\sin(4\pi f_0 t) \qquad (70)
\end{aligned}
$$

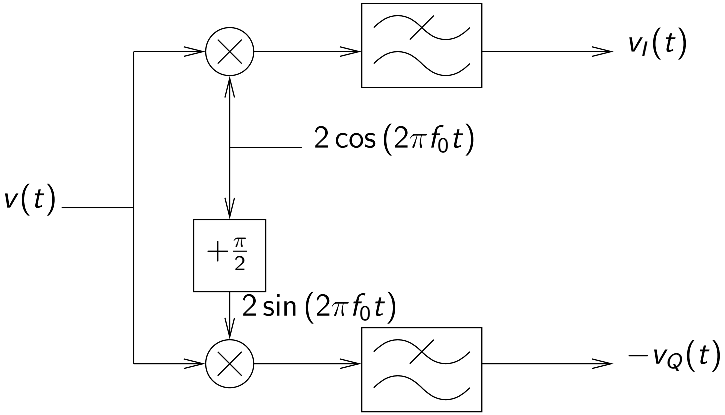
And after a low-pass filter,

$$v(t) \times 2\cos(2\pi f_0 t) \longrightarrow \text{lowpass filter} \longrightarrow v_I(t) \qquad (71)$$

Likewise,

$$v(t) \times 2\sin(2\pi f_0 t) \longrightarrow \text{lowpass filter} \longrightarrow -v_Q(t) \qquad (72)$$

A practical method to compute Rice components III



$$e_v(t) = v_I(t) + jv_Q(t) = a_v(t)\, e^{j\phi_v(t)} \tag{73}$$

$$\begin{tikzcd}
v(t) \arrow[r, leftrightarrow, "v_I(t)\cos(.) - v_Q(t)\sin(.)"] \arrow[d, shift left, "\otimes\left(\delta(t) + \frac{j}{\pi t}\right)"] & v_I(t) + jv_Q(t) \arrow[d, leftrightarrow, "\equiv"] & \\
v_a(t) \arrow[u, shift left, "\mathrm{Re}(.)"] \arrow[r, shift left, "\times e^{-2\pi j f_0 t}"] & e_v(t) \arrow[l, shift left, "\times e^{+2\pi j f_0 t}"] \arrow[r, leftrightarrow, "\equiv"] & a_v(t)e^{j\phi_v(t)}
\end{tikzcd}$$

By substituting $e_v(t)$ by its amplitude + phase description, we have

$$v(t) = \mathrm{Re}\left(e_v(t)e^{2\pi j f_0 t}\right) \tag{74}$$

$$= \mathrm{Re}\left(a_v(t)\, e^{j\phi_v(t)} e^{2\pi j f_0 t}\right) \tag{75}$$

$$= a_v(t) \cos\left(2\pi f_0 t + \phi_v(t)\right) \tag{76}$$

with, by definition,

$$a_v(t) = \sqrt{v_I^2(t) + v_Q^2(t)} \tag{77}$$

$$\phi_v(t) = \tan^{-1} \frac{v_Q(t)}{v_I(t)} \tag{78}$$

Amplitude/phase representation of the complex envelope III

Conclusions for $v(t) = a_v(t) \cos(2\pi f_0 t + \phi_v(t))$

1. the amplitude $a_v(t)$ of the complex envelope is the envelope of the original signal $v(t)$.
2. the instantaneous phase of $v(t)$ is given by the phase of the complex envelope.

Any bandpass signal can be seen as a signal modulated both in amplitude and in phase.

Assume a bandpass filter, around f_0, whose response filter is

$$h(t) = \mathrm{Re}\left(e_h(t)e^{2\pi j f_0 t}\right) \tag{79}$$

The filtered signal is given by

$$\begin{aligned} y(t) &= v(t) \otimes h(t) && (80) \\ &= \int_{-\infty}^{+\infty} h(\lambda) v(t-\lambda) d\lambda && (81) \end{aligned}$$

Thesis:

$$e_y(t) = \frac{1}{2} e_h(t) \otimes e_v(t) \tag{82}$$

Assume a bandpass filter, around f_0, whose response filter is

$$h(t) = \mathrm{Re}\left(e_h(t)e^{2\pi j f_0 t}\right) \tag{79}$$

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Thesis:

$$e_y(t) = \frac{1}{2}\, e_h(t) \otimes e_v(t) \tag{82}$$

Proof I

$$y(t) = v(t) \otimes h(t) = \int_{-\infty}^{+\infty} h(\lambda) v(t-\lambda) d\lambda \tag{83}$$

To get rid of Re (...), we use

$$\mathrm{Re}\,(a + jb) = \frac{a+jb}{2} + \frac{a-jb}{2} = \frac{a+jb}{2} + \left(\frac{a+jb}{2}\right)^* \tag{84}$$

So, $v(t-\lambda) = \frac{1}{2}\left(e_v(t-\lambda)e^{2\pi j f_0 t} e^{-2\pi j f_0 \lambda} + e_v^*(t-\lambda) e^{-2\pi j f_0 t} e^{2\pi j f_0 \lambda}\right)$ and $h(\lambda) = \frac{1}{2}\left(e_h(\lambda) e^{2\pi j f_0 \lambda} + e_h^*(\lambda) e^{-2\pi j f_0 \lambda}\right)$

Therefore,

$$\begin{aligned} y(t) &= \frac{1}{4} e^{2\pi j f_0 t} \int_{-\infty}^{+\infty} e_h(\lambda) e_v(t-\lambda) d\lambda + \frac{1}{4} e^{-2\pi j f_0 t} \int_{-\infty}^{+\infty} e_h^*(\lambda) e_v^*(t-\lambda) d\lambda \\ &+ \frac{1}{4} e^{-2\pi j f_0 t} \int_{-\infty}^{+\infty} e_h(\lambda) e_v^*(t-\lambda) e^{4\pi j f_0 \lambda} d\lambda \\ &+ \frac{1}{4} e^{2\pi j f_0 t} \int_{-\infty}^{+\infty} e_h^*(\lambda) e_v(t-\lambda) e^{-4\pi j f_0 \lambda} d\lambda \end{aligned}$$

$$y(t) = \frac{1}{2}\mathrm{Re}\left(\int_{-\infty}^{+\infty} e_h(\lambda) e_v(t-\lambda) d\lambda\, e^{2\pi j f_0 t}\right) \tag{85}$$

$$+ \frac{1}{2}\mathrm{Re}\left(\int_{-\infty}^{+\infty} e_h^*(\lambda) e_v(t-\lambda) \underline{e^{-4\pi j f_0 \lambda}} d\lambda\, e^{2\pi j f_0 t}\right) \tag{86}$$

But, $e_h^*(\lambda) e_v(t-\lambda)$ is low frequency and $\underline{e^{-4\pi j f_0 \lambda}}$ is high frequency. Therefore,

$$\int_{-\infty}^{+\infty} e_h^*(\lambda) e_v(t-\lambda) \underline{e^{-4\pi j f_0 \lambda}} d\lambda \approx 0 \tag{87}$$

Finally,

$$y(t) = \frac{1}{2}\mathrm{Re}\left(\left(e_h(t) \otimes e_v(t)\right) e^{2\pi j f_0 t}\right) \tag{88}$$

or

$$\boxed{e_y(t) = \frac{1}{2} e_h(t) \otimes e_v(t)} \tag{89}$$

Bandpass filtering

Use of $e_y(t) = \frac{1}{2}\, e_h(t) \otimes e_v(t)$

- the output is a bandpass signal too ($\Rightarrow$ it has a baseband equivalent).
- we have a way to filter a bandpass signal by its baseband equivalent. This is really helpful when we work with digital signals (because the sampling frequency is much lower).
- why do we have the $\frac{1}{2}$ factor?

Analytic signal of a stochastic process

Two signals can be considered:

1. the *stochastic process*: $X(t)$
2. its *auto-correlation function* $\Gamma_{XX}(\tau)$ or its *power spectrum* $\gamma_X(f)$

The theory of equivalent baseband signals applies both to $X(t)$ and to $\Gamma_{XX}(\tau)$, but they are very different concepts.

Remember that when a stochastic process $X(t)$ passes through a (linear) filter, its power spectrum is multiplied by $\|\mathcal{H}(f)\|^2$.

$X(t)$ is related to its complex envelope by

$$X(t) = \mathrm{Re}\left(e_X(t)\, e^{2\pi j f_0 t}\right) \tag{90}$$

$X(t)$ being a stochastic process, its complex envelope $e_X(t)$ is also a stochastic process.
In general, $X(t)$ *is not stationary* because its mean is time dependent.

Solution: introduction of a random phase Θ uniformly distributed over $[0, 2\pi[$

$$X(t) = \mathrm{Re}\left(e_X(t)\, e^{j(2\pi f_0 t + \Theta)}\right) \tag{91}$$

Rice decomposition of a stochastic process I

Likewise to developments for deterministic signals, the *in-phase* and *quadrature* components of a stochastic process can be expressed as

$$e_X(t) = X_I(t) + j\,X_Q(t) \tag{92}$$

Rice components of the $X(t)$ stochastic process are then obtained by

$$\begin{aligned} X(t) &= \mathrm{Re}\left(e_X(t)\,e^{2\pi j f_0 t}\right) && (93)\\ &= \mathrm{Re}\left((X_I(t) + j\,X_Q(t))\,e^{2\pi j f_0 t}\right) && (94)\\ &= X_I(t)\cos(2\pi f_0 t) - X_Q(t)\sin(2\pi f_0 t) && (95) \end{aligned}$$

By analogy with the developments for deterministic signals, we build the analytic signal by filtering $X(t)$ with the following filter $\mathcal{H}(f)$ that removes all components for negative frequencies:

$$\mathcal{H}(f) = \begin{cases} 0 & \text{if} \quad f < 0 \\ 2 & \text{if} \quad f \geq 0 \end{cases} \tag{96}$$

Rice decomposition of a stochastic process II

By applying Wiener-Kintchine, the power spectral density of the analytic signal is given by

$$\begin{aligned}\gamma_{X_a}(f) &= \|\mathcal{H}(f)\|^2\,\gamma_X(f) && (97)\\ &= \begin{cases} 4\,\gamma_X(f) & \text{if} \quad f \geq 0 \\ 0 & \text{if} \quad f < 0 \end{cases} && (98)\end{aligned}$$

Note that it can also be written as $\gamma_{X_a}(f) = 2\mathcal{H}(f)\,\gamma_X(f)$.

Power spectrum

$$\gamma_{e_X}(f) = \gamma_{X_a}(f + f_0) \tag{99}$$

$$\begin{aligned}
\Gamma_{XX}(\tau) &= \frac{1}{2}\mathrm{Re}\left(\Gamma_{X_a X_a}(\tau)\right) && (100)\\
&= \frac{1}{2}\mathrm{Re}\left(\Gamma_{e_X e_X}(\tau)\ e^{2\pi j f_0 \tau}\right) && (101)
\end{aligned}$$

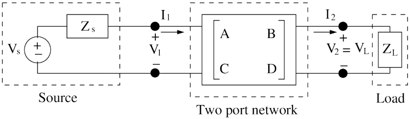
It is *shown in a later chapter* that, after stationarization, we get

$$\gamma_X(f) = \frac{\gamma_{e_X}(f - f_0) + \gamma^*_{e_X}(-f - f_0)}{4} \tag{102}$$

Important practical result: $\gamma_X(f)$ can be derived from $\gamma_{e_X}()$.

Outline

Steps



1. *Noise* *signal*
   1. Understand the origin of noise (thermal noise)
   2. Look for a model of noise (in terms of the power spectral density of a stochastic voltage)
   3. Find a subsequent model for the noise generated by a one-port circuit ($\frac{1}{2}k_B T$)
2. *Noise in* *systems*
   1. Define a practical way to derive the amount of noise when dealing with a two-port circuit (noise figure, F_0)
   2. Find formulas to calculate the amount of noise accumulated in a cascade of two-port circuits (F_0 for the cascade)
3. *Engineering*
   1. Define good "practical" rules for a cascade

Noise in telecommunications systems I

Noise is a recurrent issue/problem in telecommunications systems. Remember for example the following theorem:

Theorem (Shannon-Hartley)

Channel capacity C (conditions for the error rate $P_e \to 0$)

$$C\left[\frac{b}{s}\right] = B\,\log_2\left(1 + \frac{S}{N}\right) \tag{103}$$

where

- *B is the channel bandwidth in Hz*
- *$\frac{S}{N}$ the signal-to-noise ratio (in watt/watt, not in dB).*

Therefore, we need to understand how to deal with noise.

Noise in telecommunications systems II

There are several *physical* sources of noise:

- *thermal noise*
- shot noise
- ...

There are also "*system*" sources of noise

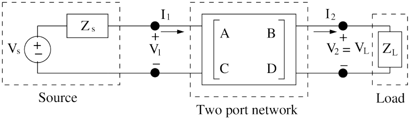
- quantization noise
- intermodulation noise
- crosstalk
- interference noise
- ...

Noise in telecommunications systems III

Steps towards a solution for dealing with noise in telecommunications systems

1. model the most common noise: *thermal noise* for electronic circuits.
2. the principles of thermal noise can also be used to model noises that are not of thermal origin (such as white noises).

Modeling noise in electronic circuits: outline



1. Characterization of a one-port circuit (source)
   - Available noise power
   - Noise temperature of a one-port circuit
   - Signal to noise ratio
2. Characterization of a two-port circuit (channel, amplifiers, filters, etc.)
   - Gain
   - Noise factor
   - Figure of merit
   - Effective noise temperature
   - Particular case: attenuator
3. Cascade of two-port circuits (a complete chain)

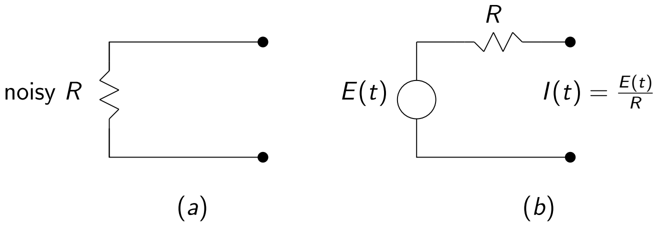


Figure: (a) Physical circuit with noise, (b) Thévenin *equivalent* circuit for a resistor considered as a noise generator.

A conductive element with two endpoints ($\equiv$ one-port circuit) may be characterized by:

- its resistance R.
- Free electrons have some random motion depending on the temperature $T \Rightarrow$ noise voltage source $E(t)$.

The noise voltage source, $E(t)$, is a stochastic process.

Thermal noise: towards a model for $\mathrm{E(t)}$ I

A natural source of noise is thermal noise, caused by the motion of free electrons in conducting material.

- Because a voltage that would be *measured* at the output of a resistance R is produced by many free electrons, by the central limit theorem, the probability density function of the voltage's amplitude can be modeled by a *Gaussian*:
  - The thermal noise has a *zero mean* (can be shown analytically and experimentally).

Thermal noise: towards a model for $\mathrm{E(t)}$ II

- The voltage $E(t)$ is a stochastic process.
  - It can be shown, experimentally, that the *autocorrelation function* of thermal noise is well modeled by

$$\Gamma_{EE}(\tau) = k_B T R \frac{e^{-|\tau|/t_0}}{t_0} \tag{104}$$

    where
    - $k_B = 1.38 \times 10^{-23}$ [J/K] is Boltzmann's constant.
    - $T = 273.15 + C$ is the absolute temperature of the resistor (in Kelvin); C is the temperature in Celsius.
    - $t_0 = 10^{-12}$ [s] is the average time between collisions of electrons.

Thermal noise: towards a model for $\mathrm{E(t)}$ III

As the autocorrelation function is

$$\Gamma_{EE}(\tau) = k_B TR \frac{e^{-|\tau|/t_0}}{t_0} \tag{105}$$

the power spectral density $\gamma_E(f)$ is

$$\gamma_E(f) = \frac{2k_B TR}{1 + (2\pi f t_0)^2} \tag{106}$$

Empirically, at room temperature, for low frequencies ($< 1000\,[\mathrm{GHz}]$), the noise PSD is almost flat so that we may take

$$\boxed{\gamma_E(f) \simeq 2k_B TR} \tag{107}$$

Noise power spectral density: $\gamma_E(f) \simeq 2k_B TR$

Consider a thermal noise with a noise temperature of $T = 290\,[\mathrm{K}]$ and $R = 1\,[\Omega]$, then we have

$$\gamma_E(f) = 2 \times 1.38 \times 10^{-23} \times 290 \times 1 = 8 \times 10^{-21} \left[\frac{\mathrm{W}}{\mathrm{Hz}}\right] \tag{108}$$

Note that $\gamma_E(f)$ is the power due to thermal noise; it is not the power available at the output!

Calculating the power of $E(t)$

For a B bandwidth, a resistor in a short circuit dissipates a noise power of (non sinusoidal signal):

$$P = \int_{-\infty}^{\infty} \gamma_E(f)df/R = 2\int_{0}^{\infty} \gamma_E(f)df/R = 4k_B TB \qquad (109)$$

In the following, we calculate the *available noise power*.

Source impedance:

$$Z_s(f) = R_s(f) + jX_s(f) \tag{110}$$

Load impedance:

$$Z_L(f) \tag{111}$$

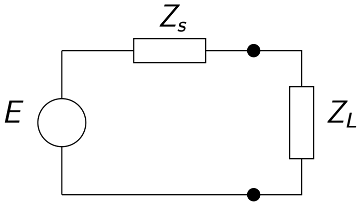


Figure: Thermal source with a load Z_L.

Definition (Available power)

The *available power* is the maximum power that can be drawn from a source.

Theorem (Maximum power transfer)

The maximum power transfer occurs when the load impedance is equal to the conjugate of the source impedance (matched impedances):

$$Z_L(f) = Z_S^*(f) \tag{112}$$

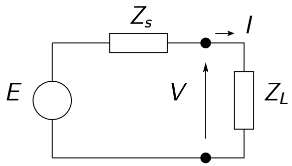
If impedances are almost purely resistive, then

$$R_L = R_S \tag{113}$$

When the signals are sinusoidal, the power provided by a source S at the output of the dipole, P_{pS}, is given by (*ergodicity* property)

$$P_{pS} = \lim_{T\to\infty} \frac{1}{T}\int_0^T v(t)i(t)dt \tag{114}$$
$$= \frac{1}{2}\mathrm{Re}\left(\widehat{V}\widehat{I}^*\right) \tag{115}$$

where $\widehat{V}$ and $\widehat{I}$ are phasors defined with *peak values*, instead of *root mean square* (*rms*) values, and therefore there is a $\frac{1}{2}$ factor.



In a load Z_L, we have (voltage divider):

$$\widehat{V} = \frac{Z_L}{Z_s + Z_L}\widehat{E}$$

and

$$\widehat{I} = \frac{\widehat{E}}{Z_s + Z_L}$$

Therefore,

$$P_{pS} = \frac{1}{2}\mathrm{Re}\left(\widehat{V}\widehat{I}^*\right) = \frac{1}{2}\mathrm{Re}\left(\frac{Z_L\widehat{E}}{Z_s + Z_L}\,\frac{\widehat{E}^*}{(Z_s + Z_L)^*}\right) = \frac{\mathrm{Re}\,(Z_L)\,\widehat{E}^2}{2\left\|Z_s + Z_L\right\|^2} \tag{116}$$

So, the *available* power (that is when $Z_L(f) = Z_S^*(f)$, so that $Z_s + Z_L = 2\,\mathrm{Re}\,(Z_s)$) from the *Source* is

$$P_{aS} = \frac{\mathrm{Re}\,(Z_L)\,\widehat{E}^2}{2\,\|Z_s + Z_L\|^2} = \frac{\mathrm{Re}\,(Z_L)\,\widehat{E}^2}{8\,\mathrm{Re}\,(Z_s)^2} = \frac{\widehat{E}^2}{8\,\mathrm{Re}\,(Z_s)} \tag{117}$$

The case of sinusoidal signals IV

Summary

The available power in the load is

$$P_{aS} = \frac{\widehat{E}^2}{8\,\mathrm{Re}\,(Z_s)} \tag{118}$$

By definition, the effective power produced by the source is (for an open circuit, matched impedance, so that $Z_s + Z_L = 2 \times \mathrm{Re}\,(Z_s)$)

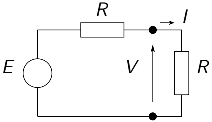
$$P_S = \frac{1}{2}\widehat{E}\widehat{I}^* = \frac{1}{2}\widehat{E}\frac{\widehat{E}^*}{(Z_s + Z_L)^*} = \frac{\widehat{E}^2}{4\,\mathrm{Re}\,(Z_s)} \tag{119}$$

It follows then that the power delivered by the Thévenin generator and the power dissipated in the generator's Thévenin resistance are the same.

Available noise power (matched impedance)

For an arbitrary random noise source, the provided noise power is (we make use of the *ergodicity* property)

$$P_{pN} = \lim_{T \to +\infty} \frac{1}{T} \int_0^T V(t) I(t) dt \tag{120}$$



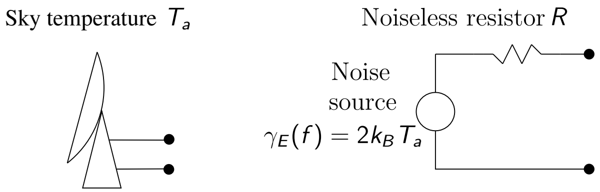
For the particular case of thermal noise

The available power spectral density is, by applying Wiener-Kintchine:

$$\gamma_{aN}(f) = \|\mathcal{H}(f)\|^2 \gamma_E(f) = \left(\frac{R}{R+R}\right)^2 2k_B T = \frac{k_B T}{2} \tag{121}$$

Remember that this assumes *impedances are matched* ($Z_s = Z_L^*$); otherwise, the result would depend on the impedances!

Noise at the output of an antenna pointing towards the sky



Example

Contributions to T_a:

- the sky contributes to 10 [K] (3 [K] of residual temperature of the Big Bang + 7 [K] due to atmospheric absorption).
- the ground contribution is typically 0.1 of 290 [K] (contributions of secondary lobes looking at the ground)

Thus,

$$T_a = 0.9 \times 10 + 0.1 \times 290 = 38\,[\mathrm{K}] \tag{122}$$

The noise power spectral density (PSD) is then

$$\gamma_{aN}(f) = \frac{k_B T}{2} = 2.6 \times 10^{-22} \left[\frac{\mathrm{W}}{\mathrm{Hz}}\right] \tag{123}$$

A note on matched impedances

There exist two ways to match loads:

1. $Z_S^* = Z_L$ (conjugate matching). This ensures the maximum transfer of power (that is the "*available power*").
2. $Z_c = Z_L$, where Z_c denotes the characteristic impedance of a transmission line. Z_c is the ratio of the amplitudes of voltage and current of a single wave propagating along the line. When $Z_c = Z_L$, there is no reflection.

In practice, what should we do when we connect a circuit to a line?

- $Z_S^* = Z_L$ is mandatory. It is the **matching condition for this chapter**.
- Luckily, $Z_c = R_c + jL_c$ and $L_c \ll R_c$, so that $Z_c \simeq R_c$.

We can almost fulfill both conditions simultaneously.

Summary I

	Arbitrary load	Matched load
Sinusoidal signals	$P_{pS} = \frac{1}{2}\text{Re}\left(\widehat{V}\widehat{I}^*\right) = \frac{\widehat{E}^2\text{Re}(Z_L)}{2\|Z_s+Z_L\|^2}$	$P_{aS} = \frac{\widehat{E}^2}{8\text{Re}(Z_s)}$
Stochastic processes	$P_{pN} = \lim_{T\to+\infty} \frac{1}{T}\int_0^T V(t)I(t)dt$	$\gamma_{aN}(f) = \frac{\gamma_E(f)}{4\text{Re}(Z_s)}$
Thermal noise		$\gamma_{aN}(f) = \frac{k_B T}{2}$

Table: Power provided by a one-port circuit.

Theorem (Available power)

The available power from a thermal source for a bandwidth B is

$$P_N = \int_{-\infty}^{+\infty} \gamma_{aN}(f)\, df = 2 \times \int_{f_0-\frac{B}{2}}^{f_0+\frac{B}{2}} \frac{k_B T}{2}\, df = k_B TB \qquad (124)$$

Summary II

Conclusions:

We have a model for the power of (thermal) noise

$$P_N = k_B T B \qquad (125)$$

It *does not depend on the impedance* (because we assume impedance matching). It depends only on

- the temperature T
- the bandwidth B

Noise power

Consider a thermal noise with a noise temperature of $T = 290\,[\mathrm{K}]$ and $B = 10\,[\mathrm{MHz}]$, then we have

$$P_N(f) = 1.38 \times 10^{-23} \times 290 \times 10^7 = 40 \times 10^{-15}\ [\mathrm{W}] \qquad (126)$$

Where does noise matter? [transmitter? channel? receiver?]

Summary III

$$P_N = k_B T B \tag{127}$$

As the noise power increases with the bandwidth, *we may want to reduce the bandwidth*.
However, the *capacity* is given by

$$C = B \log_2 \left(1 + \frac{S}{N}\right) \tag{128}$$

If S is fixed and we only have a white noise, then

$$C = B \log_2 \left(1 + \frac{S}{k_B T B}\right) \tag{129}$$

To increase the capacity C, there is no alternative other than to increase the bandwidth $\Rightarrow$ trade-off !

Definition (Noise temperature [at a given frequency])

The *noise temperature at a given frequency* is the absolute temperature that an impedance should have to produce, by thermal effect, for a given frequency, a noise power spectral density equal to that of the circuit.

By definition thus,

$$\gamma_{aN}(f) = \frac{k_B T(f)}{2} \tag{130}$$

Definition (Frequency average for $T(f)$ and bandwidth)

The maximum temperature, denoted T, the noise temperature of the dipole and the bandwidth are defined such that

$$P_{aN} = k_B T B \tag{131}$$

Noise temperature for a one-port circuit [generic model]

Definition (Noise temperature [at a given frequency])

The *noise temperature at a given frequency* is the absolute temperature that an impedance should have to produce, by thermal effect, for a given frequency, a noise power spectral density equal to that of the circuit.

By definition thus,

$$\gamma_{aN}(f) = \frac{k_B T(f)}{2} \tag{130}$$

Definition (Frequency average for $T(f)$ and bandwidth)

The maximum temperature, denoted T, the noise temperature of the dipole and the bandwidth are defined such that

$$P_{aN} = k_B T B \tag{131}$$

Signal to noise ratio of a one-port circuit

Definition

The Signal to Noise ratio (S/N) of a dipole is defined as the ratio between the available power of the signal and the noise power

$$\frac{S}{N} = \frac{P_{aS}}{P_{aN}} \tag{132}$$

Warning: in the following S denotes the Signal and not the noise source!

By convention, if the signal is modulated, the power of the useful signal is defined as follows (it is just a reference to compare techniques):

- for amplitude or angular modulations, we consider the *power of the carrier* (it is then a carrier to noise ratio $\frac{C}{N}$),
- for suppressed carrier amplitude modulation techniques, we take the *mean power of the modulating signal*, and
- for impulse coded modulation techniques, it is the *peak power*.

Characterization of a two-port circuit (quadripole) I

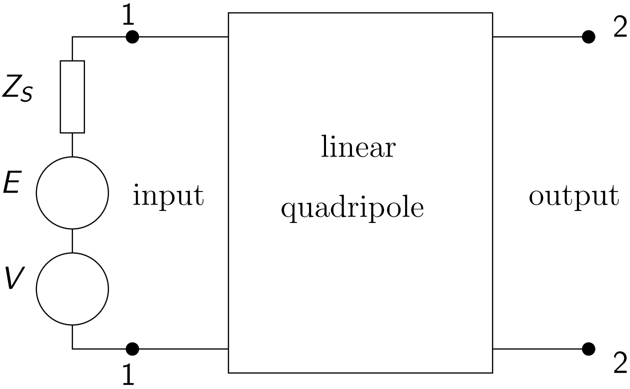


Figure: Scheme of a two-port circuit.

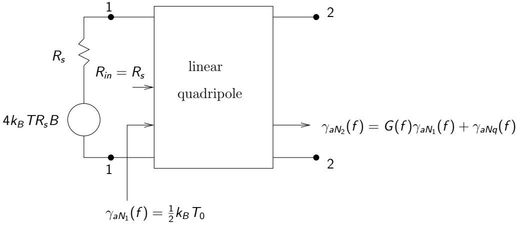
Steps:

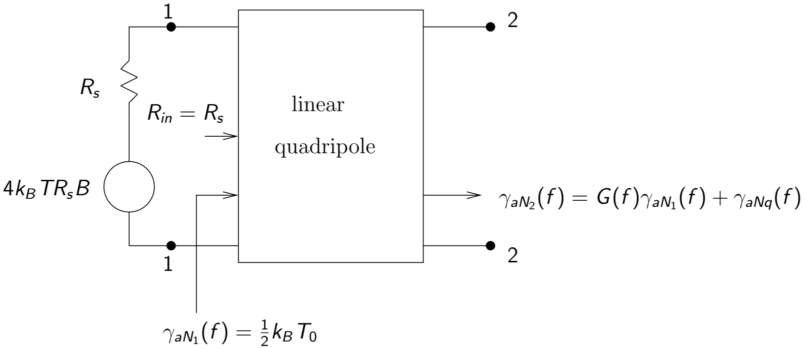
- Notion of *gain*?
- Characterization of the amount of internal noise of the quadripole by means of the normalized notion of *noise figure*
- *Equivalent* circuits
- *Figure of merit* (*not* normalized)
- *Effective noise temperature*
- Special case: purely resistive *attenuator*

Hypothesis: loads are *matched* (complex conjugate) at the *input* and at the *output*

Definition (Noise Figure (NF), F_0)

Assuming fixed internal impedances, the *(spot) noise figure* of a two-port circuit, for a given frequency f, denoted $F_0(f)$, is the ratio between
(1) the noise power spectral density at the output of the quadripole, for the appropriate frequency f, when the noise temperature of the generating one-port put at the input is normalized to be $T_0 = 290\,[K]$, and
(2) only the contribution of the generating input source, at a frequency f, to the noise power spectral density at the output.





So, by definition,

$$F_0(f) = \frac{\gamma_{aN_2}(f)}{G(f)\gamma_{aN_1}(f)} = \frac{G(f)\gamma_{aN_1}(f) + \gamma_{aNq}(f)}{G(f)\gamma_{aN_1}(f)} > 1 \qquad (133)$$

Equivalent circuit

$$F_0(f) = \frac{G(f)\gamma_{aN_1}(f) + \gamma_{aNq}(f)}{G(f)\gamma_{aN_1}(f)} \Longleftrightarrow \gamma_{aNq}(f) = \left[\left(F_0(f) - 1\right)\gamma_{aN_1}(f)\right] G(f)$$

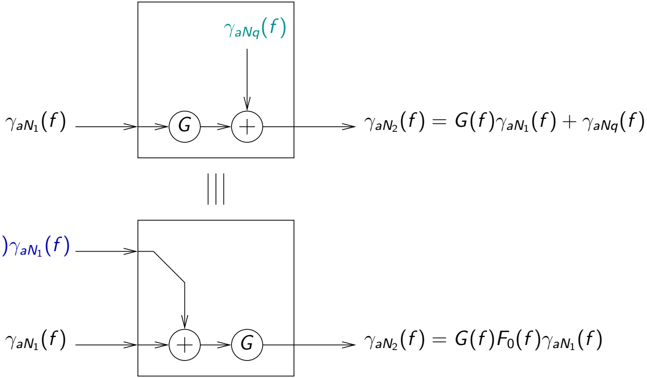


Figure: Scheme of a noisy quadripole and an equivalent circuit (we model the internal noise by $\left(F_0(f) - 1\right)\gamma_{aN_1}(f)$ at the entrance).

Interpretation of the notion of noise figure

The input signal to noise ratio is given by

$$\left(\frac{S}{N}\right)_{\text{in}} = \frac{\gamma_{\text{in}}(f)}{\gamma_{aN_1}(f)} \tag{134}$$

At the output of the two-port circuit, we have

$$\left(\frac{S}{N}\right)_{\text{out}} = \frac{\gamma_{\text{out}}(f)}{\gamma_{aN_2}(f)} \tag{135}$$

Therefore,

$$\frac{\left(\frac{S}{N}\right)_{\text{in}}}{\left(\frac{S}{N}\right)_{\text{out}}} = \frac{\gamma_{\text{in}}(f)}{\gamma_{aN_1}(f)}\,\frac{\gamma_{aN_2}(f)}{\gamma_{\text{out}}(f)} \tag{136}$$

As $\gamma_{\text{out}}(f) = G(f)\gamma_{\text{in}}(f)$, this ratio becomes

$$\frac{\left(\frac{S}{N}\right)_{\text{in}}}{\left(\frac{S}{N}\right)_{\text{out}}} = \frac{\gamma_{aN_2}(f)}{\gamma_{aN_1}(f)G(f)} = \frac{G(f)F_0(f)\gamma_{aN_1}(f)}{\gamma_{aN_1}(f)G(f)} = F_0(f) \tag{137}$$

$\boxed{F_0 \text{ therefore expresses the signal to noise ratio degradation.}}$

What if the input temperature $T_s \neq T_0$?
This leads to define a different notion $\Rightarrow$ the *figure of merit F*.

Link between F and F_0 (F_0 is provided by the manufacturer)?
Remember that, considering $\gamma_{aN_1}(f)\,|_{T=T_0} = \frac{1}{2}k_B T_0$:

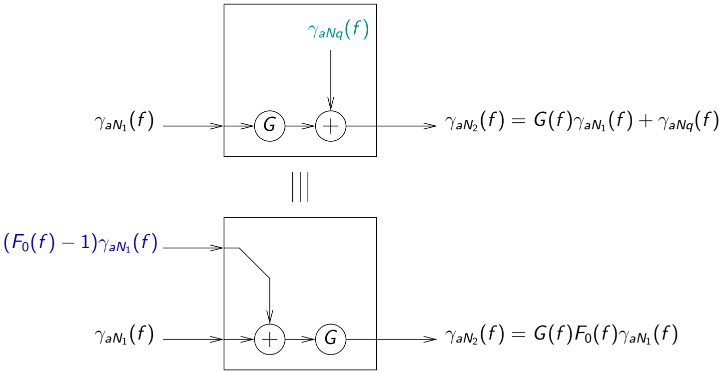
$$\gamma_{aNq}(f) = (F_0 - 1)\,\frac{1}{2}k_B T_0 G(f) \tag{138}$$

The internal noise of a two-port circuit is independent of the input temperature (the last is just a convention). Therefore, another temperature T_s then leads to another figure of merit F. It is derived as follows

$$\gamma_{aNq}(f) = (F_0 - 1)\,\frac{1}{2}k_B T_0 G(f) = (F - 1)\,\frac{1}{2}k_B T_s G(f) \tag{139}$$

and, finally,

$$\boxed{F = 1 + \frac{T_0}{T_s}(F_0 - 1)} \tag{140}$$



$$\gamma_{aN_2}(f) = \frac{1}{2}k_B T_0 G(f) + \gamma_{aNq}(f) = \frac{1}{2}k_B \left[T_0 + (F_0 - 1)T_0\right] G(f)$$

Effective noise temperature of a two-port circuit II

Definition (*Effective (input-)noise temperature*)

$$T_e = (F_0 - 1)\,T_0 \tag{141}$$

It is the additional temperature required for an input source to produce the same available power at the output.

Note that:

$$T_e = (F_0 - 1)\,T_0 \;\;\Leftrightarrow\;\; F_0 = 1 + \frac{T_e}{T_0} \tag{142}$$

Noisy two-port circuit

Consider an effective noise temperature $T_e = 120\,[\mathrm{K}]$, then

$$F_0 = 1 + \frac{120}{290} = 1.41 = 1.5\,[\mathrm{dB}] \tag{143}$$

Let us take an **attenuator** (for example a purely resistive circuit or a lossy transmission line) at temperature T_0.

(1) Assuming matched input and output impedances, the *available noise power at the output* is

$$\gamma_{aN_2}(f) = \frac{1}{2} k_B T_0 \tag{144}$$

(2) But that the attenuator is characterized by its effective noise temperature T_e, then the output noise power is

$$\gamma_{aN_2}(f) = \frac{1}{2} k_B \left(T_0 + T_e\right) \frac{1}{L} \tag{145}$$

By combining these two expressions: $T_e = (L-1)T_0$. So that,

$$F_0 = 1 + \frac{T_e}{T_0} = 1 + \frac{(L-1)\,T_0}{T_0} = L \tag{146}$$

Conclusion: attenuators have a noise figure F_0 equal to their attenuation ratio L when their physical temperature equals T_0.

Attenuator with a "Gain" $G = 1/L$ or Loss L

Let us take an **attenuator** (for example a purely resistive circuit or a lossy transmission line) at temperature T_0.

(1) Assuming matched input and output impedances, the *available noise power at the output* is

$$\gamma_{aN_2}(f) = \frac{1}{2} k_B T_0 \tag{144}$$

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Conclusion: attenuators have a noise figure F_0 equal to their attenuation ratio L when their physical temperature equals T_0.

What about the attenuator L at other temperatures?

Let us consider $T_s \neq T_0$ and calculate F.

There are two possible ways to calculate F:

1. same reasoning as previously: it is impossible to discriminate the output of the two-port circuit from the input one-port circuit, so that
$$\gamma_{aN_2}(f) = \frac{1}{2} k_B T_S \quad = \quad \gamma_{aN_2}(f) = \frac{1}{2} k_B \left(T_S + T_e\right) \frac{1}{L} \tag{147}$$
and $F = L$.
2. Remember that F_0 represents the signal to noise degradation
$$F_0 = \left(\frac{S}{N}\right)_{\text{in}} / \left(\frac{S}{N}\right)_{\text{out}} = L \tag{148}$$
But, as $L = S_{\text{in}}/S_{\text{out}}$, we have $N_{\text{in}} = N_{\text{out}}$.

Conclusion: for an attenuator with a factor L, the amount of noise is always unaffected, so that
$$F_0 = F = L \tag{149}$$

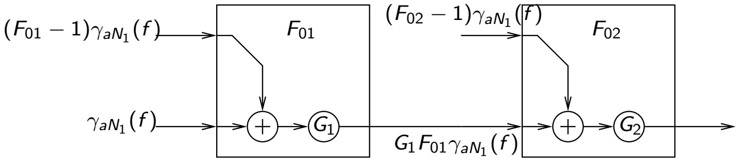


Figure: Cascading two-port elements.

For a two-port network with 2 stages,

$$
\begin{aligned}
F_0 &= \frac{\gamma_{aN_1}(f)G_1G_2 + (F_{01}-1)\,\gamma_{aN_1}(f)G_1G_2 + (F_{02}-1)\,\gamma_{aN_1}(f)G_2}{\gamma_{aN_1}(f)G_1G_2} \\
&= 1 + (F_{01}-1) + \frac{(F_{02}-1)}{G_1} \qquad (150) \\
&= F_{01} + \frac{(F_{02}-1)}{G_1} \qquad (151)
\end{aligned}
$$

Noise figure of two-port networks II

For a two-port network with n stages,

$$F_0 = F_{01} + \frac{F_{02} - 1}{G_1} + \frac{F_{03} - 1}{G_1 G_2} + \cdots = F_{01} + \sum_{i=2}^{n} \frac{F_{0i} - 1}{\prod_{j=1}^{i-1} G_j} \quad (152)$$

Likewise,

$$T_e = T_{e1} + \frac{T_{e2}}{G_1} + \frac{T_{e3}}{G_1 G_2} + \cdots = T_{e1} + \sum_{i=2}^{n} \frac{T_{ei}}{\prod_{j=1}^{i-1} G_j} \quad (153)$$

$$F_0 = F_{01} + \frac{F_{02} - 1}{G_1} + \frac{F_{03} - 1}{G_1 G_2} + \cdots \tag{154}$$

Two consequences:

1. the noise figure always increases with an additional stage.
2. the overall noise figure of a receiver is primarily set by the noise figure of its first amplifying stage.

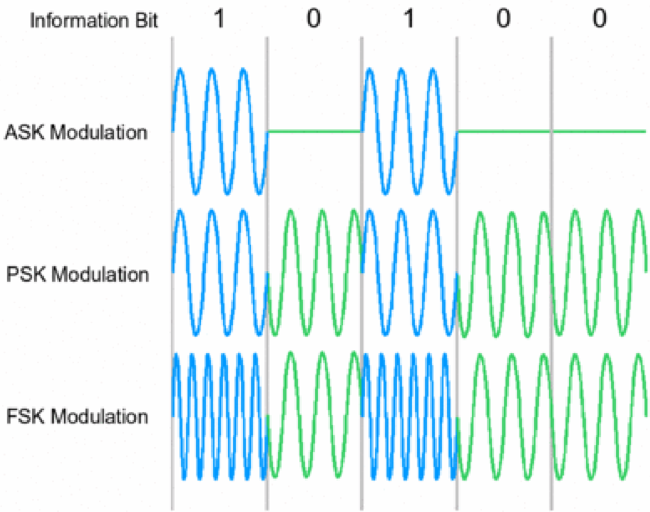
Therefore, the first stage amplifier is often a *Low-Noise Amplifier* (LNA). Then, the overall receiver noise figure is

$$F_{\text{receiver}} \simeq F_{\text{LNA}} + \frac{F_{\text{others}} - 1}{G_{\text{LNA}}} \tag{155}$$

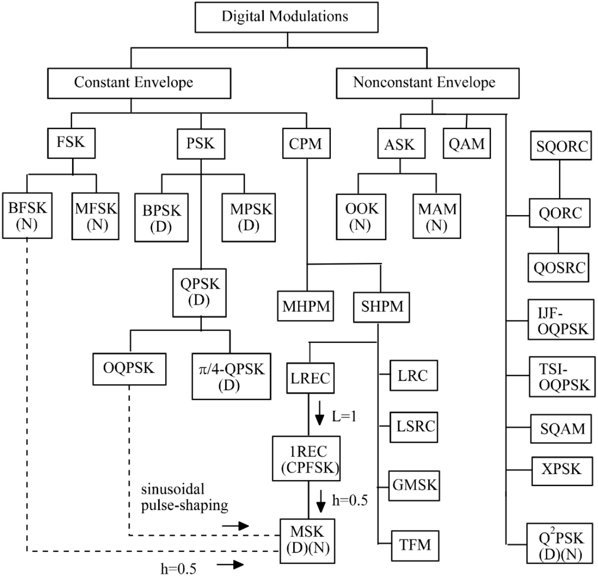
Outline

1. Criteria ruling the selection of a modulation scheme
2. Definition and *typology* of digital modulations
3. Classic linear modulations
   - Description
   - Determination of the power spectrum
   - Amplitude modulation (ASK)
   - Phase modulation (PSK)
   - Quadrature modulation (QPSK)
4. Offset modulations
   - Description
   - Determination of the power spectrum
   - Offset quadrature modulation (OQPSK)
   - Minimum shift modulation techniques (MSK)

Introduction to digital modulation



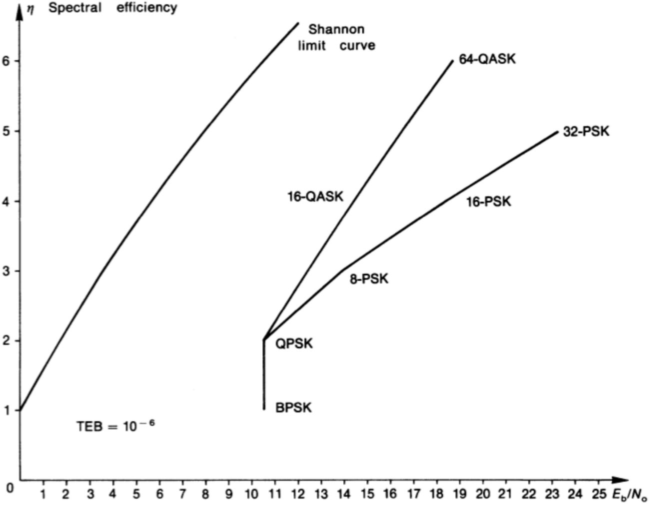
A subset of digital modulation techniques



Criteria for choosing a modulation scheme

1. *Resistance* to distortion and perturbation. That includes:
   - resistance to additive white Gaussian *noise*. Noise usually results in a *bit error rate* P_e, expressed in terms of the E_b/N_0 ratio.
   - sensitivity to *interference* (multipath, other users, etc.).
   - sensitivity to *imperfect filters*. This is associated to the phenomenon of *intersymbol interference*.
   - sensitivity to *non-linearities*.
2. *Spectral* occupancy:
   1. spectral efficiency η, expressed in bit per second per Hertz $\left[\frac{\text{b/s}}{\text{Hz}}\right]$, which measures the bit rate that can be transmitted per unit of frequency bandwidth for a given modulation.
   2. asymptotic behavior, defined by the values of the spectral density for frequencies relatively distant from the carrier frequency.
3. *Simplicity* of implementation.

Spectral efficiency of linear modulation techniques



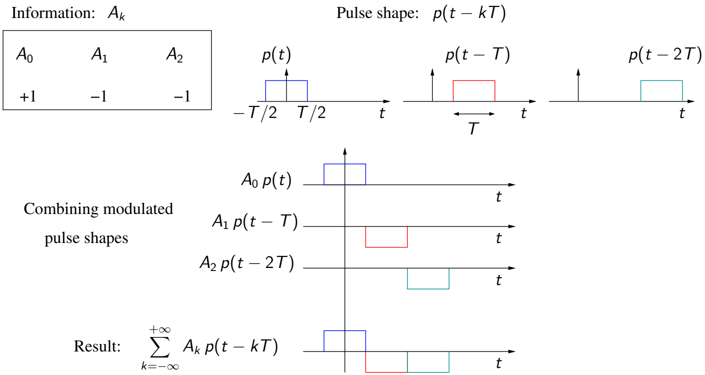
$$C = B\,\log_2\left(1 + \frac{E_b R_b}{B\,N_0}\right) \quad \xrightarrow{\text{max: } R_b \to C} \quad \eta \simeq \frac{R_b}{B} = \log_2\left(1 + \eta\,\frac{E_b}{N_0}\right)$$

Understanding digital modulations

We will make an intensive use of alternative representations

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
v(t) \arrow[r, leftrightarrow, "v_I(t)\cos(.) - v_Q(t)\sin(.)"] \arrow[d, shift left, "\otimes\left(\delta(t) + \frac{j}{\pi t}\right)"] & v_I(t) + jv_Q(t) \arrow[d, leftrightarrow, "\equiv"] & \\
v_a(t) \arrow[u, shift left, "\mathrm{Re}(.)"] \arrow[r, shift left, "\times e^{-2\pi j f_0 t}"] & e_v(t) \arrow[l, shift left, "\times e^{+2\pi j f_0 t}"] \arrow[r, leftrightarrow, "\equiv"] & a_v(t)e^{j\phi_v(t)}
\end{tikzcd}
$$

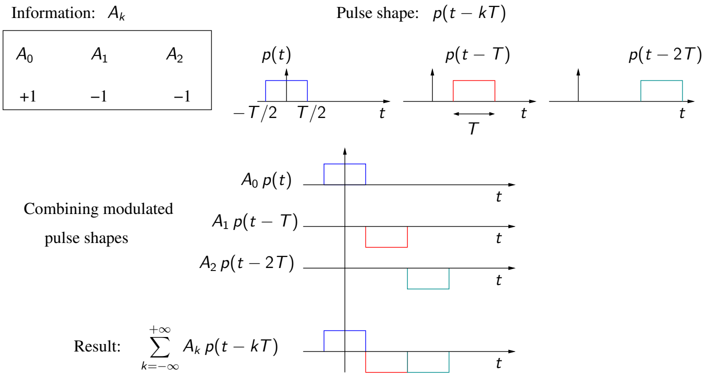
Reminder: building a digital signal $\sum_{k=-\infty}^{+\infty} A_k\, p(t-kT)$



Main characteristics of digital signals:

1. With digital signals, the fundamental unit is a time slot T.
2. One information symbol A_k per time slot T (no overlap). $p(t-kT)$ **thus also acts as a time window!**
3. Same pulse shape for each time slot (eases the task of the receiver).

Reminder: building a digital signal $\sum_{k=-\infty}^{+\infty} A_k\, p(t-kT)$



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   $p(t-kT)$ **thus also acts as a time window!**
3. Same pulse shape for each time slot (eases the task of the receiver).

Definition and typology of digital modulations I

Definition (General expression for digital modulations (based on the complex envelope))

$$s(t) = \mathrm{Re}\left(\psi\left[m(t)\right] e^{j(2\pi f_c t + \varphi_c)}\right) \tag{156}$$

The complex function $\psi\left[m(t)\right]$, which is related to the modulating waveform $m(t)$, defines the type of modulation. It is also the complex envelope $e_s(t)$ of the modulated signal $s(t)$.
Depending on the form of $\psi(.) = \psi_I(.) + j\,\psi_Q(.)$, we generally distinguish:

- **linear modulations** for which $\psi\left[m(t)\right]$ is a linear function of $m(t)$.
- **angular modulations** for which $\psi\left[m(t)\right]$ has the form of

$$\psi\left[m(t)\right] = e^{j\varphi[m(t)]} \tag{157}$$

where $\varphi\left[m(t)\right]$ is a linear function of $m(t)$.

The modulated signal can also be expressed as

$$s(t) = \psi_I\left[m(t)\right] \cos\left(2\pi f_c t + \varphi_c\right) - \psi_Q\left[m(t)\right] \sin\left(2\pi f_c t + \varphi_c\right) \tag{158}$$

and

$$s(t) = \left\|\psi\left[m(t)\right]\right\| \cos\left(2\pi f_c t + \varphi_c + \arg\psi\left[m(t)\right]\right) \tag{159}$$

In the following, we will focus on modulations that can be written as

$$s(t) = \mathrm{Re}\left(\left(\sum_{k=-\infty}^{+\infty} d_k(t)\, e^{j(\theta_k - 2\pi f_c k T)}\right) e^{j(2\pi f_c t + \varphi_c)}\right) \tag{160}$$

Two types of linear modulations will be studied:

1. *"classic" modulations*, for which $\theta_k = 2\pi f_c k T$
2. *offset modulations*, for which $\theta_k = 2\pi f_c k T + k\frac{\pi}{2}$

Usually, one takes $\varphi_c = 0$.

Description

Classic linear modulations are such that $\theta_k = 2\pi f_c kT$. Therefore, we have that

$$s(t) = \text{Re}\left(e_s(t)\, e^{j(2\pi f_c t + \varphi_c)}\right) \tag{161}$$

The complex envelope takes the form

$$\begin{aligned} e_s(t) &= \sum_{k=-\infty}^{+\infty} d_k(t) && (162) \\ &= \sum_{k=-\infty}^{+\infty} D_k\, p_k(t-kT) && (163) \end{aligned}$$

where $D_k = A_k + jB_k$ is complex and A_k, B_k are two real random variables.

In most cases, the pulse shape of $p_k(t-kT)$ is the same for each symbol k. Therefore, $p_k(.)$ becomes $p()$.

The complex envelope can also be written as $e_s(t) = s_I(t) + js_Q(t)$, so that

$$s_I(t) = \sum_{k=-\infty}^{+\infty} A_k \, p(t - kT) \qquad (164)$$

$$s_Q(t) = \sum_{k=-\infty}^{+\infty} B_k \, p(t - kT) \qquad (165)$$

resulting in

$$s(t) = s_I(t) \cos(2\pi f_c t + \varphi_c) - s_Q(t) \sin(2\pi f_c t + \varphi_c) \qquad (166)$$

and, by replacing s_I and s_Q by their value,

$$s(t) = \left[\sum_{k=-\infty}^{+\infty} A_k \, p(t - kT) \right] \cos(2\pi f_c t + \varphi_c) - \left[\sum_{k=-\infty}^{+\infty} B_k \, p(t - kT) \right] \sin(2\pi f_c t + \varphi_c)$$

Power spectral density of a modulated signal?
The modulated signal is a stochastic process $S(t)$ that can be written, taking $\varphi_c = 0$, as

$$S(t) = \mathrm{Re}\left(M(t)\, e^{j2\pi f_c t}\right) \tag{167}$$

where $M(t)$ is a complex stochastic process (such as the complex envelope $e_s(t)$ in our case).

But is $S(t)$ stationary?
Obviously, we take $M(t)$ stationary. However, even then

$$\mu_S = E\left\{S(t)\right\} = \mathrm{Re}\left(\mu_M\, e^{j2\pi f_c t}\right) \tag{168}$$

is time-dependent (not constant), unless $\mu_M = 0$.
Therefore, as such, we cannot calculate $\gamma_S(f)$.

Derivation of the power spectral density II

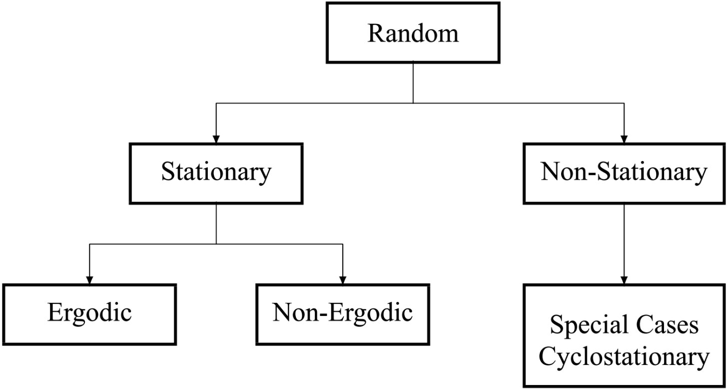


Figure: Taxonomy of random signals.

Stationarization

The $S(t)$ process is *not stationary* because its mean is time-dependent. *We have to "stationarize" the signal*.

For that purpose, we add the random phase Θ whose probability density function (pdf) is uniformly distributed over $[0, 2\pi[$ (in other words, $\text{pdf}_{\Theta}(\theta) = \frac{1}{2\pi}$ for $\theta \in [0, 2\pi[$ and 0 outside):

$$S(t) = \text{Re}\left(M(t)\, e^{j(2\pi f_c t + \Theta)}\right) \tag{169}$$

Derivation of the power spectral density IV

Mean of $S(t) = \mathbf{Re}\left(M(t)\, e^{j(2\pi f_c t+\Theta)}\right)$?

Note that $M(t)$ and Θ are independent.

For the computation, we do

1. $\mathrm{Re}\,(a+jb)$ is replaced by $\mathrm{Re}\,(a+jb) = \frac{a+jb}{2} + \frac{a-jb}{2}$
2. Then, we take the expectation of both terms: the expectation of a sum is the sum of the expectations.
3. Then, for the first term, we have

$$\begin{aligned}\frac{1}{2}E\left\{M(t)\, e^{j(2\pi f_c t+\Theta)}\right\} &= \frac{1}{2}E\left\{M(t)\right\}E\left\{e^{j(2\pi f_c t+\Theta)}\right\}\\ &= \frac{1}{2}\mu_M \int_0^{2\pi} e^{j(2\pi f_c t+\theta)}\frac{1}{2\pi}d\theta\\ &= \frac{1}{2}\mu_M \times 0 = 0\end{aligned}$$

$$\mu_S = E\left\{S(t)\right\} = 0 \tag{170}$$

Autocorrelation function of $S(t)$?

$$\Gamma_{SS}\left(t,\, t-\tau\right) = E\left\{S(t)\, S(t-\tau)\right\} \tag{171}$$

As

$$S(t) = \mathrm{Re}\left(M(t)\, e^{j(2\pi f_c t + \Theta)}\right) \tag{172}$$

$$= \frac{1}{2}\left[M(t)\, e^{j(2\pi f_c t + \Theta)} + M^*(t)\, e^{-j(2\pi f_c t + \Theta)}\right] \tag{173}$$

we have

$$\begin{aligned} S(t)\, S(t-\tau) \;&=\; \frac{1}{2}\left[M(t)\, e^{j(2\pi f_c t + \Theta)} + M^*(t)\, e^{-j(2\pi f_c t + \Theta)}\right] \\ &\times\; \frac{1}{2}\left[M(t-\tau)\, e^{j(2\pi f_c (t-\tau) + \Theta)} + M^*(t-\tau)\, e^{-j(2\pi f_c (t-\tau) + \Theta)}\right] \end{aligned} \tag{174}$$

Derivation of the power spectral density VI

$$
\begin{aligned}
S(t)\, S(t-\tau) &= \frac{1}{4} M(t)\, e^{j(2\pi f_c t+\Theta)} M(t-\tau)\, e^{j(2\pi f_c (t-\tau)+\Theta)} && (175)\\
&+ \frac{1}{4} M(t)\, e^{j(2\pi f_c t+\Theta)} M^*(t-\tau)\, e^{-j(2\pi f_c (t-\tau)+\Theta)} && (176)\\
&+ \frac{1}{4} M^*(t)\, e^{-j(2\pi f_c t+\Theta)} M(t-\tau)\, e^{j(2\pi f_c (t-\tau)+\Theta)} && (177)\\
&+ \frac{1}{4} M^*(t)\, e^{-j(2\pi f_c t+\Theta)} M^*(t-\tau)\, e^{-j(2\pi f_c (t-\tau)+\Theta)}
\end{aligned}
$$

$$
\begin{aligned}
S(t)\, S(t-\tau) &= \frac{1}{4} M(t)\, M(t-\tau)\, e^{j(2\pi f_c (2t-\tau)+2\Theta)} && (178)\\
&+ \frac{1}{4} M(t)\, M^*(t-\tau)\, e^{j2\pi f_c \tau} && (179)\\
&+ \frac{1}{4} M^*(t)\, M(t-\tau)\, e^{-j2\pi f_c \tau} && (180)\\
&+ \frac{1}{4} M^*(t)\, M^*(t-\tau)\, e^{-j(2\pi f_c (2t-\tau)+2\Theta)} && (181)
\end{aligned}
$$

Because, $\Gamma_{SS}(t,\, t-\tau) = E\{S(t)\, S(t-\tau)\}$, the terms of $E\{.\}$ containing 2Θ are null. Therefore, we are left with

$$\begin{aligned}
\Gamma_{SS}(t,\, t-\tau) &= \frac{1}{4} E\left\{M(t)\, M^*(t-\tau)\, e^{j2\pi f_c \tau} + M^*(t)\, M(t-\tau)\, e^{-j2\pi f_c \tau}\right\} \\
&= \frac{1}{4} E\left\{2\,\mathrm{Re}\left(M(t)\, M^*(t-\tau)\, e^{j2\pi f_c \tau}\right)\right\} && (182) \\
&= \frac{1}{2} \mathrm{Re}\left(E\left\{M(t)\, M^*(t-\tau)\, e^{j2\pi f_c \tau}\right\}\right) && (183) \\
&= \frac{1}{2} \mathrm{Re}\left(\Gamma_{MM}(t,\, t-\tau)\; e^{j2\pi f_c \tau}\right) && (184)
\end{aligned}$$

Finally, because

$$\Gamma_{SS}(\tau) = \frac{1}{4}\left[\Gamma_{MM}(\tau)\; e^{j2\pi f_c \tau} + \Gamma_{MM}(\tau)^*\; e^{-j2\pi f_c \tau}\right] \tag{185}$$

and $x^*(t) \leftrightarrow \mathcal{X}^*(-f)$, we have that

$$\gamma_S(f) = \frac{\gamma_M(f-f_c) + \gamma_M^*(-f-f_c)}{4} \tag{186}$$

where $\gamma_M(f)$ is the power spectral density of $M(t)$.

Derivation of the power spectral density VIII

Power spectral density of a modulation signal for classic linear modulation techniques

If γ_M is real:

$$\gamma_S(f) = \frac{\gamma_M(f - f_c) + \gamma_M(f + f_c)}{4} \tag{187}$$

If γ_M is complex:

$$\gamma_S(f) = \frac{\gamma_M(f - f_c) + \gamma_M^*(-f - f_c)}{4} \tag{188}$$

Calculation of the power spectrum of the complex envelope (for classic linear modulation techniques)

The complex envelope of the modulated signal is $(e_s(t) = M(t))$

$$M(t) = \sum_{k=-\infty}^{+\infty} D_k\, p(t - kT) \tag{189}$$

The sequence of complex random variables D_k is characterized by

- mean: $\mu_D = E\{D_k\}$
- variance: $\sigma_D^2 = E\{(D_k - \mu_D)(D_k - \mu_D)^*\}$
- autocorrelation function: $\Gamma_{DD}(k, k-l) = E\{D_k D_{k-l}^*\}$
- covariance function: $C_{DD}(k, k-l) = E\{(D_k - \mu_D)(D_{k-l} - \mu_D)^*\}$

After the stationarization of D_k, the PSD of the baseband complex envelope is (see first course in telecommunications)

$$\gamma_M(f) = \frac{\|\mathcal{P}(f)\|^2}{T}\left[\sigma_D^2 + \|\mu_D\|^2 \sum_{m=-\infty}^{+\infty} \frac{1}{T}\delta\left(f - \frac{m}{T}\right)\right] \tag{190}$$

In **conclusion** ($\gamma_M(f)$ being real in this case):

$$\gamma_S(f) = \frac{\gamma_M(f - f_c) + \gamma_M(f + f_c)}{4} \tag{191}$$

Definition (Complex envelope of the Amplitude Shift Keying (ASK))

$$e_s(t) = \sum_{k=-\infty}^{+\infty} A_k \, p(t - kT) \tag{192}$$

Common choice for the shaping pulse function over $[0, T]$:

$$p(t) = \mathrm{rect}_{[0,T]}(t) \tag{193}$$

Goal of the following slides: we need to find the envelope $a(t)$ and the phase $\varphi(t)$ of the modulated signal. These two signals can be derived from

$$e_s(t) = a(t)\, e^{j\varphi(t)} \tag{194}$$

Amplitude Shift Keying (amplitude modulation) II

Trick: $\pm A_k$ can be expressed as $\pm A_k = \|A_k\| \, e^{\left(1-\text{sign}(A_k)\right)\frac{\pi}{2}j}$

Let us verify this:

- A_k is positive ($A_k = \|A_k\|$):

$$A_k = \|A_k\| \, e^{\left(1-\text{sign}(A_k)\right)\frac{\pi}{2}j} = \|A_k\| \, e^{(1-1)\frac{\pi}{2}j} = \|A_k\| \, e^{0\,\frac{\pi}{2}j} = \|A_k\| \quad (195)$$

- A_k is negative ($A_k = - \|A_k\|$):

$$A_k = \|A_k\| \, e^{(1-(-1))\frac{\pi}{2}j} = \|A_k\| \, e^{2\,\frac{\pi}{2}j} = \|A_k\| \, e^{\pi j} = - \|A_k\| \quad (196)$$

So, we have

$$e_s(t) = \sum_{k=-\infty}^{+\infty} A_k \, p(t - kT) \quad (197)$$

$$= \sum_{k=-\infty}^{+\infty} \|A_k\| \, e^{\left(1-\text{sign}(A_k)\right)\frac{\pi}{2}j} \, p(t - kT) \quad (198)$$

Amplitude Shift Keying (amplitude modulation) III

Therefore, the envelope $a(t)$ and the phase $\varphi(t)$ of the modulated signal are given by

$$a(t) = \sum_{k=-\infty}^{+\infty} \|A_k\| \, \mathrm{rect}_{[0,T]}(t-kT) \tag{199}$$

$$\varphi(t) = \sum_{k=-\infty}^{+\infty} \frac{\pi}{2}\left(1-\mathrm{sign}\left(A_k\right)\right) \mathrm{rect}_{[0,T]}(t-kT) \tag{200}$$

Note the presence of the **time windowing function** $\mathrm{rect}_{[0,T]}(t-kT)$ in these expressions. **Why?**

Power spectral density of the ASK-2

Hypothesis: both signals $\pm A$ have an equal probability!

- The mean μ_A of the random variable A_k is equal to 0.
- The variance is given by $\sigma_A^2 = E\left\{A_k^2\right\} = A^2$.
- $p(t) = \mathrm{rect}_{[0,T]}(t)$. Thus, its Fourier transform is

$$\mathcal{P}(f) = e^{-j2\pi f \frac{T}{2}}\, T\, sinc(fT) \tag{201}$$

Therefore, the power spectrum of the complex envelop is $\left(\gamma_{e_s}(f) = \frac{\|\mathcal{P}(f)\|^2}{T}\left[\sigma_A^2 + \|\mu_A\|^2 \sum_{m=-\infty}^{+\infty} \frac{1}{T}\delta\left(f - \frac{m}{T}\right)\right]\right)$:

$$\gamma_{e_s}(f) = A^2\, T\, sinc^2\,(fT) \tag{202}$$

and that of the ASK-2 modulated signal is

$$\gamma_s(f) = \frac{A^2\, T\left\{sinc^2\left[(f - f_c)\, T\right] + sinc^2\left[(f + f_c)\, T\right]\right\}}{4} \tag{203}$$

The PSD of ASK-2 is

$$\gamma_s(f) = \frac{A^2 T \left\{ sinc^2 \left[(f - f_c)\, T \right] + sinc^2 \left[(f + f_c)\, T \right] \right\}}{4} \qquad (204)$$

At fixed bitrate R_b:

Technique	bandwidth	spectral efficiency η
Baseband (NRZ)	$W = 0.6\, R_b$	$\eta = \frac{R_b}{0.6\, R_b} \simeq 1.6$
ASK-2	$B = 2 \times 0.6\, R_b = 1.2\, R_b$	$\eta = \frac{R_b}{1.2\, R_b} \simeq 0.8$

Constellation or state diagram: definition I

We have:

$$s(t) = s_I(t) \cos(2\pi f_c t) - s_Q(t) \sin(2\pi f_c t) \quad (205)$$

and, for the complex envelope:

$$e_s(t) = s_I(t) + j s_Q(t) \quad (206)$$

The alternative is

$$s(t) = a_s(t) \cos(2\pi f_c t + \phi_s(t)) \quad (207)$$

with

$$a_s(t) = \sqrt{s_I^2(t) + s_Q^2(t)} \quad (208)$$

and

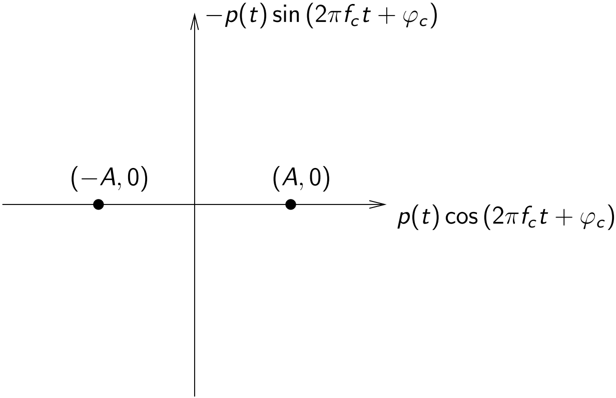
$$\phi_s(t) = \tan^{-1} \frac{s_Q(t)}{s_I(t)} \quad (209)$$

Definition (Constellation diagram)

The plot of $e_s(t)$ in a diagram whose axis units are $(\cos(2\pi f_c t), -\sin(2\pi f_c t))$ defines the *constellation diagram*.

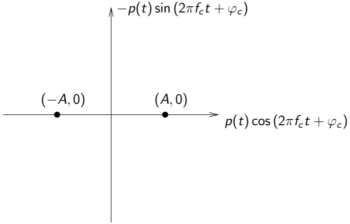
Constellation or state diagram: definition II

Constellation diagram of a ASK-2 or BPSK modulation



The presence of $p(t)$ reinforces the idea that there is only one symbol per time slot, but it is often dropped. Likewise, φ_c is often taken equal to 0.

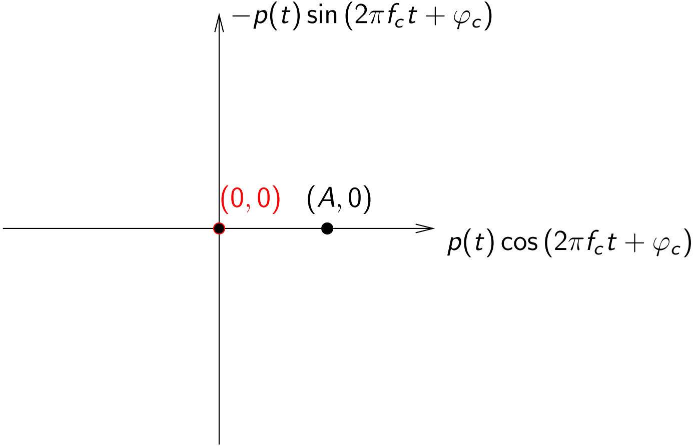
Constellation diagram: purposes



- Representation of the possible states in the complex plane.
- See how the state diagram is used (here, we immediately see that the sin() axis is not used).
- The distance between states is essential for finding the P_e. Closer states mean less resistance to noise.
- See the paths from one state to another (trajectories).

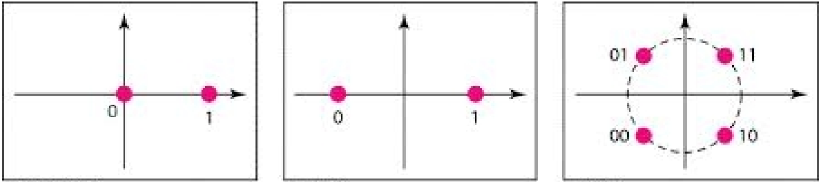
Note that we move from one state to another state at the rhythm of the symbol rate (not the bit rate!)

Constellation diagram of an On-Off Shift Keying (OOK)



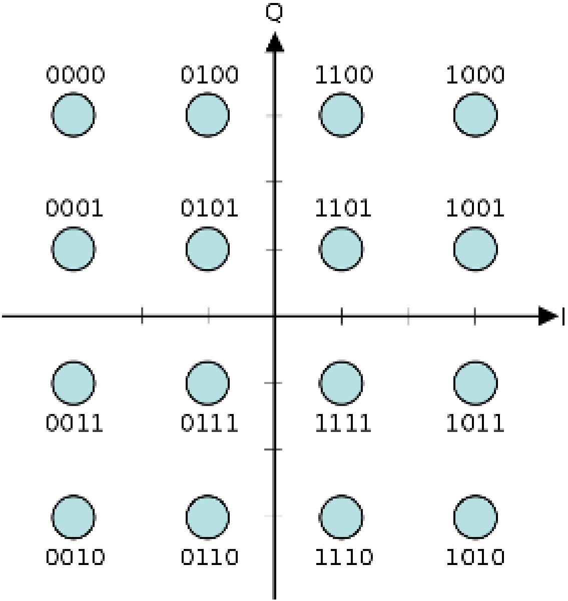
Infrared signals are usually sent using On-Off Shift Keying (because it is hard to determine the phase of an infrared signal).

Constellation diagrams



OOK, ASK-2≡BPSK≡PSK-2, and QPSK.

16-QAM constellation diagram

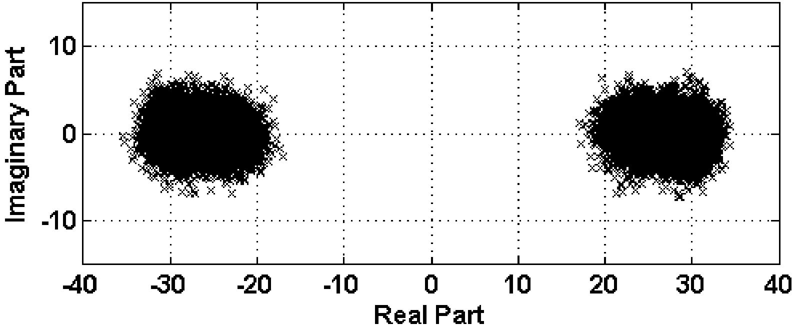


Example of a noisy (real) constellation diagram (ASK-2)

Noise can also be expressed according to its Rice's decomposition:

$$n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \tag{210}$$

Therefore, for a noisy modulated ASK-2 modulation, we will observe



Definition (Phase Shift Keying modulation (PSK))

$$s(t) = A \sum_{k=-\infty}^{+\infty} \mathrm{rect}_{[0,T]}(t - kT) \cos(2\pi f_c t + \varphi_c + \psi_k) \quad (211)$$

where ψ_k is a random variable that:

1. remains constant over the $[kT, (k+1)T[$ interval,
2. takes a value among N possible values:

$$\psi_k \in \left\{ \psi \,\middle|\, \psi = \varphi_0 + i\frac{2\pi}{N},\ i = 0, \ldots, N-1 \right\} \quad (212)$$

The modulated signal is (using $\cos(a+b) = \cos a \cos b - \sin a \sin b$)

$$\begin{aligned}
s(t) &= A \sum_{k=-\infty}^{+\infty} \mathrm{rect}_{[0,T]}(t-kT)\left[\cos(2\pi f_c t + \varphi_c)\cos\psi_k - \sin(2\pi f_c t + \varphi_c)\sin\psi_k\right] \\
&= \sum_{k=-\infty}^{+\infty} \Big(\big[A\cos\psi_k \,\mathrm{rect}_{[0,T]}(t-kT)\big]\cos(2\pi f_c t + \varphi_c) && (213) \\
&\qquad -\big[A\sin\psi_k \,\mathrm{rect}_{[0,T]}(t-kT)\big]\sin(2\pi f_c t + \varphi_c)\Big) && (214) \\
&= s_I(t)\cos(2\pi f_c t + \varphi_c) - s_Q(t)\sin(2\pi f_c t + \varphi_c) && (215)
\end{aligned}$$

We then derive its complex envelope

$$\begin{aligned}
e_s(t) &= s_I(t) + j\, s_Q(t) && (216) \\
&= A \sum_{k=-\infty}^{+\infty} \mathrm{rect}_{[0,T]}(t-kT)\left(\cos\psi_k + j\sin\psi_k\right) && (217)
\end{aligned}$$

Phase Shift Keying modulation (PSK) III

$$e_s(t) = A \sum_{k=-\infty}^{+\infty} \mathrm{rect}_{[0,T]}(t - kT)\,(\cos \psi_k + j \sin \psi_k) \tag{218}$$

$$= A \sum_{k=-\infty}^{+\infty} \mathrm{rect}_{[0,T]}(t - kT)\, e^{j\psi_k} \tag{219}$$

Conclusions for a PSK modulation:

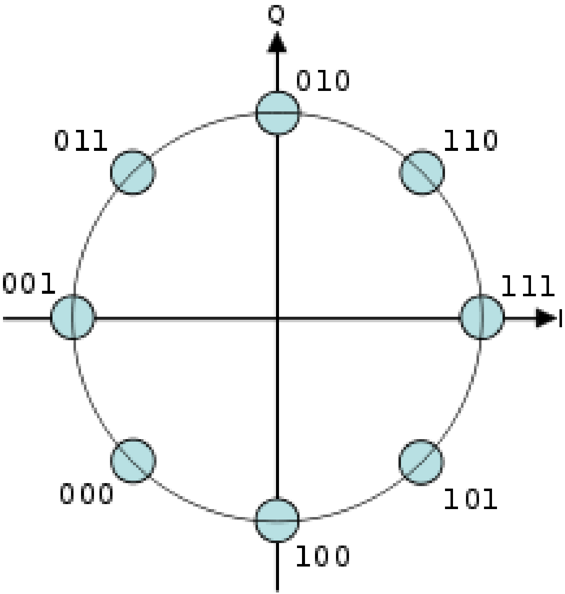
Therefore, the envelope and phase of the modulated signal $s(t)$ are:

$$a(t) = A \sum_{k=-\infty}^{+\infty} \mathrm{rect}_{[0,T]}(t - kT) = A \tag{220}$$

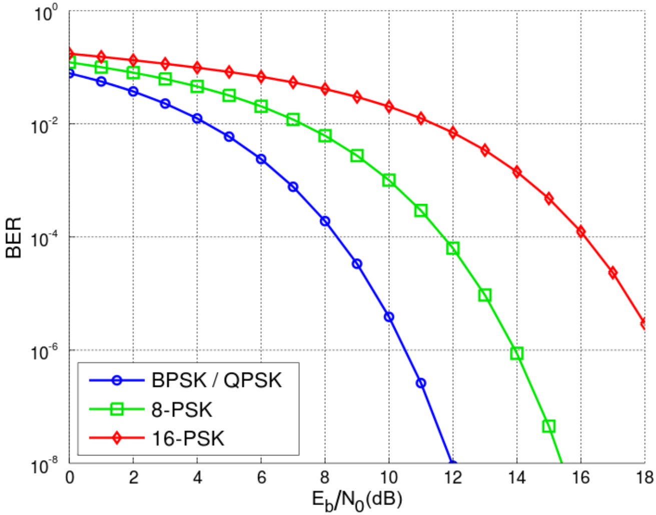
$$\varphi(t) = \sum_{k=-\infty}^{+\infty} \psi_k\, \mathrm{rect}_{[0,T]}(t - kT) \tag{221}$$

The envelope of a PSK modulated signal is constant.

8-PSK constellation diagram



Bit error rates when increasing the number of states



The PSD of ASK-2 is

$$\gamma_s(f) = \frac{A^2 T \left\{ sinc^2 \left[(f - f_c) T \right] + sinc^2 \left[(f + f_c) T \right] \right\}}{4} \tag{222}$$

At fixed bitrate R_b:

modulation	bandwidth	spectral efficiency η
ASK-2	$1.2\, R_b$	$\eta = \frac{R_b}{1.2\, R_b} \simeq 0.8$
BPSK ($\equiv$ PSK-2)	$1.2\, R_b$	$\eta = \frac{R_b}{1.2\, R_b} \simeq 0.8$

Digital modulation

1. Criteria ruling the selection of a modulation scheme
2. Definition and *typology* of digital modulations
3. Classic linear modulations
   - Description
   - Determination of the power spectrum
   - Amplitude modulation (ASK)
   - Phase modulation (PSK)
   - *Quadrature modulation (QPSK)*
4. *Offset modulations*
   - *Description*
   - *Determination of the power spectrum*
   - *Offset quadrature modulation (OQPSK)*
   - *Minimum shift modulation techniques (MSK)*

Quadrature Phase Shift Keying modulation (QPSK) I

Why would we be using quadrature modulation?

1. Starting point: for quadrature modulation, we would like to use the $\cos(2\pi f_c t)$ axis AND the $\sin(2\pi f_c t)$ axis.
2. There exist techniques capable to recover $m_1(t)$ and $m_2(t)$ separately from a modulated signal such as

$$s(t) = m_1(t)\cos(2\pi f_c t) + m_2(t)\sin(2\pi f_c t) \tag{223}$$

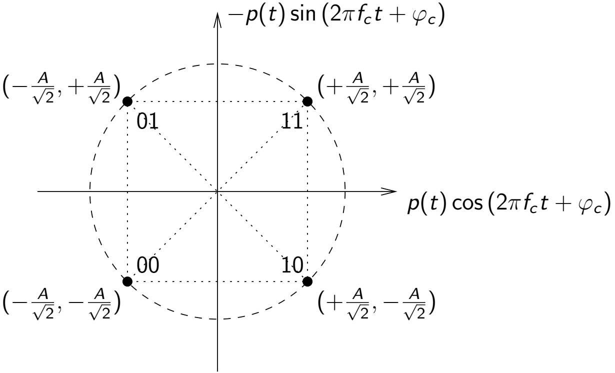
3. Benefit: reduce the used bandwidth ($m_1(t) \neq m_2(t)$) or reinforce the signal ($m_1(t) = m_2(t)$).

Definition (QPSK)

A Quadrature Phase Shift Keying (QPSK) typically uses different phases (4 in this particular example):

$$D_k \in \left\{ Ae^{-j\frac{3\pi}{4}}, Ae^{-j\frac{\pi}{4}}, Ae^{j\frac{\pi}{4}}, Ae^{j\frac{3\pi}{4}} \right\} \tag{224}$$

This corresponds to the following constellation diagram



Side note: this diagram is equivalent to a constellation diagram whose states are located on the axes (diagram rotated by $\frac{\pi}{4}$)!

Quadrature Phase Shift Keying modulation (QPSK) III

For the development, we introduce a new binary random sequence!

Let us consider a binary source generated by a regularly time-spaced series of Dirac delta functions:

$$I(t) = \sum_{k=-\infty}^{+\infty} I_k \, \delta(t - kT_b) \tag{225}$$

where $I_k = -1$ or $+1$ corresponds to a binary symbol 0 or 1, respectively. It is a sequence whose symbols are normalized to an amplitude of 1.

QPSK: decomposition into 2 sequences I

Starting from $I(t)$, we split the original sequence in two sequences with a "*slower*" bitrate (*even* I_{2k} and *odd* I_{2k+1} bits):

$$s_I(t) = \frac{A}{\sqrt{2}} \sum_{k=-\infty}^{+\infty} I_{2k}\, p(t-kT) = \sum_{k=-\infty}^{+\infty} A_k\, p(t-kT) \quad (226)$$

$$s_Q(t) = \frac{A}{\sqrt{2}} \sum_{k=-\infty}^{+\infty} I_{2k+1}\, p(t-kT) = \sum_{k=-\infty}^{+\infty} B_k\, p(t-kT)$$

where

- $T = 2T_b$. T is twice longer than T_b!
- $p(t)$ is a T-long rectangular pulse: $p(t) = \mathrm{rect}_{[0,T]}(t)$
- $A_k = I_{2k}\frac{A}{\sqrt{2}}$ (*even* index bits) and $B_k = I_{2k+1}\frac{A}{\sqrt{2}}$ (*odd* index bits)

Note that the *bit* rate R_b is twice the *symbol* rate R_S ($R_S = R_b/2$)

QPSK: decomposition into 2 sequences II

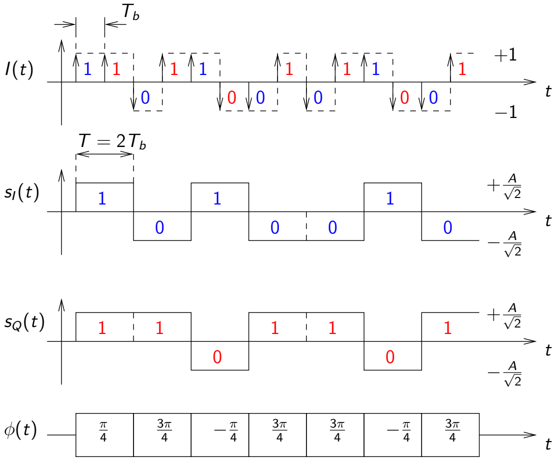


Figure: Construction of in-phase and quadrature signals of a QPSK (remember that $I_k = -1$ for a 0 bit and $I_k = +1$ for a 1 bit).

Complex envelope, amplitude, and phase of a QPSK I

$$
\begin{aligned}
e_s(t) &= s_I(t) + j\, s_Q(t) = \sum_{k=-\infty}^{+\infty} (A_k + j\, B_k)\, \mathrm{rect}_{[0,T]}(t - kT) \\
&= \frac{A}{\sqrt{2}} \sum_{k=-\infty}^{+\infty} (I_{2k} + j\, I_{2k+1})\, \mathrm{rect}_{[0,T]}(t - kT) \qquad (227)
\end{aligned}
$$

$$
\begin{aligned}
a(t) &= \sqrt{s_I^2(t) + s_Q^2(t)} \qquad (228) \\
&= \frac{A}{\sqrt{2}} \sum_{k=-\infty}^{+\infty} \sqrt{I_{2k}^2 + I_{2k+1}^2}\, \mathrm{rect}_{[0,T]}(t - kT) \qquad (229) \\
&= A \sum_{k=-\infty}^{+\infty} \mathrm{rect}_{[0,T]}(t - kT) = A \qquad (230)
\end{aligned}
$$

$\Rightarrow$ The amplitude of a QPSK is constant.

Complex envelope, amplitude, and phase of a QPSK II

Phase?

$$\varphi(t) = \tan^{-1}\left(\frac{s_Q(t)}{s_I(t)}\right) \tag{231}$$

$$= \sum_{k=-\infty}^{+\infty} \mathrm{rect}_{[0,T]}(t-kT)\tan^{-1}\left(\frac{I_{2k+1}}{I_{2k}}\right) \tag{232}$$

Conclusions about the phase of a QPSK modulated signal:

- knowing that $I_{2k},\ I_{2k+1} = \pm 1$, we have $\tan^{-1}\left(\frac{I_{2k+1}}{I_{2k}}\right) = \pm\frac{\pi}{4}$.
- but there is an ambiguity between $\frac{1}{1}$ or $\frac{-1}{-1}$, and $\frac{-1}{1}$ or $\frac{1}{-1}$.

QPSK: illustration

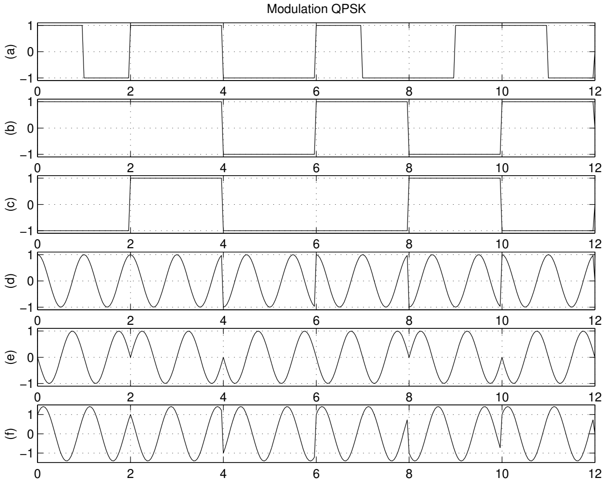
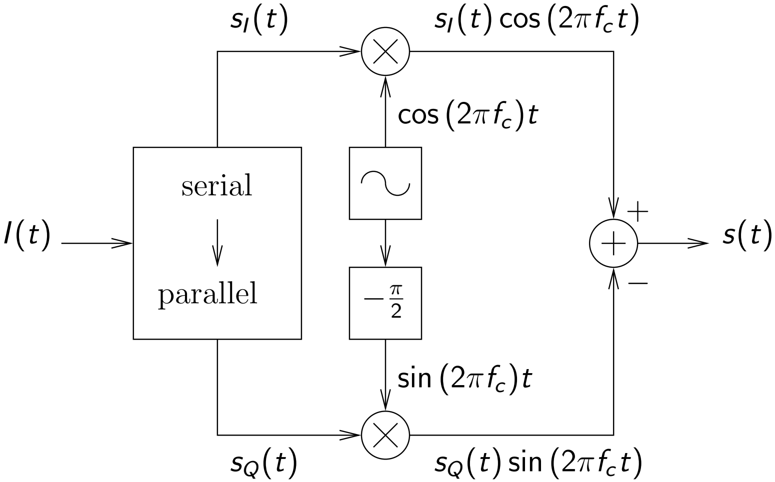
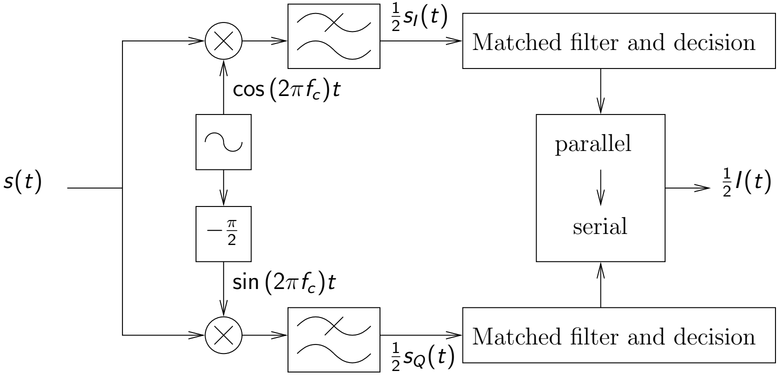


Figure: Illustration for the QPSK modulation: (a) binary input sequence $I(t)$, (b) in-phase signal $s_I(t)$, (c) quadrature signal $s_Q(t)$, (d) $s_I(t)\cos(2\pi f_c t)$, (e) $s_Q(t)\sin(2\pi f_c t)$, and (f) modulated signal $s(t)$.

QPSK modulator



QPSK demodulator



QPSK-4: calculation of the power spectral density

Reasoning:

1. the two signals $s_I(t)$ and $s_Q(t)$ that modulate either a $\cos(2\pi f_c t)$ or a $\sin(2\pi f_c t)$ are independent.
2. the power spectral density of a two-state amplitude modulated signal (ASK-2), such as $s_I(t)\cos(2\pi f_c t)$ is known to be

$$\gamma_{s_I}(f) = \frac{B^2 T \left\{ sinc^2\left[(f - f_c)\, T\right] + sinc^2\left[(f + f_c)\, T\right]\right\}}{4} \tag{233}$$

   where the amplitude B is, in this case, $A/\sqrt{2}$.
3. the power spectral density of a sum of two independent stochastic processes is the sum of the power spectral densities.

Taking $T = 2T_b$,

$$\gamma_s(f) = \frac{A^2 T_b \left\{ sinc^2\left[(f - f_c)\, 2T_b\right] + sinc^2\left[(f + f_c)\, 2T_b\right]\right\}}{2} \tag{234}$$

For the same bitrate, it needs half the bandwidth of that of a PSK.

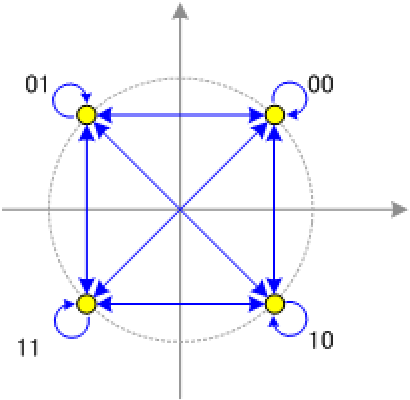
We use the bitrate R_b and the symbol rate R_S.

modulation	bandwidth	spectral efficiency η
ASK-2	$1.2\,R_b$	$\eta = \frac{R_b}{1.2\,R_b} \simeq 0.8$
BPSK ($\equiv$ PSK-2)	$1.2\,R_b$	$\eta = \frac{R_b}{1.2\,R_b} \simeq 0.8$
QPSK	$1.2\,R_S = 0.6\,R_b$	$\eta = \frac{R_b}{0.6\,R_b} \simeq 1.6$
ASK-2^N or PSK-2^N	$1.2\,R_S = 1.2\,\frac{R_b}{N}$	$\eta = \frac{R_b}{1.2\,R_b}\,N \simeq 0.8\,N$

Concatenating N bits in larger symbols has the following consequences:

- (+) it increases the spectral efficiency.
- (–) for the same power level, it decreases the average distance between symbols, which leads to an increase in the Bit Error Rate (BER or P_e).
  - Solution: use pre-coding to change the input sequence. Probabilities to move between neighboring states should be less than that of distant states in the constellation diagram.

QPSK trajectories



- Possible transitions: self, close neighbors, opposite state (transition crosses the origin)
- Time between state changes: $2T_b$ (changes occur at the speed of the symbol rate R_S)

General form for linear modulation techniques:

$$s(t) = \mathrm{Re}\left(e^{j(2\pi f_c t + \varphi_c)} \sum_{k=-\infty}^{+\infty} d_k(t)\, e^{j(\theta_k - 2\pi f_c k T_b)} \right) \tag{235}$$

Offset modulations have $\theta_k = 2\pi f_c k T_b + k\frac{\pi}{2}$.

Definition (Offset modulation)

The complex envelope of offset modulation techniques is:

$$e_s(t) = \sum_{k=-\infty}^{+\infty} A_k\, p(t - k T_b)\, e^{jk\frac{\pi}{2}} \tag{236}$$

So, we rotate the constellation diagram by $\frac{\pi}{2}$ after each bit (and by π after each symbol)!

$$\begin{aligned} s(t) &= \sum_{k=-\infty}^{+\infty} A_k\, p(t-kT_b)\cos\left(2\pi f_c t+\varphi_c+k\frac{\pi}{2}\right) &(237)\\ &= \left[\sum_{k=-\infty}^{+\infty} A_k\, p(t-kT_b)\cos\left(k\frac{\pi}{2}\right)\right]\cos\left(2\pi f_c t+\varphi_c\right)\\ &\quad-\left[\sum_{k=-\infty}^{+\infty} A_k\, p(t-kT_b)\sin\left(k\frac{\pi}{2}\right)\right]\sin\left(2\pi f_c t+\varphi_c\right) \end{aligned}$$

- For k odd $(1, 3, 5, \ldots) \Rightarrow \cos\left(k\frac{\pi}{2}\right) = 0$.
  One out of two cosine terms is null; remaining terms: k even.
- For k even $(0, 2, 4, \ldots) \Rightarrow \sin\left(k\frac{\pi}{2}\right) = 0$.
  One out of two sine terms is null; remaining terms: k odd.

Offset modulation techniques III

Considering that $\cos(k\pi/2) = 0$ for k odd and that $\sin(k\pi/2) = 0$ for k even, in-phase $s_I(t)$ and quadrature $s_Q(t)$ are respectively:

$$
\begin{aligned}
s_I(t) &= \sum_{k=-\infty}^{+\infty} A_k\, p(t - kT_b) \cos\left(k\frac{\pi}{2}\right) && (238) \\
&= \sum_{k'=-\infty}^{+\infty} A_{2k'}\, p\left(t - 2k'T_b\right) \cos\left(2k'\frac{\pi}{2}\right) && (239) \\
&= \sum_{k'=-\infty}^{+\infty} A_{2k'} \left(-1\right)^{k'} p\left(t - 2k'T_b\right) && (240)
\end{aligned}
$$

and

$$
\begin{aligned}
s_Q(t) &= \sum_{k=-\infty}^{+\infty} A_k\, p(t - kT_b) \sin\left(k\frac{\pi}{2}\right) && (241) \\
&= \sum_{k'=-\infty}^{+\infty} A_{2k'+1} \left(-1\right)^{k'} p\left(t - \left(2k'+1\right) T_b\right) && (242)
\end{aligned}
$$

Building an Offset Quadrature Phase Shift Keying signal
Consider a binary source generated by a regularly time-spaced series of Dirac delta functions:

$$I(t) = \sum_{k=-\infty}^{+\infty} I_k\, \delta(t - kT_b) \tag{243}$$

We construct two signals with the odd and even bits of the input sequence:

$$\begin{aligned} s_I(t) &= \frac{A}{\sqrt{2}} \sum_{k=-\infty}^{+\infty} (-1)^k\, I_{2k}\, p\,(t - 2kT_b) = \sum_{k=-\infty}^{+\infty} A_{2k}\, p\,(t - kT) \\ s_Q(t) &= \frac{A}{\sqrt{2}} \sum_{k=-\infty}^{+\infty} (-1)^k\, I_{2k+1}\, p\,(t - (2k+1)\, T_b) = \sum_{k=-\infty}^{+\infty} A_{2k+1}\, p\,(t - kT - T_b) \end{aligned} \tag{244}$$

where $p(t)$ is a unit rectangular pulse over $[0,\ 2T_b] = [0,\ T]$, $A_{2k} = \frac{A}{\sqrt{2}}(-1)^k\, I_{2k}$ and $A_{2k+1} = \frac{A}{\sqrt{2}}(-1)^k\, I_{2k+1}$, $\forall k$.

Discussion on the encoding of 0 and 1 bits

Let us consider the following bit stream: $b_0b_1b_2b_3 = 1\,1\,1\,0$
If we take $I_\alpha = -1$ or $+1$ respectively when the bit is 0 and 1, then according to $A_{2k} = \frac{A}{\sqrt{2}}(-1)^k\, I_{2k}$ and $A_{2k+1} = \frac{A}{\sqrt{2}}(-1)^k\, I_{2k+1}$, we would have the following encoding:

Bit	Value	k	Amplitude	Final value
b_0	1	0	$A_{2k} = \frac{A}{\sqrt{2}}(-1)^k\, I_{2k}$	$\frac{A}{\sqrt{2}}(-1)^0(+1) = +\frac{A}{\sqrt{2}}$
b_1	1	0	$A_{2k+1} = \frac{A}{\sqrt{2}}(-1)^k\, I_{2k+1}$	$\frac{A}{\sqrt{2}}(-1)^0(+1) = +\frac{A}{\sqrt{2}}$
b_2	1	1	$A_{2k} = \frac{A}{\sqrt{2}}(-1)^k\, I_{2k}$	$\frac{A}{\sqrt{2}}(-1)^1(+1) = -\frac{A}{\sqrt{2}}$
b_3	0	1	$A_{2k+1} = \frac{A}{\sqrt{2}}(-1)^k\, I_{2k+1}$	$\frac{A}{\sqrt{2}}(-1)^1(-1) = +\frac{A}{\sqrt{2}}$

Offset Quadrature Phase Shift Keying (OQPSK) III

Impact on the constellation diagram

$$\mathrm{b}_0\mathrm{b}_1\mathrm{b}_2\mathrm{b}_3 = 1\,1\,1\,0 \longrightarrow +\frac{A}{\sqrt{2}} + \frac{A}{\sqrt{2}} - \frac{A}{\sqrt{2}} + \frac{A}{\sqrt{2}} \tag{245}$$

Therefore, consecutive equal even bits such as 11 result in $+-$ polarities (which is due to two consecutive turns of $\frac{\pi}{2}$ between consecutive even bits).

However, for the ease of the interpretation only (on the constellation diagram and time plot), we do not rotate the diagram, which means that we will ignore $(-1)^k$ in the A_{2k} and A_{2k+1} terms.

In other words, each symbol, such as $(1, 1)$, will remain at the same location in the constellation diagram.

Offset Quadrature Phase Shift Keying (OQPSK) IV

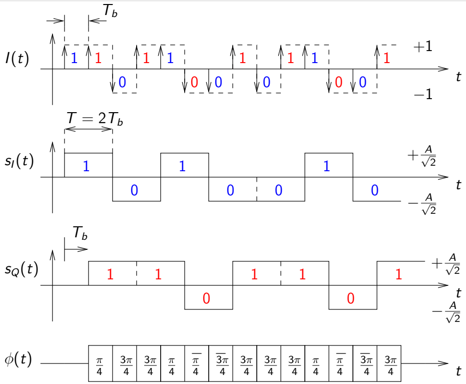


Figure: Formation of in-phase and quadrature signals of a OQPSK (for this drawing, we assume that the constellation diagram does not turn by $\frac{\pi}{2}$ for each bit).

Important note: the signal changes every T_b instead of every $2T_b$!

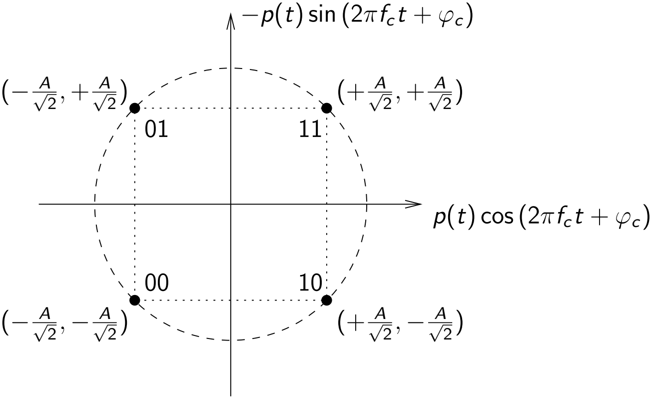


Figure: Constellation diagram of an OQPSK (note: we have frozen the $\frac{\pi}{2}$ rotation factor between consecutive bits).

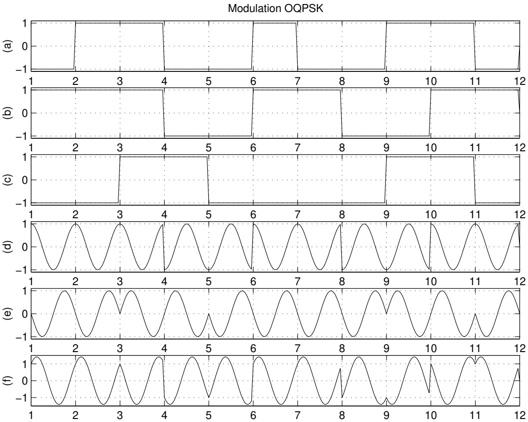


Figure: Illustration for the OQPSK modulation: (a) binary input sequence $I(t)$, (b) in-phase signal $s_I(t)$, (c) quadrature signal $s_Q(t)$, (d) $s_I(t)\cos(2\pi f_c t)$, (e) $s_Q(t)\sin(2\pi f_c t)$, and (f) modulated signal $s(t)$.

Power spectral density of the OQPSK modulation I

First guess?

- construction of two independent sequences.
- one sequence modulates the $\cos(.)$ function, the other the $\sin(.)$ function.
- the power spectrum of a sum is the sum of the power spectrum densities if the sequences are independent.

$\Rightarrow$ why should the spectral occupancy of an OQPSK be different to that of a QPSK?

Towards an analytic expression of the power spectral density of an offset modulation.

Trick: we modify the expression of the pulse function by adding the $e^{-j2\pi\frac{t}{4T_b}}$ term (later compensated in the general expression by $e^{j2\pi\frac{t}{4T_b}}$), to get:

$$p(t-kT_b)\,e^{jk\frac{\pi}{2}}\,e^{-j2\pi\frac{t}{4T_b}} = p(t-kT_b)\,e^{-j\frac{\pi}{2T_b}(t-kT_b)} \tag{246}$$

The introduction of a phase factor in the envelope has to be compensated by introducing a new carrier frequency f_c'

$$\begin{aligned} s(t) &= \mathrm{Re}\left(e_s(t)\,e^{-j2\pi\frac{t}{4T_b}}\,e^{j(2\pi f_c t+\varphi_c)}\,e^{j2\pi\frac{t}{4T_b}}\right) && (247)\\ &= \mathrm{Re}\left(e_s(t)\,e^{-j2\pi\frac{t}{4T_b}}\,e^{j\left(2\pi\left(f_c+\frac{1}{4T_b}\right)t+\varphi_c\right)}\right) && (248)\end{aligned}$$

So,

$$s(t) = \mathrm{Re}\left(e_s(t)\, e^{-j2\pi\frac{t}{4T_b}}\; e^{j\left(2\pi\left(f_c+\frac{1}{4T_b}\right)t+\varphi_c\right)}\right)$$

We introduce $z(t) = e_s(t)e^{-j2\pi\frac{t}{4T_b}}$ and $f_c' = f_c + \frac{1}{4T_b}$

$$s(t) \quad = \quad \mathrm{Re}\left(z(t)\, e^{j(2\pi f_c' t+\varphi_c)}\right) \tag{249}$$

If the power spectral density of the complex signal $z(t)$ was known, then

$$\gamma_s(f) = \frac{\gamma_z\left(f - f_c'\right) + \gamma_z^*\left(-f - f_c'\right)}{4} \tag{250}$$

$$= \frac{\gamma_z\left(f - f_c - \frac{1}{4T_b}\right) + \gamma_z^*\left(-f - f_c - \frac{1}{4T_b}\right)}{4} \tag{251}$$

It remains to calculate $\gamma_z(f)$:

$$z(t) = \sum_{k=-\infty}^{+\infty} A_k\, p(t-kT_b)\, e^{jk\frac{\pi}{2}}\, e^{-j2\pi\frac{t}{4T_b}} \tag{252}$$

$$= \sum_{k=-\infty}^{+\infty} A_k\, p(t-kT_b)\, e^{-j\frac{\pi}{2T_b}(t-kT_b)} \tag{253}$$

$$= \sum_{k=-\infty}^{+\infty} A_k\, h(t-kT_b) \tag{254}$$

where we have defined the new function $h(x)$ by

$$h(x) = p(x)\, e^{-j\frac{\pi}{2T_b}x} = p(x)\, e^{2\pi jx\left(-\frac{1}{4T_b}\right)} \tag{255}$$

$h(t)$ is in fact a new "shaping" function whose Fourier transform is derived from that of $p(t)$ by

$$\mathcal{H}(f) = \mathcal{P}\left(f - \left(-\frac{1}{4T_b}\right)\right) = \mathcal{P}\left(f + \frac{1}{4T_b}\right) \tag{256}$$

Therefore, we only need to adapt the power spectral density of $z(t)$ by considering $\mathcal{H}(f)$ in replacement of $\mathcal{P}(f)$

$$\gamma_z(f) = \frac{\left\|\mathcal{P}\left(f + \frac{1}{4T_b}\right)\right\|^2}{T_b}\left[\sigma_A^2 + \mu_A^2 \sum_{m=-\infty}^{+\infty} \frac{1}{T_b}\delta\left(f - \frac{m}{T_b}\right)\right] \tag{257}$$

in

$$\gamma_s(f) = \frac{\gamma_z\left(f - f_c - \frac{1}{4T_b}\right) + \gamma_z^*\left(-f - f_c - \frac{1}{4T_b}\right)}{4} \tag{258}$$

Obviously, the $\frac{1}{4T_b}$ component of the filter annihilates the factor $-\frac{1}{4T_b}$ in the expression of $\gamma_s(f)$ (except for the series of $\delta()$)

In **conclusion**, *the OQPSK and the QPSK have the same power spectral density* (except for the locations of $\delta()$ in some cases).

OQPSK-4: calculation of the power spectral density

The complex envelope of the modulated signal is given by

$$e_s(t) = \sum_{k=-\infty}^{+\infty} A_k \, p(t - kT_b) \, e^{jk\frac{\pi}{2}} \quad (259)$$

where $p(t)$ is the shaping function and A_k is the random variable containing the information, respectively given by

$$p(t) = \mathrm{rect}_{[0,\, 2T_b]}(t) \quad (260)$$

$$A_k \in \left\{ +\frac{A}{\sqrt{2}}, -\frac{A}{\sqrt{2}} \right\} \quad (261)$$

The power spectral density of $z(t)$ is calculated as follows:

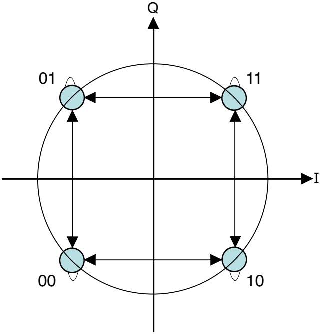
$$\gamma_z(f) = 2A^2 T_b \, sinc^2 \left[\left(f + \frac{1}{4T_b} \right) 2T_b \right] \quad (262)$$

Therefore, the power spectral density of the OQPSK modulation is

$$\gamma_s(f) = \frac{A^2 T_b}{2} \left\{ sinc^2 \left[(f - f_c) \, 2T_b \right] + sinc^2 \left[(f + f_c) \, 2T_b \right] \right\} \quad (263)$$

which is also that of the QPSK.

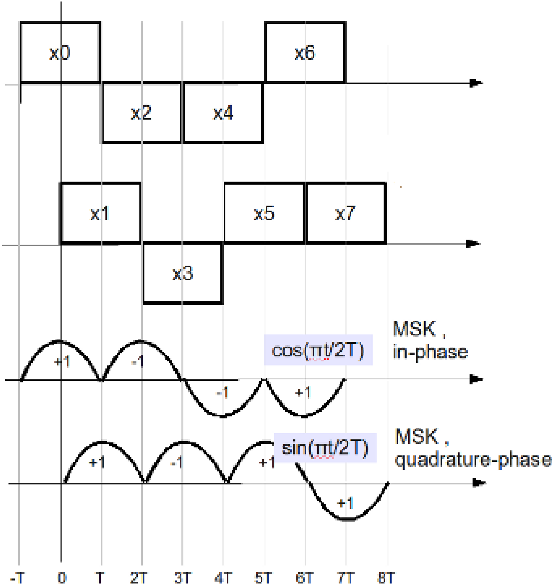
OQPSK-4 trajectories in the constellation diagram



[reminder: for this drawing, we assume that the constellation diagram does not turn by $\frac{\pi}{2}$ after each bit]

- Possible transitions: self (1 unchanged bit) or close neighbors
- Time rhythm between state changes: T_b (changes occur at the speed of the bit rate, which is twice that of the symbol rate!).

From OQPSK towards MSK: rectangular shapes are replaced by half sin shapes [in this drawing, we do not ignore the $(-1)^k$ term, that is we have a shift of π between consecutive even (odd) bits]



Minimum Shift Keying modulation (MSK) II

Definition (MSK)

For defining the Minimum Shift Keying modulation (MSK) modulation, we use the principle of the OQPSK and a **non-rectangular pulse shape**. More precisely, we take

$$p(t) = \mathrm{rect}_{[0,\,2T_b]}(t)\,\sin\left(\frac{\pi t}{2T_b}\right) \tag{264}$$

$s_I(t)$ and $s_Q(t)$ are built exactly as for the OQPSK.
$s_I(t)$ is calculated as

$$\begin{aligned}
s_I(t) &= \frac{A}{\sqrt{2}} \sum_{k=-\infty}^{+\infty} I_{2k}\,\mathrm{rect}_{[0,\,2T_b]}(t-2kT_b)\,\sin\left[\frac{\pi\,(t-2kT_b)}{2T_b}\right] \\
&= \frac{A}{\sqrt{2}} \sum_{k=-\infty}^{+\infty} I_{2k}\,\mathrm{rect}_{[0,\,2T_b]}(t-2kT_b)\,\sin\left[\frac{\pi t}{2T_b} - k\pi\right]
\end{aligned}$$

Since $\sin(a-b) = \sin a \cos b - \cos a \sin b$, we have

$$s_I(t) = \frac{A}{\sqrt{2}} \sum_{k=-\infty}^{+\infty} I_{2k}\, \mathrm{rect}_{[0,\,2T_b]}(t-2kT_b) \sin\left[\frac{\pi t}{2T_b} - k\pi\right] \tag{265}$$

$$= \frac{A}{\sqrt{2}} \sum_{k=-\infty}^{+\infty} I_{2k}\, \mathrm{rect}_{[0,\,2T_b]}(t-2kT_b) \left[\sin\left(\frac{\pi t}{2T_b}\right)\cos(k\pi) - \cos\left(\tfrac{\pi t}{2T_b}\right)\sin(k\pi)\right]$$

$$= \frac{A}{\sqrt{2}} \sum_{k=-\infty}^{+\infty} I_{2k}\, \mathrm{rect}_{[0,\,2T_b]}(t-2kT_b) \sin\left(\frac{\pi t}{2T_b}\right)\cos(k\pi) \tag{266}$$

$$= \sum_{k=-\infty}^{+\infty} \frac{A}{\sqrt{2}} I_{2k} (-1)^k\, \mathrm{rect}_{[0,\,2T_b]}(t-2kT_b) \sin\left(\frac{\pi t}{2T_b}\right) \tag{267}$$

$$= \cos\left(\frac{\pi}{2} - \frac{\pi t}{2T_b}\right) \sum_{k=-\infty}^{+\infty} \frac{A}{\sqrt{2}} I_{2k}\, (-1)^k\, \mathrm{rect}_{[0,\,2T_b]}(t-2kT_b) \tag{268}$$

$$= \cos\left(\frac{\pi t}{2T_b} - \frac{\pi}{2}\right) \sum_{k=-\infty}^{+\infty} \frac{A}{\sqrt{2}} I_{2k}\, (-1)^k\, \mathrm{rect}_{[0,\,2T_b]}(t-2kT_b) \tag{269}$$

Likewise,

$$s_Q(t) = \sin\left(\frac{\pi t}{2T_b} - \frac{\pi}{2}\right) \sum_{k=-\infty}^{+\infty} \frac{A}{\sqrt{2}}\, I_{2k+1}\, (-1)^k\, \mathrm{rect}_{[0,\,2T_b]}\left(t-(2k+1)\,T_b\right)$$

The instantaneous envelope of the modulated signal is then (we ignore the $\mathrm{rect}_{[0,\,2T_b]}(.)$ in the following expressions)

$$\begin{aligned} a(t) &= \sqrt{s_I^2(t) + s_Q^2(t)} && (270)\\ &= \sqrt{\left(\frac{A}{\sqrt{2}}\right)^2 \cos^2\left(\frac{\pi t}{2T_b} - \frac{\pi}{2}\right) + \left(\frac{A}{\sqrt{2}}\right)^2 \sin^2\left(\frac{\pi t}{2T_b} - \frac{\pi}{2}\right)} && (271)\\ &= \sqrt{\left(\frac{A}{\sqrt{2}}\right)^2} = \frac{A}{\sqrt{2}} && (272) \end{aligned}$$

The instantaneous phase is

$$\begin{aligned}\varphi(t) &= \tan^{-1}\left[\frac{s_Q(t)}{s_I(t)}\right] \qquad (273)\\ &= \tan^{-1}\left[\tan\left(\frac{\pi t}{2T_b}-\frac{\pi}{2}\right)\frac{\sum_{k=-\infty}^{+\infty}\frac{A}{\sqrt{2}}\, I_{2k+1}\,(-1)^k\,\mathrm{rect}_{[0,\,2T_b]}\left(t-(2k+1)T_b\right)}{\sum_{k=-\infty}^{+\infty}\frac{A}{\sqrt{2}}\, I_{2k}\,(-1)^k\,\mathrm{rect}_{[0,\,2T_b]}\left(t-2kT_b\right)}\right]\end{aligned}$$

$$\varphi(t) = \tan^{-1}\left[\tan\left(\frac{\pi t}{2T_b}-\frac{\pi}{2}\right)\frac{\sum_{k=-\infty}^{+\infty}\frac{A}{\sqrt{2}}\, I_{2k+1}\,(-1)^k\,\mathrm{rect}_{[0,\,2T_b]}\left(t-(2k+1)T_b\right)}{\sum_{k=-\infty}^{+\infty}\frac{A}{\sqrt{2}}\, I_{2k}\,(-1)^k\,\mathrm{rect}_{[0,\,2T_b]}\left(t-2kT_b\right)}\right]$$

Understanding the phase of an MSK

- there is an ambiguity between $\frac{1}{1}$ or $\frac{-1}{-1}$, and $\frac{-1}{1}$ or $\frac{1}{-1}$.
- the phase evolves *linearly with time*:

$$\varphi(t) = \tan^{-1}\left[\tan\left(\frac{\pi t}{2T_b} - \frac{\pi}{2}\right) \times (\pm 1)\right] \tag{274}$$

$$= \pm\left(\frac{\pi t}{2T_b} - \frac{\pi}{2}\right) \tag{275}$$

  It is continuous with time t!
- the transition rate is given by the duration T_b.
  - the time slope is $\frac{\pi}{2T_b}$ or $-\frac{\pi}{2T_b}$.
  - the phase increment after a T_b period is $\frac{\pi}{2}$ or $-\frac{\pi}{2}$

Minimum Shift Keying modulation (MSK) VII

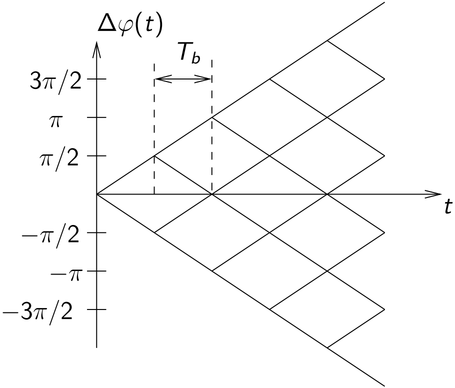


Figure: Phase trellis diagram of the MSK modulation.

Minimum Shift Keying modulation (MSK) VIII

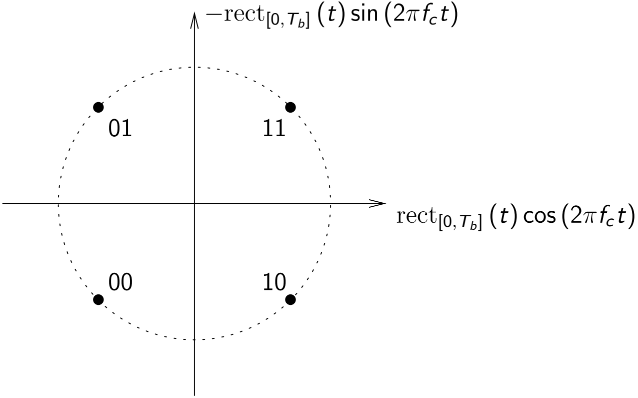


Figure: Constellation diagram of the MSK modulation.

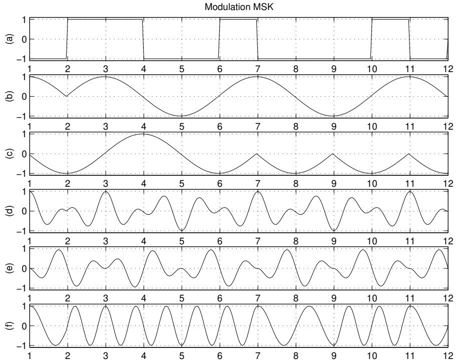


Figure: Illustration for the MSK modulation: (a) binary input sequence $I(t)$, (b) in-phase signal $s_I(t)$, (c) quadrature signal $s_Q(t)$, (d) $s_I(t)\cos(2\pi f_c t)$, (e) $s_Q(t)\sin(2\pi f_c t)$, and (f) modulated signal $s(t)$.

Which family of modulation techniques does the MSK belong to?

Because the envelope is constant and the phase evolves as (over a period of T_b)

$$\Delta\varphi(t) = \pm\frac{\pi t}{2T_b} \tag{276}$$

we can write that

$$\begin{aligned} s(t) &= A\cos\left(2\pi f_c t \pm \frac{\pi t}{2T_b}\right) && (277)\\ &= A\cos\left[2\pi\left(f_c \pm \frac{1}{4T_b}\right)t\right] && (278) \end{aligned}$$

This illustrates that, in fact, the MSK modulation is a "pure" frequency modulation (whose frequency excursion is limited to $\Delta f = \frac{1}{4T_b}$).

Power spectral density of the MSK I

The complex envelope of the modulated signal is again given by

$$e_s(t) = \sum_{k=-\infty}^{+\infty} A_k \, p(t - kT_b) \, e^{jk\frac{\pi}{2}} \tag{279}$$

where $p(t)$ is the shaping function and A_k is the random variable containing the information, respectively given by

$$\begin{aligned} p(t) &= \mathrm{rect}_{[0,2T_b]}(t) \sin\left(\frac{\pi t}{2T_b}\right) && (280) \\ A_k &\in \{+A, -A\} && (281) \end{aligned}$$

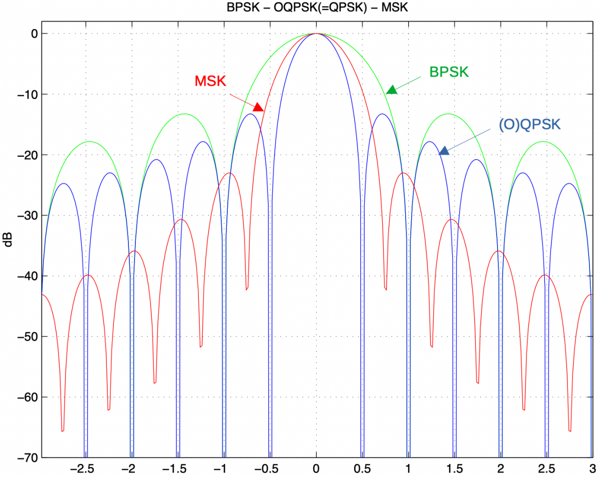
After considering the Fourier transform of $\mathcal{H}(f) = \mathcal{P}\left(f + \frac{1}{4T_b}\right)$,

$$\gamma_z(f) = \frac{16\,A^2\,T_b}{\pi^2} \left\{ \frac{\cos\left[2\pi\left(f + \frac{1}{4T_b}\right) T_b\right]}{1 - 16\left(f + \frac{1}{4T_b}\right)^2 T_b^2} \right\}^2 \tag{282}$$

This yields the power spectral density of the MSK modulation:

$$\gamma_s(f) = \frac{4\,A^2\,T_b}{\pi}\left\{\left(\frac{\cos\left[2\pi\left(f-f_c\right)\,T_b\right]}{1-16\left(f-f_c\right)^{2}\,T_b^2}\right)^2+\left(\frac{\cos\left[2\pi\left(f+f_c\right)\,T_b\right]}{1-16\left(f+f_c\right)^{2}\,T_b^2}\right)^2\right\} \tag{283}$$

Power spectral density of the MSK III



The modulation technique used in the GSM standard is an MSK variant; it is the **Gaussian Minimum Shift Keying** (GMSK) for which the pulse shaping function is a Gaussian.

Outline

1. Motivation
   - Utilization
   - Techniques for spread spectrum
2. Direct sequence spreading
   - Principles
   - Generation of pseudo-random sequences
   - Principles of baseband transmission
   - BPSK modulated spreading
3. Performance study
   - Error probability
   - Interference margin
   - Capacity

Application of spread spectrum systems

Spread-spectrum involves the transmission of a signal in a frequency bandwidth substantially greater than the information bandwidth to achieve a particular operational advantage.

Purposes	Military use	Commercial use
Anti-jamming	$\surd$	$\surd$
Multiple access	$\surd$	$\surd$
Detection harvesting	$\surd$	
Message privacy	$\surd$	$\surd$
Selective calling	$\surd$	$\surd$
Identification	$\surd$	$\surd$
Navigation	$\surd$	$\surd$
Multipath mitigation	$\surd$	$\surd$
Low radiated flux density	$\surd$	$\surd$

There exist two main spread spectrum families:

1. *Frequency Hopping*.
2. *Direct Sequence Spread Spectrum* (DSSS or DS). This technique leads to *Code-Division Multiplexing* (CDM).

Frequency Hopping Spread Spectrum I

Definition

Frequency Hopping Spread Spectrum (FHSS) is a method of transmitting radio signals by rapidly switching a carrier among many frequency channels, using a pseudo-random sequence *known* to both the transmitter and receiver.

Bandwidth. (−) The overall bandwidth required for frequency hopping is much wider than that required to transmit the same information using only one carrier frequency.

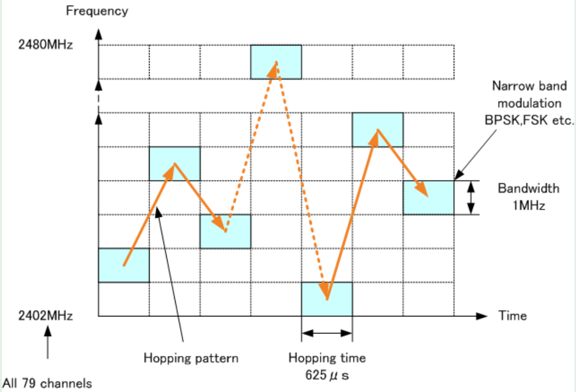
Challenge. (−) Need to synchronize the hopping sequence between the transmitter and the receiver.

Advantage. (+) Possibility to avoid being permanently locked in a “bad” frequency channel.

Frequency Hopping Spread Spectrum II

Bluetooth

- total bandwidth of 79 [MHz]
- width of individual signals: 1 [MHz]
- 1600 changes per second (hopping time of 625 [μs])



Definition

Direct Sequence *Spread Spectrum* (*DSSS* or *DS*) transmissions multiply the data being transmitted by a "noise/carrier" signal. This noise/carrier signal is a pseudo-random NRZ-like sequence of $+1$ and -1 values, also known as *spreading sequence*, at a bitrate much higher than that of the original signal.

For a *Direct Sequence spreading (DS) system*, there are two signals:

1. a binary baseband waveform $b(t)$, whose symbol rate is $R_b = 1/T_b$. This is the original sequence.
2. a pseudo-random binary spreading waveform $c(t)$, whose "chip" rate $R_c = 1/T_c$ is much faster than the symbol rate R_b ($T_c \ll T_b$). This waveform is the *spreading sequence*.

Direct Sequence spreading: principles II

Note that the two signals are aligned (synchronized).
In the following, we will consider:

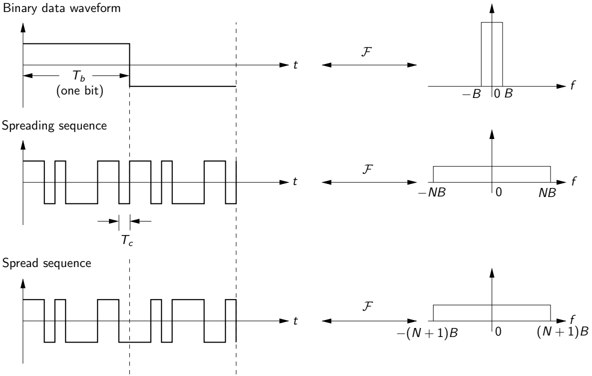
$$R_c = N R_b \iff T_b = N T_c \tag{284}$$

N is the spreading factor.

Questions:

- effects of spreading (bandwidth, Bit Error Rate P_e, etc.)?
- how do we build an appropriate spreading waveform?
  How should a spreading sequence look like?

Effects of direct sequence spreading



1. Compare the *power of the original signal* and the *power of the spread signal.*
2. What is the *bandwidth of a spread waveform*? (in *baseband* or *modulated* form)

Bandwidth? Summary

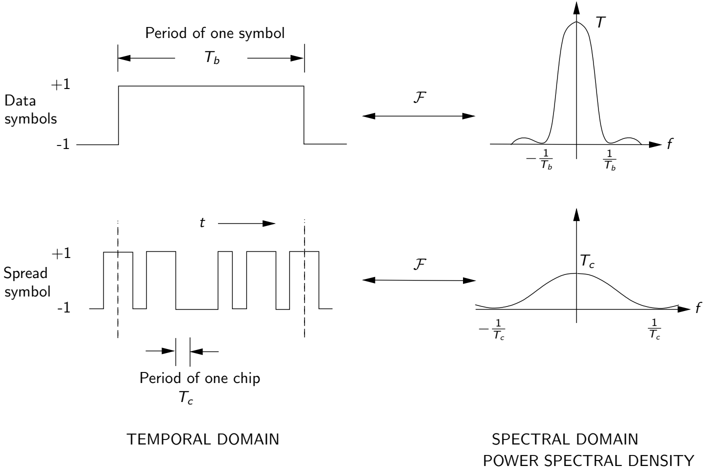
- The bandwidth of $b(t)$ is related to $1/T_b$ [Hz]
- The bandwidth of $c(t)$ is related to $1/T_c$ [Hz]

Because $T_c \ll T_b$, $c(t)$ is a wideband signal.

The resultant spectrum of the product $c(t)b(t)$ is the a convolution of two spectra.

Essentially, it will occupy a bandwidth that is practically the same as that of $c(t)$.

Power Spectral Density II



The bandwidth of a spread waveform is about N times larger.

How can we construct a pseudo-random spreading sequence?

There are digital circuits to generate sequences that "look like" random sequences.

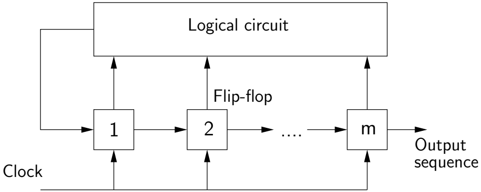


Figure: Feedback shift register.

Why should the spreading sequences be "almost" random?

- random sequences have a flat power spectral density.

Example

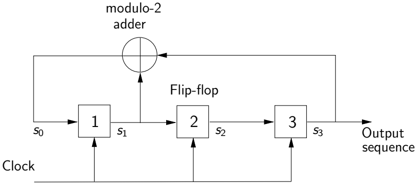


Figure: Example of a "linear" feedback shift register.

Questions:

- how does it work?
- what are the properties of the generated sequence? For example, what is the spreading factor that can be achieved with the circuits?

Consider an example I

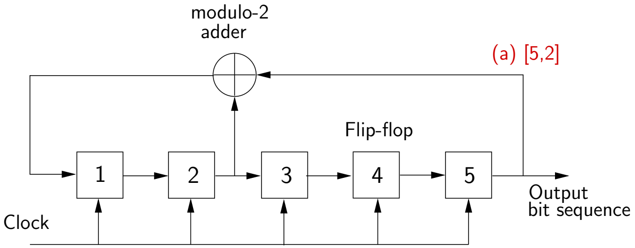


Figure: A valid configuration of feedback shift register of length $R = 5$.

- Generation of a sequence [5,2]
  - 5 registers.
  - the contents of registers 2 and 5 are XORed (eXclusive OR) to feed register 1.
- We are interested in understanding what is outputted by the circuit.

Consider an example II

2 XOR 5	Register					
Feedback bit	1	2	3	4	5	Output bit
	0	1	0	0	0	
1 $\longrightarrow$	1	0	1	0	0	0
0 $\longrightarrow$	0	**1**	0	1	**0**	0
1 $\longrightarrow$	1	0	1	0	1	**0**
1 $\longrightarrow$	1	1	0	1	0	1
1 $\longrightarrow$	1	**1**	1	0	**1**	0
0 $\longrightarrow$	0	1	1	1	0	**1**
1 $\longrightarrow$	1	0	1	1	1	0
1 $\longrightarrow$	1	**1**	0	1	**1**	1
0 $\longrightarrow$	0	1	1	0	1	**1**
0 $\longrightarrow$	0	0	1	1	0	1
...	...	...	...	...	...	...

Table: State of the registers over time (one row per clock pulse).

Consider an example III

- The length of the outputted sequence is related to the number of possible states stored in the registers.
  - Maximum of $2^5 = 32$ possible states.
- One state is forbidden.
- Finally, we have $2^5 - 1 = 31$ possible states.

Conclusions

- The generated (pseudo-random) sequence is periodic with a period of $2^R - 1$. Such sequences are named of "*maximum-length*".
- The length of the period is odd.

Choosing a maximum-length sequence I

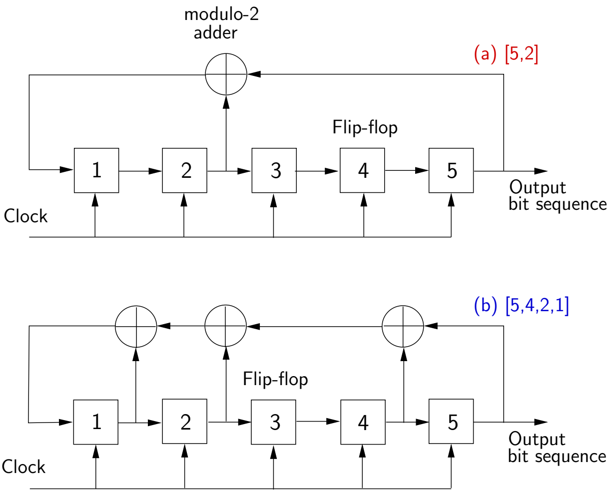


Figure: Two different configurations of feedback shift register of length $R = 5$.

Choosing a maximum-length sequence II

Shift register length R	Possible configurations for the feedback taps
2	[2,1]
3	[3,1]
4	[4,1]
5	[5,2], [5,4,3,2], [5,4,2,1]
6	[6,1], [6,5,2,1], [6,5,3,2]
7	[7,1], [7,3], [7,3,2,1], [7,4,3,2], [7,6,4,2], [7,6,3,1], [7,6,5,2], [7,6,5,4,2,1], [7,5,4,3,2,1]
8	[8,4,3,2], [8,6,5,3], [8,6,5,2], [8,5,3,1], [8,6,5,1], [8,7,6,1], [8,7,6,5,2,1], [8,6,4,3,2,1]

Figure: Maximum-length sequences of shift register lengths 2 through 8.

Properties of maximum-length sequences I

Maximal length

The all-zeros 0000... state is forbidden. So, for R flip-flops, the maximum length is exactly $N = 2^R - 1$.

Balance property

The number of 1s is always one more than the number of 0s.

Therefore, the mean of a pseudo-random sequence $c(t)$ is

$$E\{c(t)\} = \frac{1}{N} \tag{285}$$

It is not equal to 0 as for a real random sequence.

Properties of maximum-length sequences II

Correlation property

The autocorrelation function of a maximum-length sequence is periodic.

1. **[Definition]** The autocorrelation of a periodic sequence (new definition because otherwise the autocorrelation function would be infinite) is defined as

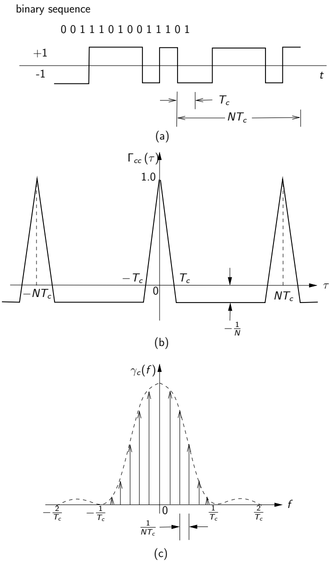
$$\Gamma_{cc}(\tau) = \frac{1}{T_b} \int_0^{T_b} E\{c(t)c(t-\tau)\}\, dt \tag{286}$$

2. **[Auto-correlation function]** The autocorrelation function of a maximum-length sequence:

$$\Gamma_{cc}(\tau) = \begin{cases} 1 - \left(1 + \frac{1}{N}\right)\frac{|\tau|}{T_c}, & |\tau| \leq T_c \\ -\frac{1}{N} & \text{elsewhere} \end{cases} \tag{287}$$

Properties of maximum-length sequences III

3. **[Periodicity of the autocorrelation function]**
   The auto-correlation function of a maximum-length sequence is periodic (with period $T_b = NT_c$).


binary sequence
0 0 1 1 1 0 1 0 0 1 1 1 0 1
+1
-1
t
T_c
NT_c
(a)
$\Gamma_{cc}(\tau)$
1.0
$-T_c$
T_c
$-NT_c$
0
NT_c
τ
$-\frac{1}{N}$
(b)
$\gamma_c(f)$
f
$-\frac{2}{T_c}$
$-\frac{1}{T_c}$
0
$\frac{1}{T_c}$
$\frac{2}{T_c}$
$\frac{1}{NT_c}$
(c)

Power spectral density of a spread signal II

$$\gamma_c(f) = \frac{1}{N^2}\delta(f) + \frac{1+N}{N^2} \sum_{n=-\infty,\, n\neq 0}^{+\infty} \left(\frac{\sin(\pi\frac{n}{N})}{\pi\frac{n}{N}}\right)^2 \delta\left(f - \frac{n}{NT_c}\right)$$

Let us compare this expression (for a maximum-length sequence) with that of a random sequence:

- for a purely random sequence (which is not periodic), the autocorrelation function would be

$$\begin{cases} 1 - \frac{|\tau|}{T_c} & |\tau| \leq T_c \\ 0 & |\tau| > T_c \end{cases} \tag{288}$$

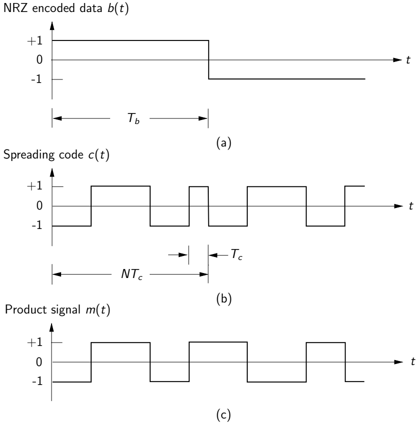
  The waveforms have the same envelope $\text{sinc}^2()$, for their power spectral densities.

- the main difference is that, for a maximum-length sequence, it consists of delta functions spaced $1/NT_c$ $[Hz]$ apart.

Spread spectrum in the presence of interference noise

1. Transmitter: the signal is spread (we know how to spread a signal)
2. In the channel, interferers add a noise on the signal.
3. Receiver: what happens to the signal and to the noise after despreading?

Let $\{b_k\}$ denote a binary data sequence, and $b(t)$ be its polar ± 1 NRZ representation.



For a baseband transmission, the transmitted signal is the product $y(t)$ obtained by

$$y(t) = c(t)b(t) \tag{289}$$

where $c(t)$ is the spreading sequence.

In the Fourier domain,

$$\mathcal{Y}(f) = \mathcal{B}(f) \otimes \mathcal{C}(f) = \int_{-\infty}^{+\infty} \mathcal{C}(\tau)\mathcal{B}(f-\tau)\, d\tau \tag{290}$$

If all signals were deterministic, the bandwidth of $\mathcal{Y}(f)$ would be almost equal to that of $\mathcal{C}(f)$.

Input of the receiver before despreading

The received signal $r(t)$ consists of the transmitted signal $y(t)$ plus an additive interference, denoted by $i(t)$,

$$\begin{aligned} r(t) &= y(t) + i(t) && (291) \\ &= c(t)b(t) + i(t) && (292) \end{aligned}$$

$i(t)$ is the signal due to other users (plus white noise). It has the form of

$$i(t) = \sum_k c_k(t) b_k(t)$$

At the receiver after despreading

Assuming a perfect synchronization between the transmitter and the receiver, the input of the receiver is built as (the synchronized sequence $c(t)$ also serves to despread the signal):

$$\begin{aligned} z(t) &= c(t)r(t) \qquad (293) \\ &= c^2(t)b(t) + c(t)i(t) \qquad (294) \end{aligned}$$

The $c(t)$ sequence alternates between -1 and $+1$. Therefore, $c^2(t)$ is equal to $+1$, for all t. This leads to

$$z(t) = b(t) + c(t)i(t) \qquad (295)$$

At the multiplier output of the receiver, we have

- $b(t)$, the original decoded signal.
- an interference signal spread by $c(t)$.

The receiver gets

$$z(t) = c^2(t)b(t) + c(t)i(t) = b(t) + c(t)i(t) \tag{296}$$

If $i(t)$ originates from another user, it is of the same type as of $b(t)$. Let assume that:

$$i(t) = c'(t)b'(t) \tag{297}$$

Then, it is important to compare:

- $c^2(t)$, whose average value is the value of the autocorrelation function for $\tau = 0$, and
- $c(t)c'(t)$, whose average is given by the cross-correlation function $\Gamma_{cc'}(\tau)$.

More about $c(t)c'(t)$

In fact, there are two possibilities for building the signal $i(t)$ of another user:

1. use a different spreading sequence $c'(t)$. Then, because we don't know the synchronization time of $i(t)$, we need to upper bound $\Gamma_{cc'}(\tau)$ to a much lower value than $\Gamma_{cc}(0)$:

$$\Gamma_{cc'}(\tau) \ll \Gamma_{cc}(0) \tag{298}$$

2. use the same spreading sequence, but delayed by some known time shift value $\triangle T$. Then, the condition becomes

$$|\Gamma_{cc}(\triangle T)| \ll \Gamma_{cc}(0) \tag{299}$$

Many systems, such as the GPS system, are designed to work with a unique sequence $\Rightarrow$ hardware simplification.

Delay	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
R_{11}	31	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
R_{12}	7	-1	-1	7	7	-1	7	-1	-1	-9	-1	-1	-1	-1	-9	-1	7	-1	-9	-9	7	-1	7	7	-1	-1	-1	-9	7	-9	7
R_{13}	7	7	7	-1	-1	-9	7	-1	-9	-1	-1	-9	-1	-1	-1	-1	7	-1	-9	-1	7	7	7	-1	7	-9	-1	-1	-1	7	-9
R_{14}	3	3	-5	-9	-5	3	-1	7	-9	3	3	3	-5	-5	-1	-9	3	-5	-1	3	-1	-1	11	7	7	-9	7	3	-9	3	7
R_{15}	-1	7	-1	-1	-1	-9	7	7	-1	-9	7	-9	-9	-9	-1	-1	7	-1	-1	7	7	-1	-1	-1	-1	7	-1	7	-9	-1	7
R_{16}	7	-1	-1	7	-1	-1	-1	7	-1	7	7	-9	-9	-9	-1	-9	7	-1	-1	-9	7	7	-1	-1	-1	-1	7	-9	-1	7	-1
R_{22}	31	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
R_{23}	7	7	-9	-1	-1	-9	-1	-1	-9	7	-1	-1	-1	-1	-9	-1	7	7	-1	-9	-1	7	-1	-1	-1	7	-9	7	-1	7	7
R_{24}	-1	7	-1	-1	-1	-9	7	7	-1	-9	7	-9	-9	-9	-1	-1	7	-1	-1	7	7	-1	-1	-1	-1	7	-1	7	-9	-1	7
R_{25}	3	-5	3	3	11	-9	-9	3	-9	-1	3	-5	-9	-1	-1	7	3	7	-5	3	-9	3	-1	-5	-1	7	7	3	3	-5	7
R_{26}	7	7	-9	7	7	-1	7	-9	-1	-1	7	-9	-9	-1	-1	-1	-1	7	7	-1	-9	-1	-9	-1	-1	7	-1	-1	-1	7	-1
R_{33}	31	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
R_{34}	7	-1	-1	7	-1	-1	-1	7	-1	7	7	-9	-9	-9	-1	-9	7	-1	-1	-9	7	7	-1	-1	-1	-1	7	-9	-1	7	-1
R_{35}	7	7	-9	7	7	-1	7	-9	-1	-1	7	-9	-9	-1	-1	-1	-1	7	7	-1	-9	-1	-9	-1	-1	7	-1	-1	-1	7	-1
R_{36}	11	3	3	-1	3	3	-1	7	3	3	3	-9	-1	-9	7	-5	3	-1	3	7	3	-9	-9	-5	-1	7	-9	-5	7	-5	-5
R_{44}	31	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
R_{45}	7	7	-9	7	-9	-1	-1	-1	7	7	-1	7	-9	-9	-1	7	-1	-9	-1	-1	-1	-1	-9	-1	-1	7	-1	7	7	-1	-1
R_{46}	7	-9	7	-1	-1	-1	-9	7	-1	7	7	7	-1	-9	-1	7	-1	-1	-1	-1	-9	-1	-1	-9	-1	7	-9	-1	-1	7	7
R_{55}	31	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
R_{56}	7	7	7	-1	7	-9	7	-1	-1	-1	7	-1	-9	-1	7	7	-1	-9	-1	-1	-1	-1	7	-9	-1	-1	-9	-1	-1	-9	7
R_{66}	31	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1

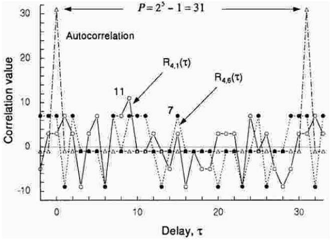


Figure: Cross-correlation functions.

Direct sequence spread BPSK modulation I

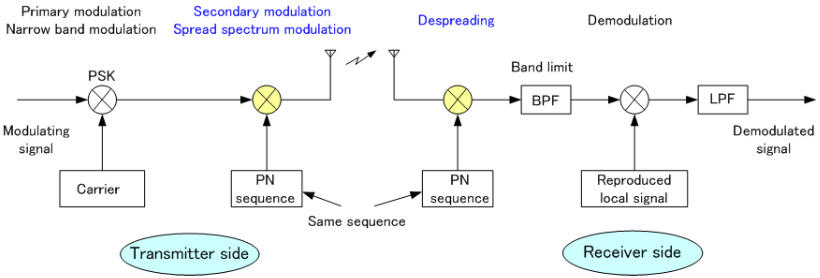
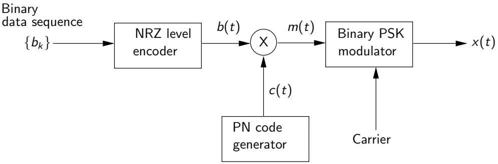
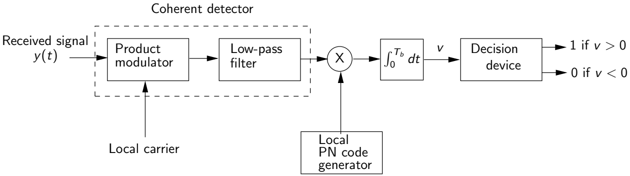


Figure: Direct sequence spread modulation: general overview.

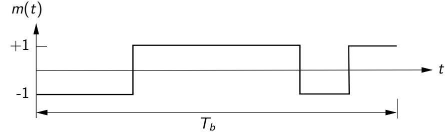


(a) transmitter

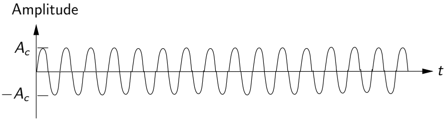


(b) receiver

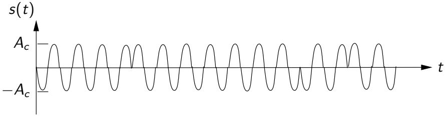
Figure: Direct sequence spread coherent phase-shift keying.



(a) ±1 NRZ encoded sequence



(b) carrier



(c) after multiplication

Figure: Spreading + BPSK modulation.

Model for analysis

From a conceptual point of view, it is possible to *interchange the order of spectrum spreading and phase modulation*. The interference, modeled by $j(t)$, limits the performance.

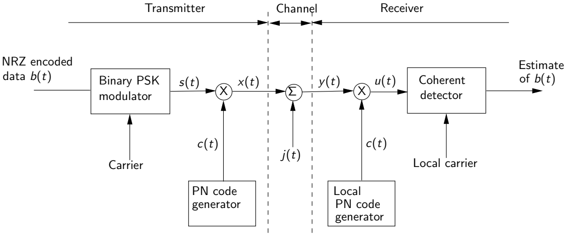


Figure: Model of direct sequence spread BPSK system.

Sources of noise in the channel of a spread spectrum system I

In a channel, we have:

1. *additive white Gaussian noise*, that models a sum of independent noise signals.
2. *small bandwidth noises* (*"colored" noises*), that are very localized in the spectrum. Note that despreading a useful signal will spread these noises.
3. *BPSK/CDMA signals of other users*.
   If M represents the number of users that share the same bandwidth, the level of noise power is estimated as $M-1$ times the nominal power of one user.

It is important to understand what happens after despreading.

Sources of noise in the channel of a spread spectrum system II

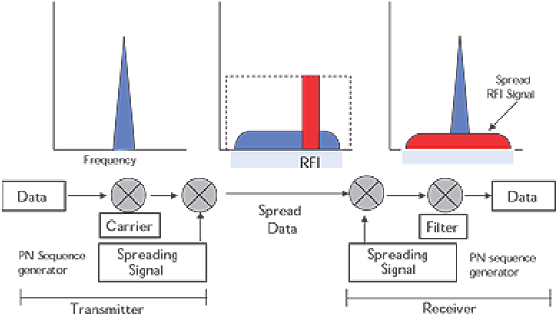


Figure: A small bandwidth noise (drawn in red) in the channel is spread by the despreading sequence of the user (drawn in blue).

Sources of noise in the channel of a spread spectrum system III

in the channel	**after despreading at the transmitter**
additive white noise	remains a white noise at the same level
small bandwidth noises ("colored" noise)	is spread by the specific spreading sequence of the user. The total amount of power is spread but remains unchanged.
signals of other users	are spread by the specific spreading sequences of the users. The total amount of power is spread but remains unchanged.

A thorough analysis of the performance leads to (*no proof provided*)

$$10\log\left(\frac{S}{N}\right)_{OUT} = 10\log\left(\frac{S}{N}\right)_{IN} + 3 + 10\log\left(\frac{T_b}{T_c}\right) \quad (300)$$

- The 3 [dB] term originates from the coherent demodulation mechanism.
- There is a gain that is proportional to

$$\frac{T_b}{T_c} = N \quad (301)$$

N is named the *processing gain*.

Performances of a coherent BPSK spread spectrum system II

Theorem (Bit Error Rate for a DS/BPSK modulation system)

$$P_e \simeq \frac{1}{2}\, erfc\left(\sqrt{\frac{E_b}{J T_c}}\right) \tag{302}$$

where J is the average interference power: $J = \frac{1}{T_b}\int_0^{T_b} j^2(t)\, dt$.

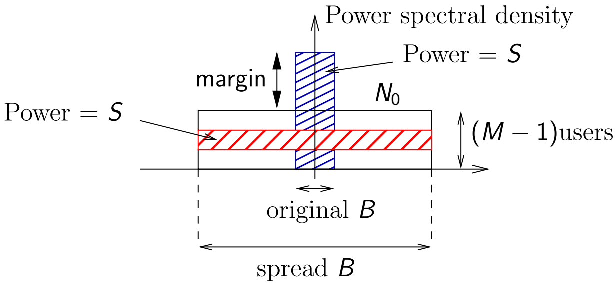
In other terms, if $N_J = \int_0^{T_b} j^2(t)\, dt$, then

$$P_e \simeq \frac{1}{2}\, erfc\left(\sqrt{\frac{T_b}{T_C}\frac{E_b}{N_J}}\right) = \frac{1}{2}\, erfc\left(\sqrt{N\,\frac{E_b}{N_J}}\right) \tag{303}$$

The N factor is obtained because:

- noise is unaffected by the despreading operation
- the useful signal is concentrated in a N times smaller bandwidth after despreading.

Performance: interference margin I



Definitions and notations:

- S [W] received power for the desired signal. It is equal to the energy per bit $\times$ the bitrate ($S = E_b \times R_b$),
- E_b [J = W $\times$ s] received energy per bit for the desired signal,
- $R_b = \frac{1}{T_b}$ [Hz] data bitrate,
- B [Hz] spread bandwidth in Hz,
- J [W] received power for the jamming signals,
- N_0 [W/Hz] equivalent noise power spectral density.

Case 1: **absence** of spreading

The signal to noise (jamming) ratio is given by

$$\frac{S}{J} = \frac{E_b R_b}{N_0 B} \tag{304}$$

For a system that is not spread in bandwidth: $B \approx R_b$.
$\left(\frac{E_b}{N_0}\right)$ is then equal to the signal to noise ratio:

$$\frac{S}{J} = \frac{E_b}{N_0} \tag{305}$$

Equivalently,

$$\frac{J}{S} = \frac{1}{E_b/N_0} \tag{306}$$

and the minimal $\frac{S}{J}$ is set by the required $\frac{E_b}{N_0}$:

$$\frac{J}{S}\,[\mathrm{dB}] = -\left(\frac{E_b}{N_0}\right)_{req}[\mathrm{dB}] \tag{307}$$

Performance: interference margin III

Case 2: **with** spreading

The ratio of the equivalent "noise" power J to S is given by

$$\frac{J}{S} = \frac{N_0 B}{E_b R_b} = \frac{B/R_b}{E_b/N_0} \tag{308}$$

When the value of $\frac{E_b}{N_0}$ is set to that required for acceptable performance of the communication system $\left(\frac{E_b}{N_0}\right)_{req}$, then the ratio $\frac{J}{S}$ bears the interpretation of a jamming margin:

$$\frac{J}{S}\,[\mathrm{dB}] = \mathrm{margin}\,[\mathrm{dB}] = \frac{B}{R_b}\,[\mathrm{dB}] - \left(\frac{E_b}{N_0}\right)_{req}\,[\mathrm{dB}] \tag{309}$$

The quantity $\frac{B}{R_b}$ is called the spread-spectrum processing gain. It is equal to ($B \approx R_c$)

$$\frac{B}{R_b} \approx \frac{R_c}{R_b} = \frac{T_b}{T_c} = N \tag{310}$$

Capacity of a spread spectrum system I

Theorem

The *capacity* *(=number of users) of a multi-user spread spectrum system is the ratio of the spread bandwidth to the data rate:*

$$M = \alpha \frac{B}{R_b} = \alpha N \tag{311}$$

To calculate its value, we consider a unique cell.

- The carrier power is

$$C = S = R_b E_b \tag{312}$$

- Likewise, the interference power I, at the base station receiver, is

$$I = BN_0 \tag{313}$$

 where B is the transmission bandwidth.

Thus, the *carrier to interference power ratio* for a particular mobile user at the base station is given by

$$\frac{C}{I} = \frac{R_b E_b}{B N_0} = \frac{E_b/N_0}{B/R_b} \tag{314}$$

Let M denote the number of users. Assuming that all users have the same power level at the base station, then the total interference power for one user is caused by $M-1$ interferers

$$I = C(M-1) \tag{315}$$

which leads to

$$\frac{C}{I} = \frac{1}{M-1} \tag{316}$$

By combining the expressions (314) and (316) for $\frac{C}{I}$, we derive that

$$M = \frac{B}{R_b}\frac{1}{E_b/N_0} + 1 \simeq \frac{B}{R_b}\frac{1}{E_b/N_0} = N\frac{1}{E_b/N_0} \tag{317}$$

1. *Duty-cycle* α of a full-duplex voice conversations is 35%. In practice, we take 50%.
   Consequence:
   - *the power sent by each user drops to* $\alpha R_b E_b$

$$M \simeq \frac{B}{\alpha R_b} \frac{1}{E_b/N_0} = \frac{B}{R} \frac{1}{E_b/N_0} \frac{1}{\alpha} \tag{318}$$

2. *Antenna directivity*. Cells are divided into 3 sectors (with 15% overlap) and they provide a "gain".
   Consequence:
   - The number of possible users is multiplied by the number of available frequencies (3).
   - Because a user in between two cells must chose one single frequency, we have 15% loss. Therefore, we have a "gain" of $G = 3 \times 0.85 = 2.55$:

$$M \simeq \frac{B}{R_b} \frac{1}{E_b/N_0} \frac{1}{\alpha} G \tag{319}$$

Additional factors II

3. *The re-usability factor of frequencies* F_e is less than 1 ($F_e \simeq 0.65$ for the IS-95 mobile system):

$$F_e = \frac{\text{Effective surface covered by a frequency}}{\text{Total geographic surface}} \tag{320}$$

Final estimation of users per cell

$$M \simeq \frac{B}{R_b}\,\frac{1}{E_b/N_0}\,\frac{1}{\alpha}\,G\,F_e \tag{321}$$

Additional factors III

Example

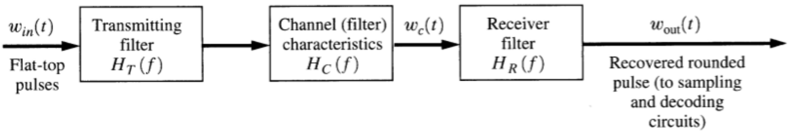
For the IS-95 radio mobile system: $\frac{B}{R_b} = 128$, $\frac{E_b}{N_0} = 7\,[\mathrm{dB}] = 5$, $\alpha = 0.5$, $G = 2.55$, and $F_e = 0.65$.
These parameters yield a capacity of

$$M = 128 \times \frac{1}{5} \times \frac{1}{0.5} \times 2.55 \times 0.65 = 85 \tag{322}$$

Outline

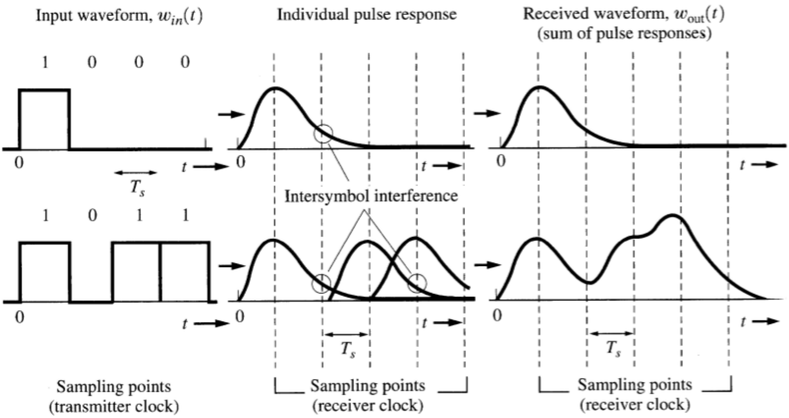
Channels for digital communications and intersymbol interference (ISI)

- Context
- Nyquist's filtering
  1. Ideal channel
  2. Nyquist's criterion
  3. Raised cosine roll-off filtering



- In a communication chain, we have:
  - a transmitting filter (why?)
  - a channel that, if linear, is described by a transfer function
  - a receiver filter (why?)

One should "optimize" the shape of the transmitting and the receiver filters with respect to the channel characteristics.



The pulse of each may be smeared into adjacent time slots: this causes InterSymbol Interference (ISI).

Theorem (Interpolation formula of Whittaker)

Let $y(t)$ be a signal band limited to $]-W, W[$. Take the $\{y[nT_b]\}$ set of samples regularly spaced by $T_b = 1/f_b$. Then the $y(t)$ function can be obtained by (with $f_b = 2W$)

$$y(t) = \sum_{n=-\infty}^{+\infty} y[nT_b] \, sinc\left(\frac{t - nT_b}{T_b}\right) \tag{323}$$

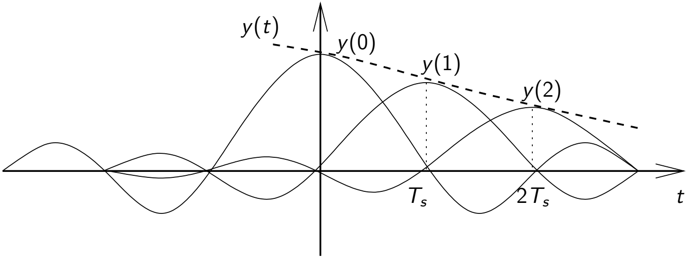


Figure: Whittaker's reconstruction scheme (interpolation).

Interpretation of Whittaker's formula

- $y(t)$: is an analog signal (theoretically, perfectly reconstructed)
- $y[nT_b]$: are samples taken every T_b
- $\text{sinc}\left(\frac{t-nT_b}{T_b}\right) = p(t)$ is a real shape (*pulse*) whose spectrum is strictly limited to $[-W, W]$, which is unfeasible in practice.

What about imperfections in the channel?

- Noise? $\Rightarrow$ matched "filter"
- Pulse shape? Some pulse shapes are impossible to build.
- Multi-path, obstacles, etc.
  This is a problem from an operational point of view.
  - $\Rightarrow$ *multiple time-shifted versions* of the signal.
  - $\Rightarrow$ need to "*compensate*" the effects of the channel.

Choice for a "better" pulse shape $p(t)$ than sinc $\left(\frac{t-nT_b}{T_b}\right)$

Constraints:

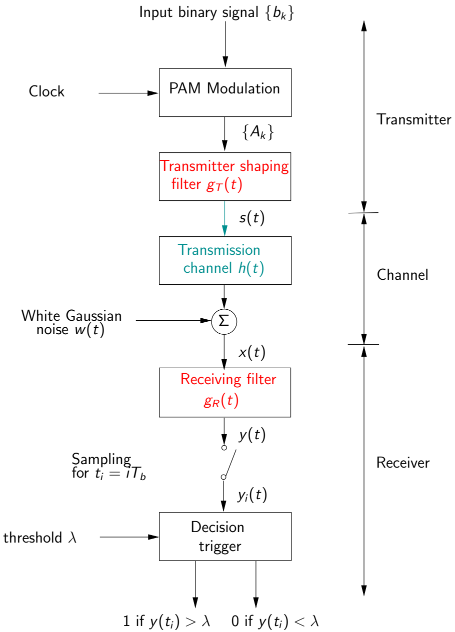
1. A pulse $p(t)$ should be *time limited*. We expect that $p(t) = 0$ except for $t \in [\tau_1, \tau_2]$.
2. $\mathcal{P}(f)$ should be a *bandpass signal*, whose content is mainly concentrated inside of $[-\frac{1}{2T_b}, \frac{1}{2T_b}]$.
3. When there is no noise, we should be able to reconstruct the signal perfectly.

Note that 1 and 2 are incompatible.

Therefore, we are looking for a trade-off:

1. $p(t)$ should be close to 0 outside its main interval, and
2. $\mathcal{P}(f)$ should be almost 0 outside its main bandwidth.

Definition



Input binary signal $\{b_k\}$
Clock
PAM Modulation
$\{A_k\}$
Transmitter shaping filter $g_T(t)$
$s(t)$
Transmission channel $h(t)$
White Gaussian noise $w(t)$
Σ
$x(t)$
Receiving filter $g_R(t)$
$y(t)$
Sampling for $t_i = iT_b$
$y_i(t)$
threshold λ
Decision trigger
1 if $y(t_i) > \lambda$
0 if $y(t_i) < \lambda$
Transmitter
Channel
Receiver

Model I

At the output of the transmission chain, we can write that

$$y(t) = \sum_{k=-\infty}^{+\infty} A_k \delta(t - kT_b) \otimes g_T(t) \otimes h(t) \otimes g_R(t) + w(t) \otimes g_R(t) \tag{324}$$

where $\otimes$ denotes the convolution.

We adopt the following notations (we take μ such that $p(0) = 1$):

$$\begin{aligned} \mu p(t) &= g_T(t) \otimes h(t) \otimes g_R(t) && (325) \\ n(t) &= w(t) \otimes g_R(t) && (326) \end{aligned}$$

In the frequency domain,

$$\mu \mathcal{P}(f) = \mathcal{G}_T(f) \mathcal{H}(f) \mathcal{G}_R(f) \tag{327}$$

As $w(t)$ is modeled as an additive zero mean white Gaussian noise, $w(t) \otimes g_R(t)$ is of the same type. It is denoted $n(t)$ hereafter.

$$
\begin{aligned}
y(t) &= \sum_{k=-\infty}^{+\infty} A_k \delta(t - kT_b) \otimes \mu p(t) + w(t) \otimes g_R(t) && (328) \\
&= \mu \sum_{k=-\infty}^{+\infty} A_k p(t - kT_b) + n(t) && (329)
\end{aligned}
$$

Once sampled at $t_i = iT_b$, the output signal $y(t)$ becomes

$$
\begin{aligned}
y[t_i] &= \mu \sum_{k=-\infty}^{+\infty} A_k p[(i-k)T_b] + n[t_i] && (330) \\
&= \mu A_i + \sum_{\substack{k=-\infty \\ k \neq i}}^{+\infty} A_k p[(i-k)T_b] + n[t_i] && (331)
\end{aligned}
$$

Analysis of $y[t_i] = \mu A_i + \sum_{\substack{k=-\infty \\ k \neq i}}^{+\infty} A_k p[(i-k)T_b] + n[t_i]$

- Useful signal: μA_i
- Noise:
  - $\sum_{\substack{k=-\infty \\ k \neq i}}^{+\infty} A_k p[(i-k)T_b]$ represents the residues of all other symbols ($k \neq i$) on the i-iest bit. It is the source of a phenomenon called *"intersymbol interference"*. This is an artificial noise that should be minimized or even nullified.
  - the noise effect due to $n[t_i]$ is "reduced" via the matching filter.

Minimizing the intersymbol interference increases the signal to noise ratio (and decreases the bit error rate P_e).

Nyquist's criterion

Goal: choose the best pulse shape $p(t)$ to minimize, or suppress, intersymbol interferences.

This is achievable if $p(t)$ verifies (noise-free channel)

$$p[(i-k)T_b] = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases} \tag{332}$$

where $p[0] = 1$ (by normalization).

Let us determine the best pulse shape $p(t)$.
To compute it, we inject the constraints on $p[(i-k)T_b]$ in the Fourier transform of a sampled version of $p(t)$.

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Sampled version of $p(t)$

Consider $p(t)$, represented by a series of its samples $\{p[mT_b]\}$, for $m = 0, \pm 1, \pm 2, \ldots$ Then, the signal

$$p_s(t) = \sum_{m=-\infty}^{+\infty} p[mT_b]\delta(t - mT_b) \tag{333}$$

is a sampled version of $p(t)$ (by definition of the sampling process).

Its Fourier transform is given by

$$\mathcal{P}_s(f) = f_b \sum_{m=-\infty}^{+\infty} \mathcal{P}(f - mf_b) \tag{334}$$

where $f_b = 1/T_b$ is the bitrate (or *rhythm*) expressed in [b/s].

Towards Nyquist's criterion II

We have two expressions for the Fourier transform $\mathcal{P}_s(f)$

1. Expression [1]: $\mathcal{P}_s(f) = f_b \sum_{m=-\infty}^{+\infty} \mathcal{P}(f - mf_b)$
2. Expression [2]: based on the definition, it can be written as

$$\mathcal{P}_s(f) = \int_{-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \left[p[mT_b]\delta(t - mT_b)\right] e^{-2\pi jft}\, dt \qquad (335)$$

but, as the sum reduces to the $m = 0$ term (according to the $p[(i-k)T_b] = 0$ if $i \neq k$ constraint), we may write

$$\mathcal{P}_s(f) = \int_{-\infty}^{+\infty} p[0]\delta(t)\, e^{-2\pi jtf}\, dt = e^{-2\pi j0f} = 1 \qquad (336)$$

By combining [1] and [2]:

$$\sum_{m=-\infty}^{+\infty} \mathcal{P}(f - mf_b) = \frac{1}{f_b} = T_b \qquad (337)$$

Towards Nyquist's criterion III

This leads to Nyquist's criterion for an ideal, noiseless, baseband transmission.

Theorem (Nyquist's criterion)

The Fourier *transform $\mathcal{P}(f)$ of the pulse shaping function $p(t)$ removes all intersymbol interferences due to other samples taken every T_b seconds if*

$$\sum_{m=-\infty}^{+\infty} \mathcal{P}(f - mf_b) = T_b \tag{338}$$

Note that the expression $\mathcal{P}(f)$ encompasses the *transmitting filter*, the *channel*, and the *receiver filter*.

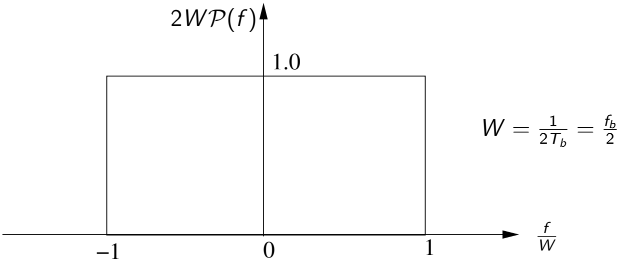
"Ideal" channel

The easiest way to satisfy Nyquist's criterion consists to take a rectangular shape for $\mathcal{P}(f)$ (in the spectral domain):

$$\begin{aligned} \mathcal{P}(f) &= \begin{cases} \frac{1}{2W} & -W < f < W \\ 0 & |f| > W \end{cases} && (339) \\ &= \frac{1}{2W}\mathrm{rect}_{[-W,+W]}(f) && (340) \end{aligned}$$

where the bandwidth W is

$$W = \frac{f_b}{2} = \frac{1}{2T_b} \quad (341)$$



Alternative pulse shapes I

There are several reasons that prevent us to use a time sinc filter (or rectangular frequency filter):

1. such a filter requires $\mathcal{P}(f)$ to be constant over $[-W, +W]$ and 0 outside.
2. the function $p(t)$ decreases as $1/|t|$ for large $|t|$ values; it decreases slowly, which means that many samples interfere with each others.

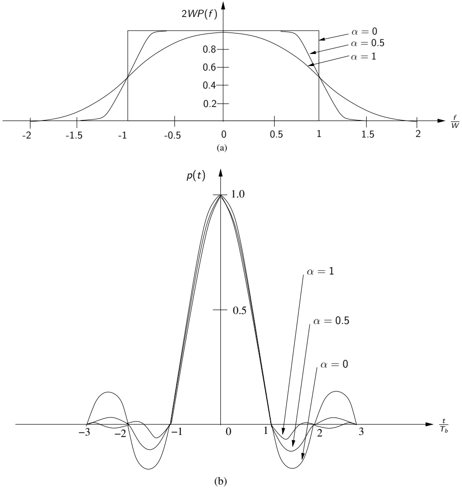
Solution: raised cosine-pulse, whose spectrum is given by

$$\mathcal{P}(f) = \begin{cases} \frac{1}{2W} & 0 \leq |f| < f_1 \\ \frac{1}{4W}\left\{1 - \sin\left[\frac{\pi(|f|-W)}{2W-2f_1}\right]\right\} & f_1 \leq |f| \leq 2W - f_1 \\ 0 & |f| \geq 2W - f_1 \end{cases} \tag{342}$$

The constant f_1 and bandwidth parameter W are related by (α is the *roll-off* factor)

$$\alpha = 1 - \frac{f_1}{W} \tag{343}$$

Alternative pulse shapes II



The bandwidth B_T is therefore given by: $B_T = 2W - f_1 = W(1 + \alpha)$

Practical remarks

$$\mu\mathcal{P}(f) = \mathcal{G}_T(f)\mathcal{H}(f)\mathcal{G}_R(f) \tag{344}$$

It is the complete chain that needs to be optimized, but $\mathcal{H}(f)$ is unknown!

Solution:

1. estimate $\mathcal{H}(f)$: $\Rightarrow \widehat{\mathcal{H}}(f)$.
2. compensate for the presence $\mathcal{H}(f)$ by taking the inverse of $\widehat{\mathcal{H}}(f)$.

In conclusion, after compensation, we should have

$$\mu\mathcal{P}(f) = \mathcal{G}_T(f)\mathcal{G}_R(f) \tag{345}$$

But how to chose the transmitter and the receiver filters?

In a real communication environment:

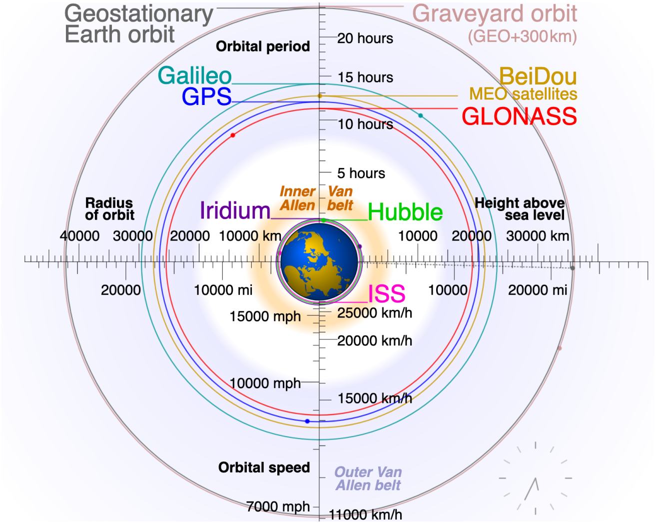
- we need a *transmitter filter*, whose purpose consists to avoid the appearance of a spectral content outside the useful bandwidth.
- there must be a *receiver filer*. It eliminates out-of-band noise.
- $\mathcal{P}(f)$ is therefore constrained to be partly implemented at the transmitter and at the receiver.

$\Longrightarrow$ we can equally divide $\mathcal{P}(f)$ over the transmitter and the receiver (such filters are named *half*-Nyquist)

Link with the notion of ideal channel for the transmission of an analog signal in a channel (*delay* and *no amplitude distortion*):

$$\mathcal{H}(f) = Ae^{-2\pi j f \tau} \tag{346}$$

Outline


Geostationary
Earth orbit
Graveyard orbit
(GEO+300km)
Orbital period
20 hours
15 hours
10 hours
5 hours
Galileo
GPS
BeiDou
MEO satellites
GLONASS
Inner Van
Allen belt
Iridium
Hubble
Radius
of orbit
40000 30000 20000 10000 km
Height above
sea level
10000 20000 30000 km
20000 10000 mi
10000 20000 mi
ISS
15000 mph
25000 km/h
20000 km/h
10000 mph
15000 km/h
Orbital speed
Outer Van
Allen belt
7000 mph
11000 km/h

GPS = Global Positioning System

Definition

Network of satellites that sent signals permanently to allow the *positioning* on the earth surface by *trilateration* measurements (sometimes, incorrectly, named as *triangulation*).

- Started by the U.S. Department of Defense
- First operational satellite in 1978
- Full constellation since 1994
- Precision of 6 to 12 [m] since May 2000

Constraints:

∗ need to be able to receive the transmitted signals.

∗ does not work inside of buildings or underground.

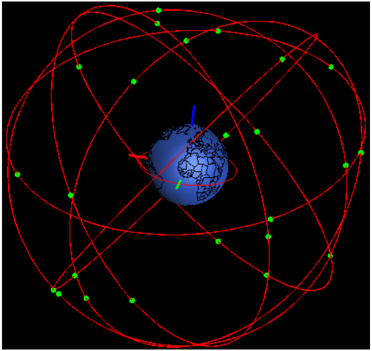
Architecture of the GPS network

3 segments:

- *Space* segment
  - comprising a constellation of 24 satellites (with 21 in service)
- *Control* segment
  - comprising ground stations
  - goal: correct the position of satellites
- *User* segment

Satellite trajectories

- Located on high orbits (but sub-geostationary)
- Revolution period of 12 hours
- Transmitting power of 20 to 50 [W]
- 6 planes with a 55° angle with the equator, spaced by 60° and with 4 satellites per plane (24 satellites in total)

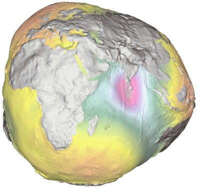


Geoid and reference ellipsoid

Definition (Geoid)

The geoid is essentially the figure of the earth abstracted from its topographical features. It is an idealized equilibrium surface of sea water, the mean sea level surface in the absence of currents, air pressure variations, etc., and continued under the continental masses.

- Because of the non-uniform repartition of masses, the earth is not a sphere.
- The geoid is close to an ellipsoid whose long axis is equal to 6400 [km].



Principles of localization/positioning

1. A distance d is related to a propagation time by t

$$d = c \times t \tag{347}$$

where $c = 3 \times 10^8$ [m/s] is the speed of light.

2. Trilateration (or even multilateration).

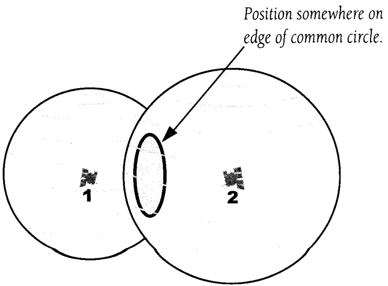


Figure: Positioning by means of intersecting the loci of 2 or 3 satellites.

Basic equations for finding user position

$$\rho_1 = \sqrt{(x_1 - x_u)^2 + (y_1 - y_u)^2 + (z_1 - z_u)^2} \qquad (348)$$

$$\rho_2 = \sqrt{(x_2 - x_u)^2 + (y_2 - y_u)^2 + (z_2 - z_u)^2} \qquad (349)$$

$$\rho_3 = \sqrt{(x_3 - x_u)^2 + (y_3 - y_u)^2 + (z_3 - z_u)^2} \qquad (350)$$

- Second order equations $\rightarrow$ difficulties
- Resolution with linearization and iterative approach

- In principle, 3 distances ρ_i suffice. But, to minimize errors due to inaccuracies, 4 distances are needed (and thus 4 satellites in line of sight).

Measurement of pseudo-range

Every satellite i sends a signal at a certain time t_{si}. It is received by the user at time t_u. The distance between the user and the satellite is

$$\rho_{iT} = c \times (t_u - t_{si}) \tag{351}$$

where c is the speed of light.
In practice, both times are biased t_u (inaccuracies)

$$t'_u = t_u + b_{ut} \tag{352}$$

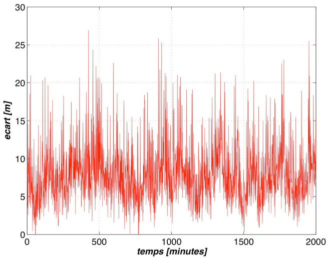
and, for t_{si},

$$t'_{si} = t_{si} + \triangle b_i \tag{353}$$

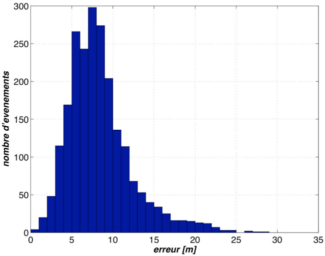
But there are other sources of errors ($\Delta D_i =$ satellite position error on range, $\Delta T_i =$ tropospheric delay error, $v_i =$ receiver measurement noise error, $\triangle v_i =$ relativistic time correction), leading to

$$\rho_i = \rho_{iT} + \Delta D_i - c \times (\triangle b_i - b_{ut}) + c \times (\Delta T_i + \Delta I_i + v_i + \triangle v_i) \tag{354}$$

Error and precision



Errors over time



Histogram of errors

Figure: Statistical errors on the distance measurements.

Signals and frequencies

A GPS signal comprises 2 spectral components: $L1$ (Link 1), $L2$

- $L1 = 1575.42\,[\text{MHz}] = 154 \times 10.23\,[\text{MHz}]$ is used for consumer market receivers.
- $L2 = 1227.6\,[\text{MHz}] = 120 \times 10.23\,[\text{MHz}]$ reserved for military applications.

The frequencies are very accurate as their reference is an atomic frequency standard.

	P	C/A
L1	$-133\,[\text{dBm}]$	$-130\,[\text{dBm}]$
L2	$-136\,[\text{dBm}]$	$-136\,[\text{dBm}]$

Table: Minimal required power levels at the receiver.

Signal and frame

Each satellite sends three types of information:

1. data
   1. type of satellite
   2. maintenance data
   3. precise calculus of the orbit (precision < 1 m), etc.
2. a precision code P: this allows a precise calculus of delays. Civilian users have access to a degraded version of P.
3. a coarse (or clear) acquisition code C/A: allows an approximated calculus of delays. Usually acquired first.

Signal sent by a satellite

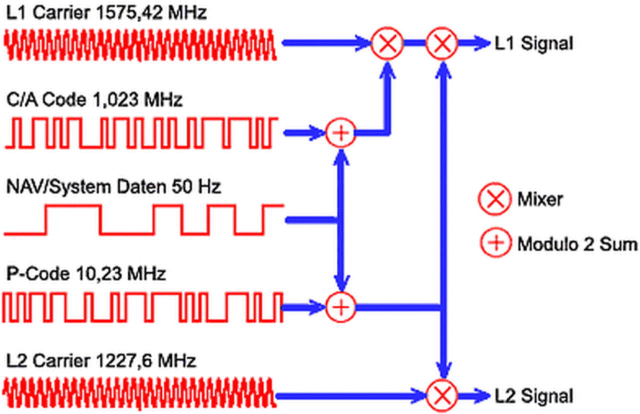


Figure: Building the signal sent by GPS satellites.

Precision code P

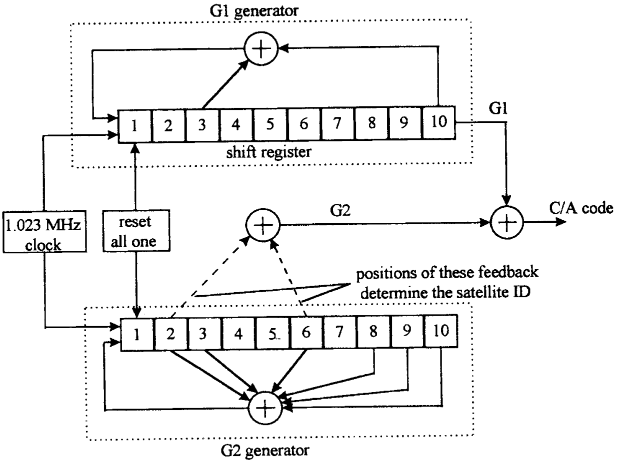
- Bi-phase modulating signal → BPSK modulation
- Bitrate of 10.23 [Mb/s] → bandwidth as defined by the main lobe size: 20.46 [MHz].
- $T_c = 97.8\,[ns]$
- Generated from two pseudorandom noise codes
  - sequence 1: $15,345,000$ chips, $1.5\,[\mathrm{s}]$ long period
  - sequence 2: $15,345,037$ chips
- As these numbers are relative prime, they have no common factors between them. Therefore, the *code length* generated by these codes is

$$1.5 \times 15,345,037 = 23,017,555.5\,[\mathrm{s}] \tag{355}$$

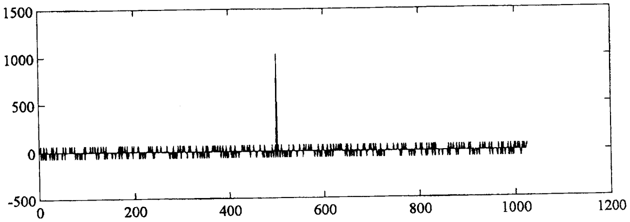
which is slightly longer than 38 weeks.

However, *the actual code length is 1 week as the code is reset every week*. This code can be divided into 37 different P codes and each satellite can use a different portion of the code (identification).

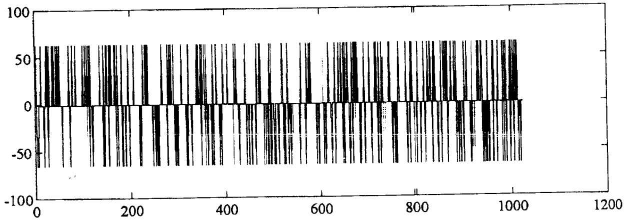
Code generation



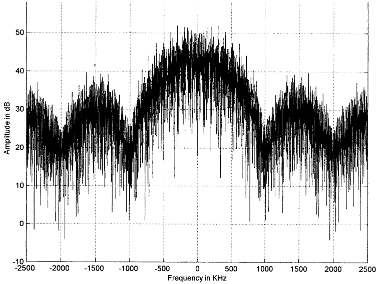
Autocorrelation et cross-correlation of satellite codes



(a) Autocorrelation of satellite 19.



(b) Cross correlation of satellites 19 and 31.



- Bi-phase NRZ code → modulation BPSK
- Bitrate of 1.023 [Mb/s] → width of the main lobe 2.046 [MHz].
  - → the bandwidth is thus fixed by the P code, not the C/A code.
- $T_c = 977.5\,[ns]$
- Generated by means of a pseudorandom sequence of 1023 chips → period of $1\,[ms]$

Better precision with the help of ground stations: Differential GPS

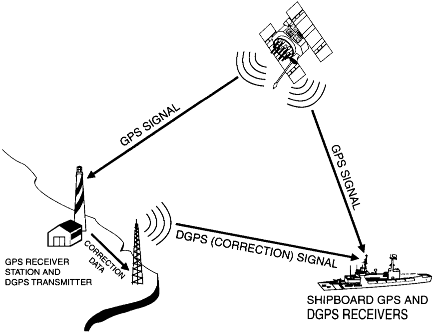
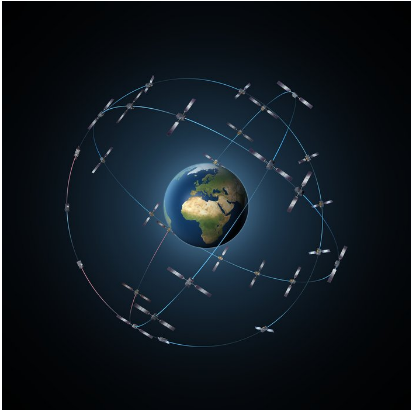


Figure: Working of the DGPS.

- FM band
- Precision up to 1 to 5 [m]

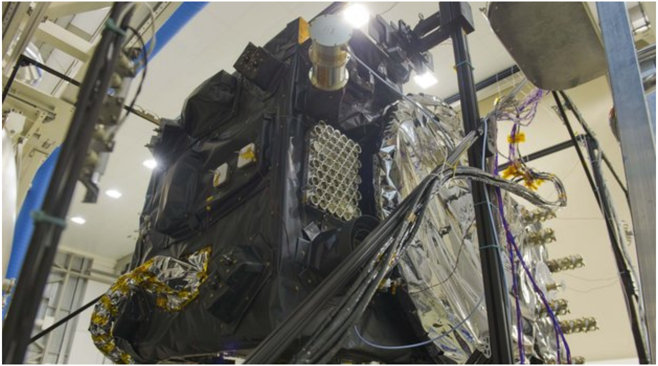
Galileo

- Orbital altitude: 23,222 [km] (MEO - Medium Earth Orbit)
- 3 orbital planes, 56° inclination, separated by 120° longitude
- Constellation of 30 satellites (with working 24 [3x8] satellites and 6 [3x2] spares)

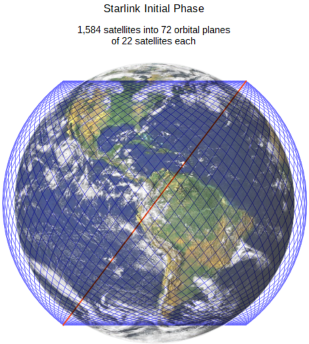


Deployment of Galileo

- First launches: 2 satellites in October 2011, 2 satellites in October 2012. These were test satellites.
- First Full Operational Capability satellite launched in November 2013.
- August 2014, two more satellites (but ... injected on a wrong orbit).
- October 2022: 23 satellites fully operational, 1 unavailable, and 4 not usable.



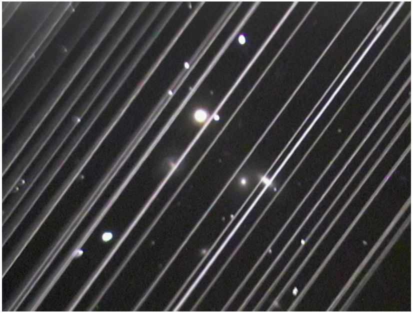
Starlink



Main characteristics:

- LEO orbits (550 km for phase 1)
- 2.091 launched
- American regulator (FCC) approved 12.000 satellites
- Internet service (500.000 subscribers, October 2022)

Starlink: controversy



Main issues:

- light pollution; ground based astronomy is jeopardized (creation of trails in the sky)
- presence of space debris, danger for satellite collision
- technology not fully tested
- usefulness ?! (it's available in Belgium)

Outline

- Multiplexing: principles and access modes
- Specific techniques for multiplexing:
  1. Frequency Division Multiplexing (FDM)
  2. Time Division Multiplexing (TDM)
     - Example: signals between telephone exchanges
  3. Code Division Multiplexing (CDM)

Multiplexing

Historically, there are two major techniques:

1. *Frequency Division Multiplexing* (FDM). This technique allocates a dedicated frequency band to each signal (user).
2. *Time Division Multiplexing* (TDM). This techniques monitors the time allocated to each user. Two users cannot use the channel simultaneously.

More recent technique: *multiplexing by spread spectrum* → *Code Division Multiplexing* (CDM)

Variants for optical fibers:

- *Wave(length) Division Multiplexing* (WDM): it is equivalent to FDM (but in terms of wavelengths)
- some variants especially targeting high speed links: *Dense Wave Division Multiplexing* (DWDM).

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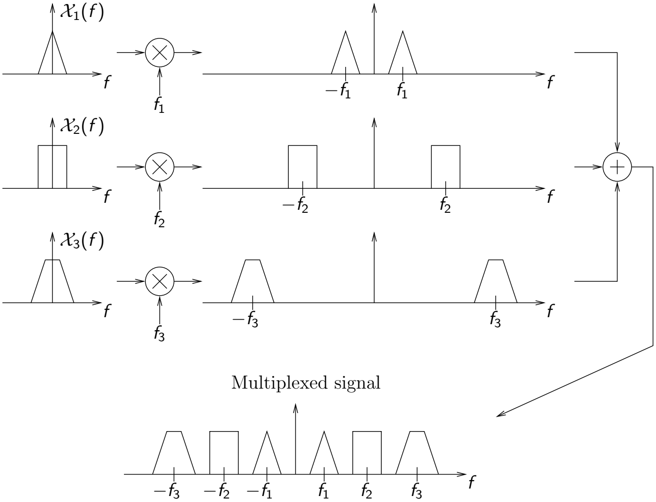
Multiple access

Multiplexing is introduced for *sharing* resources. Therefore, a multiplexing technique should propose and implement:

- techniques to access individual resources. We refer to the techniques and protocols to access a resource by the letter *A*, denoting *Access*: FDMA, TDMA, CDMA, ...
- strategies to allocate resources to users. These strategies, which are specific to a network operator, are based on performance targets: high bitrate, low delay, low congestion rate, ... These strategies are studied in other courses.

Principles of Frequency Division Multiplexing (FDM) I



Principles of Frequency Division Multiplexing (FDM) II

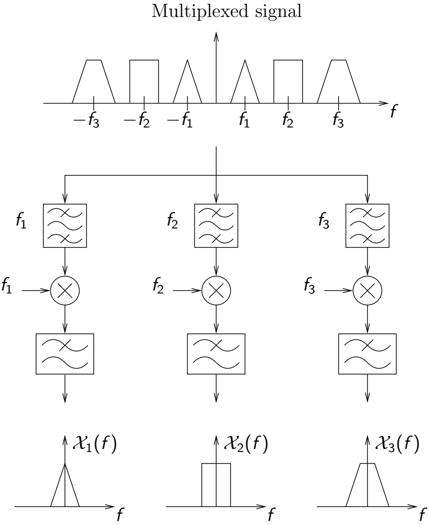


Figure: Frequency demultiplexing.

Bandwidth and access to the multiplex

$$W > \sum_i W_i \tag{356}$$

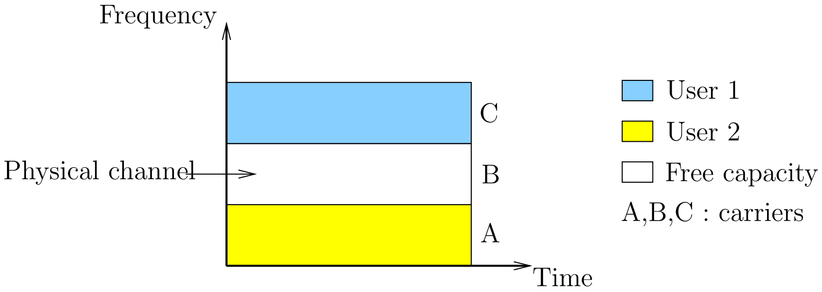
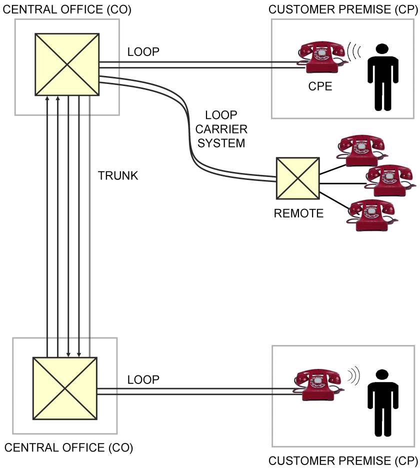
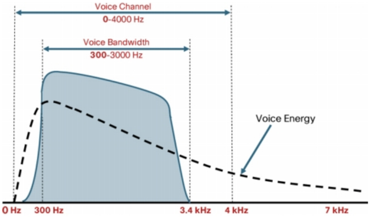


Figure: *Frequency Division Multiple Access* (FDMA): multiplexing mechanism.

Illustration: Plain Old Telephone Service (POTS)



- Wired network
- Trunks are shared
- Frequency band per user: [300 Hz, 3400 Hz]



Voice channel: [0 → 4 kHz]

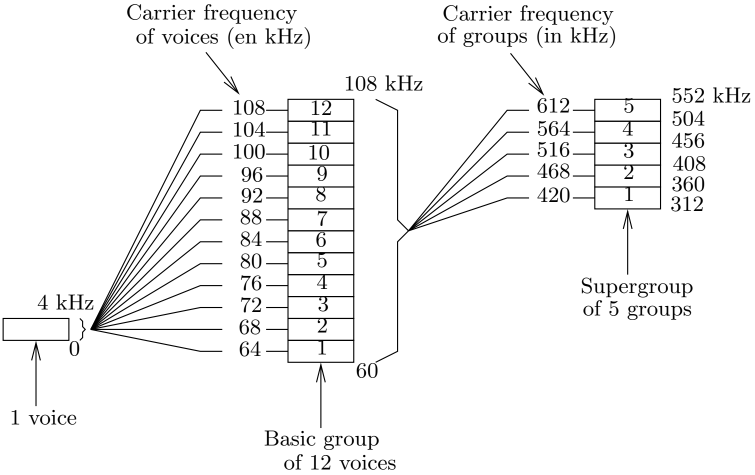


Figure: Construction of a basic group and a super group.

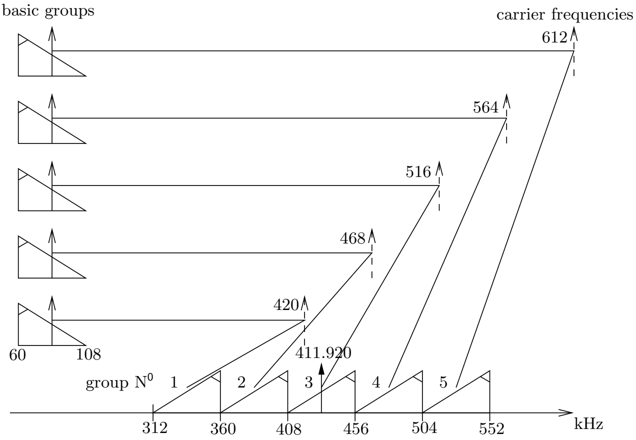


Figure: Building mechanism for super groups.

Principles of Time Division Multiplexing (TDM) I

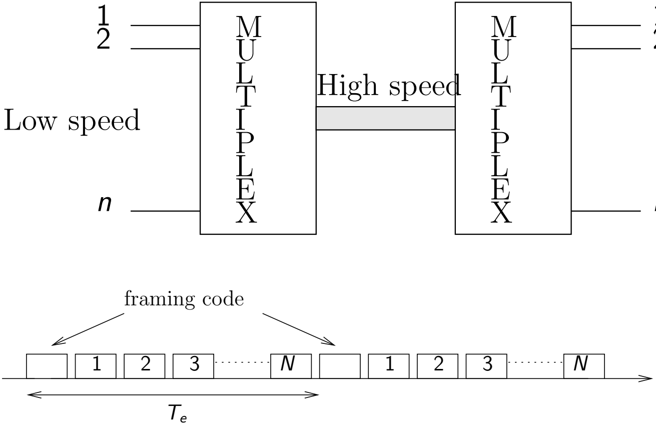
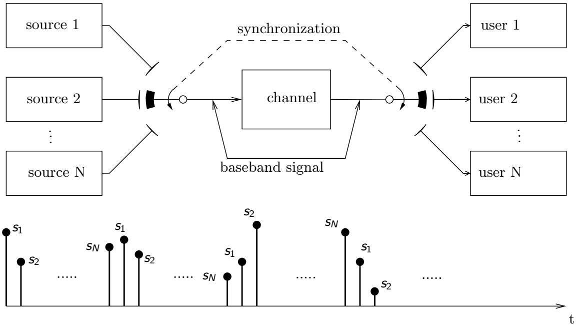


Figure: Formation process of a TDM multiplex.

The time slots for users, plus the information necessary for the multiplex itself, constitute a *frame*.



The total number of samples is $n_s = \sum_i 2W_i T$
If W is the bandwidth, we need $2W$ samples. For an interval T, we need

$$n_s = 2WT = \sum_i 2W_i T \tag{357}$$

So that,

$$W = \sum_i W_i \tag{358}$$

Access to the multiplex

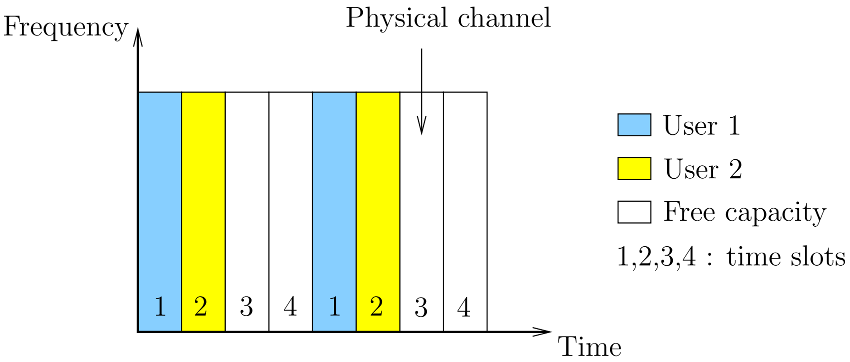


Figure: *Time Division Multiple Access* (TDMA).

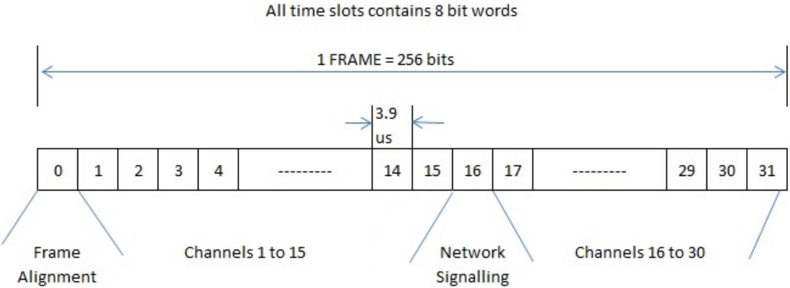
Public switch telephone network in Europe: 30 digital voice channels (E1) I

Characteristics

The European standard TDM frame for 30(+2) digital voice channels (**E1**):

- W = bandwidth of the signal $= 300 - 3400$ [Hz] and $f_s = 8$ [k sample/s].
- PCM encoding with the A (=87,6) quantization law, compression with 13 segments on 8 bits, R_v = bitrate for one digital voice channel = 64 [kb/s].
- total bitrate $= 32 \times 64$ [kb/s]=2.048 [Mb/s].
  This is the minimal bitrate for communicating with a digital public network in Europe.

Public switch telephone network in Europe: 30 digital voice channels (E1) II



- time multiplexing with 32 time slots comprising:
  - TS0 channel: framing, synchronization, alarms, CRC, etc.
  - TS1-TS15 + TS17-T31: 30 voice channels
  - TS16: signaling

Frame structure I

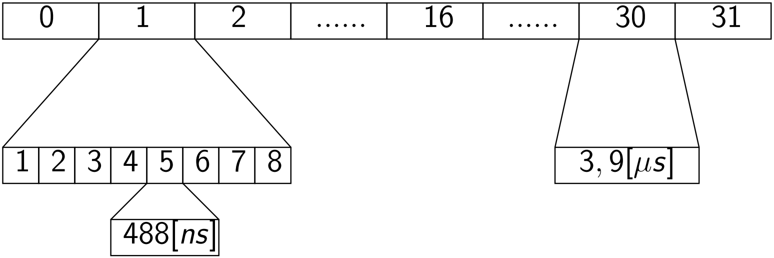
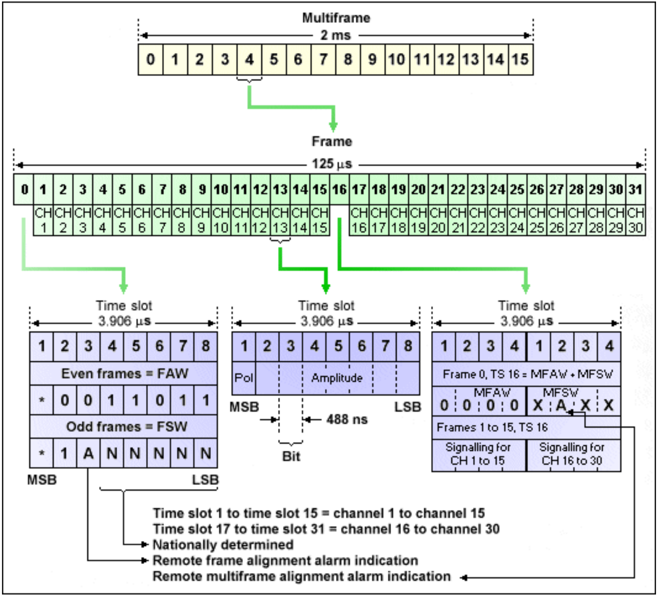


Figure: Frame structure at the bit level.

Frame structure II

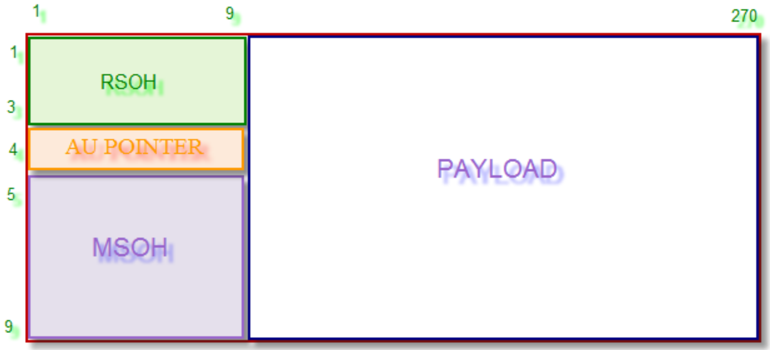


Synchronous Digital Hierarchy (SDH)

In the network, we need higher bit rates than E1 bit rates. Synchronous Digital Hierarchy (SDH) is a standardized protocol that transfer multiple digital bit streams synchronously over optical fiber.

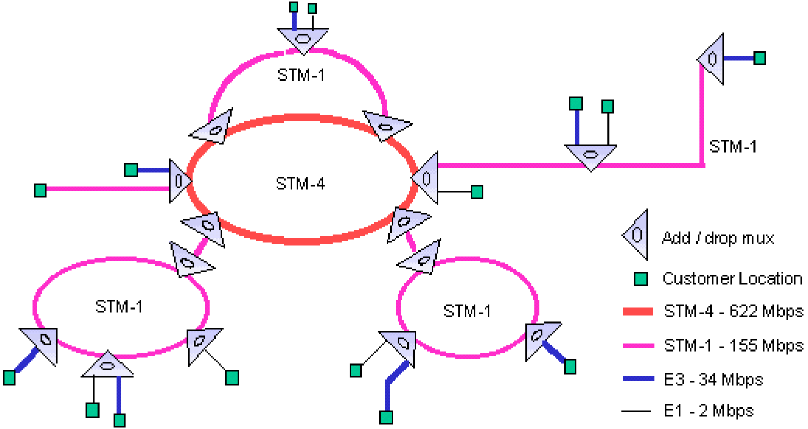
SDH level and frame format	Payload bandwidth [kb/s]	Line rate [kb/s]
STM-1	150 336	155 520
STM-4	601 344	622 080
STM-16	2 405 376	2 488 320
STM-64	9 621 504	9 953 280
STM-256	38 486 016	39 813 120

SDH frame structure



1. RSOH = regenerator section overhead
2. Pointers are used to address a Virtual Container in the payload.
3. MSOH = multiplex section overhead

Typical structure of a SDH network



- Presence of Add and Drop Multiplexers (ADM)
- Structure of a ring → redundancy in the paths

Code Division Multiplexing Access (CDMA) I

Main multiplexing techniques:

- FDMA: use some of bandwidth all of the time
- TDMA: all of the bandwidth some of the time

and ...

- CDMA: all of the bandwidth all of the time!

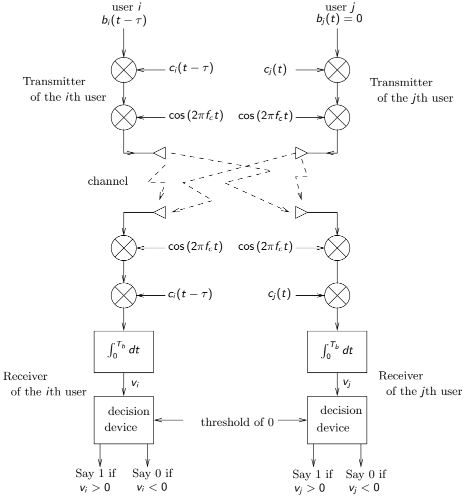


Figure: Scheme for the study the cross-correlation phenomenon.

The interference (only) due to user i evaluated at the input of the decision device in the receiver of user j is given by

$$\begin{aligned} v_j(\tau)\big|_{b_j(t)=0} &= \int_0^{T_b} b_i(t-\tau)\, c_i(t-\tau)\, c_j(t)dt &(359)\\ &= \pm \int_0^{T_b} c_j(t)c_i(t-\tau)dt &(360)\end{aligned}$$

We may rewrite this in the following form

$$v_j(\tau)\big|_{b_j(t)=0} = \pm T_b \Gamma_{ji}(\tau) \tag{361}$$

where

$$\Gamma_{ji}(\tau) = \frac{1}{T_b}\int_0^{T_b} c_j(t)c_i(t-\tau)dt \tag{362}$$

This time-averaged quantity is called the *partial cross-correlation function* of two pseudo-random sequences.

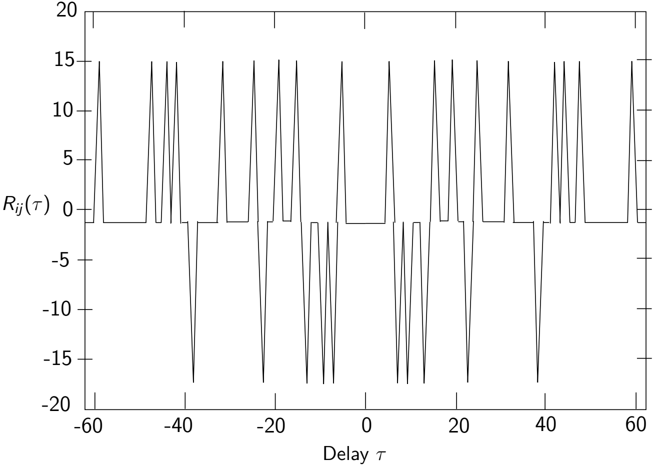


Figure: Cross-correlation of two sequences of length $N = 63$ (generated by [6, 1] and [6, 5, 2, 1]).

Role of auto-correlation and cross-correlation I

There are two techniques for multiplexing with Direct Sequence Spread Spectrum (DSSS):

- one identical spreading sequences for all the users. As long as synchronization times are kept separated by a time larger than one chip, this is not problematic
  - $\rightarrow$ the *auto-correlation function is important* to measure the performance of the system.
- different spreading sequences for all the users.
  - $\rightarrow$ the *cross-correlation function* is important to measure the performance.

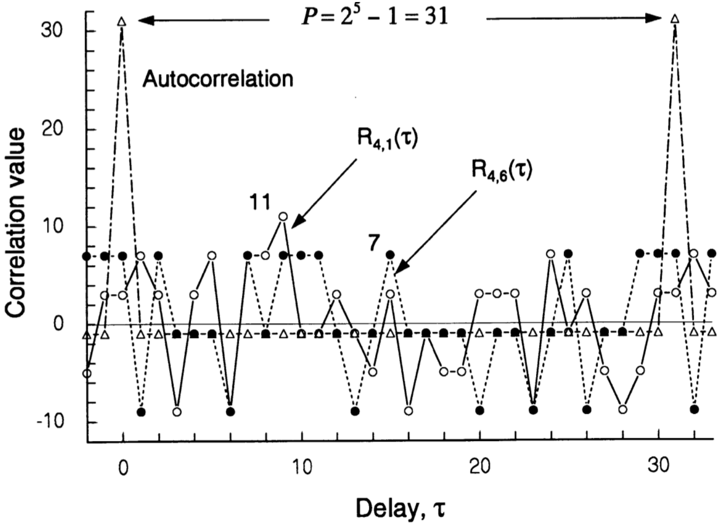


Figure: Auto-correlation and cross-correlation ($R_{4,1}(\tau)$ and $R_{4,6}(\tau)$).

Other generators for pseudo-random sequences I

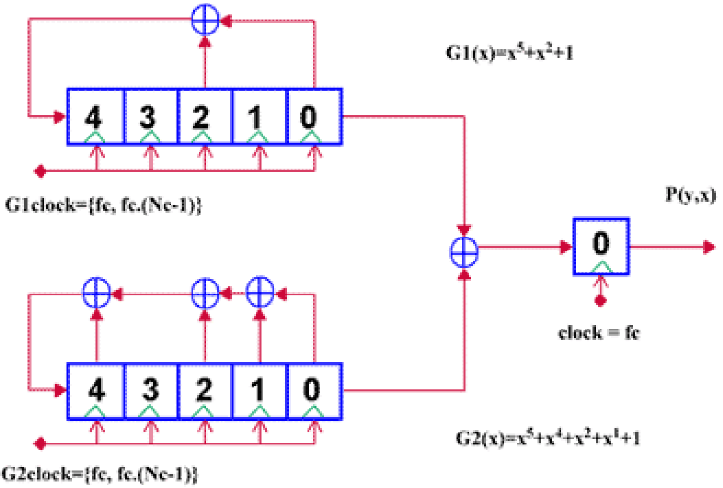


Figure: “Gold” sequence generator.

It is important to find an upper bound for the cross-correlation function. There are such results:

Theorem

It can be shown that the Gold *sequences, under certain generation conditions, satisfy the following cross-correlation function values (where n is the number of registers):*

$$|R_{1,t}(k)| \leq \begin{cases} 2^{\frac{n+1}{2}} + 1 & \text{if} \quad n \text{ is odd} \\ 2^{\frac{n+2}{2}} + 1 & \text{if} \quad n \text{ is even but } n \neq 0 \bmod 4 \end{cases} \tag{363}$$

Code Division Multiplexing (CDMA)

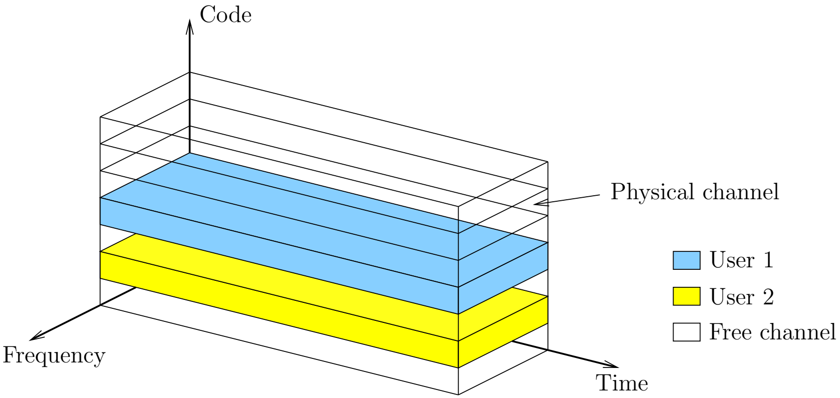


Figure: *Code Division Multiple Access* (CDMA).

Combination of multiplexing techniques

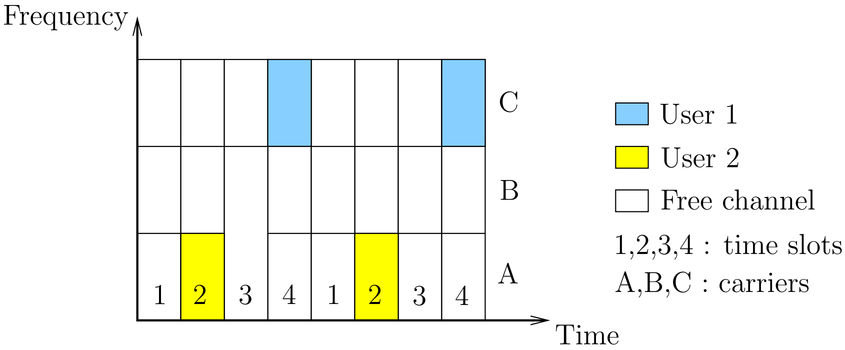
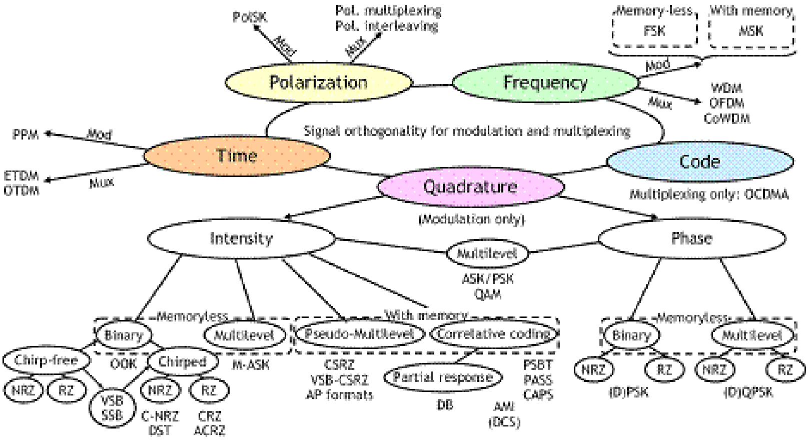


Figure: Resource sharing by combining time and frequency multiplexing (TDMA/FDMA).

Overview of *multiplexing/modulation* techniques (+ *diversity* techniques)



Space-time transmission I

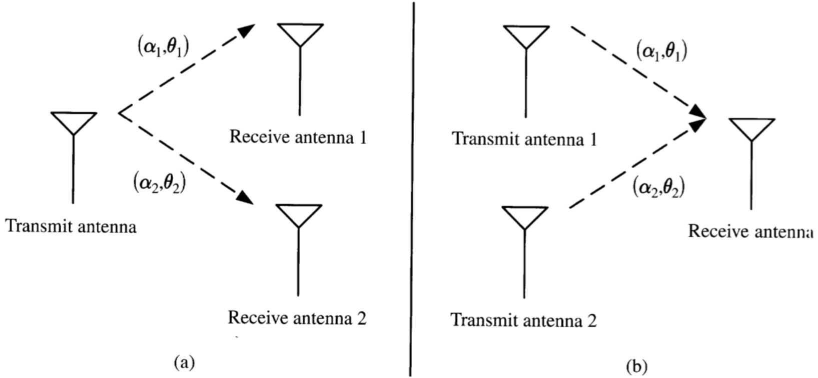
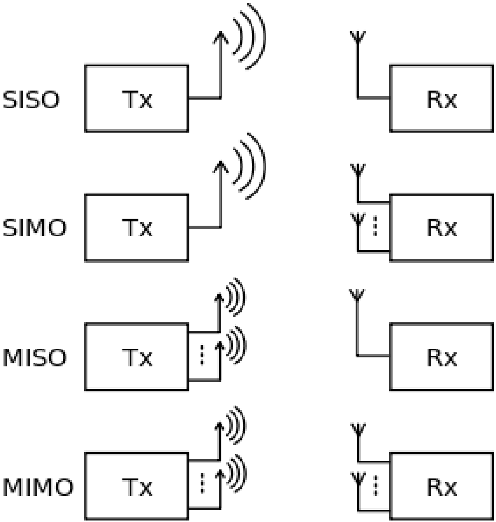


Figure: Space diversity: (a) one transmit and two receive antennas and (b) two transmit and one receive antennas.

MIMO: Multiple Inputs Multiple Outputs I



Objectives:

MIMO: Multiple Inputs Multiple Outputs II

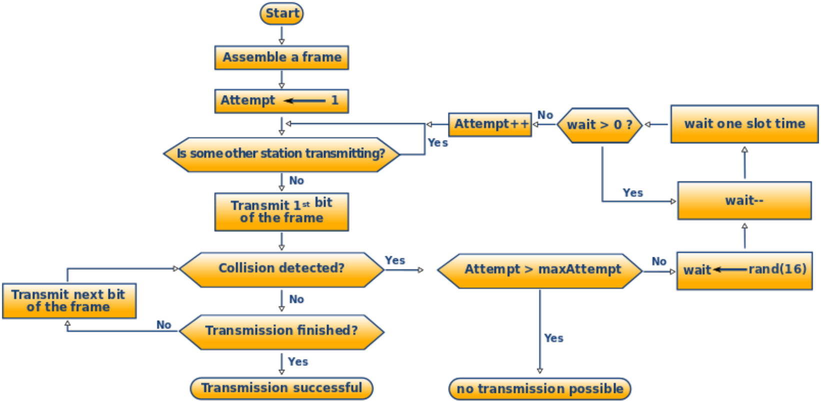
1. improve the link reliability (reduces fading)
2. increase the spectral efficiency (more bits per second per hertz of bandwidth)

One possibility:

time $\rightarrow$	$[(k-1)T_b,\ kT_b]$	$[kT_b,\ (k+1)T_b]$
antenna 1	$A_k\sqrt{E_b/T_b}\cos(2\pi f_c t)$	$-A_{k+1}\sqrt{E_b/T_b}\cos(2\pi f_c t)$
antenna 2	$A_{k+1}\sqrt{E_b/T_b}\cos(2\pi f_c t)$	$A_k\sqrt{E_b/T_b}\cos(2\pi f_c t)$

Ethernet protocol: Carrier Sense Multiple Access with Collision Detection (CSMA-CD)

- It is a protocol (not a multiplexing technique)
- Relates to the Access
- Complex state diagram



Comparison of mobile standards

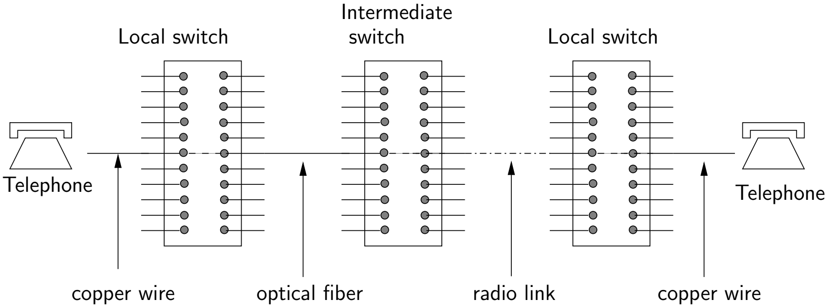
System	GSM [EUR]	DCS-1800 [EUR]	IS-54 [USA]	IS-95 (DS) [USA]
Access mode	TDMA/FDMA	TDMA/FDMA	TDMA/FDMA	CDMA/FDMA
Used bandwidth				
Upwards (MHz)	890-915	1710-1785	824-849	824-849
Downwards (MHz)	935-960	1805-1880	869-894	869-894
Distance between channels				
Upwards (kHz)	200	200	30	1250
Downwards (kHz)	200	200	30	1250
Modulation	GMSK	GMSK	$\pi/4$ DQPSK	BPSK/QPSK
Characteristics for the mobile device				
Max./Mean	1W/125mW	1W/125mW	600mW/200mW	600mW
Voice encoding	RPE-LTP	RPE-LTP	VSELP	QCELP
Voice rate (kb/s)	13	13	7,95	8 (var.)
Channel bitrate				
Upwards (kb/s)	270,833	270,833	48,6	
Downwards (kb/s)	270,833	270,833	48,6	
Frame (ms)	4,615	4,615	40	20

Outline

Telephone traffic engineering

- Characterization of a telephone network
  - Intensity
  - Carried traffic load
  - Reference load
  - Offered load
- Probabilistic analysis of calls
  - Counting process
  - Poisson distribution: definition and properties
- Probabilistic law for telephone traffic
  - Memoryless model: Erlang B statistic
  - Sizing of a trunk
  - Other models

Structure of a switched telephone network



- For economic reasons, there are fewer lines available than the number of telephones or potential callers (the cost is proportional to the number of lines).
- There is a probability that someone willing to make a call find that all the lines are in use. The call attempt is then "*blocked*".

Goal: *establish the relationship between the number of lines N of a trunk and the blocking probability.*

Intensity, traffic and load: which measure(s)?

There are two types of links in a switched telephone network:

- links that carry *telephone signals* or *data*
- links dedicated to the transmission of *signaling data*

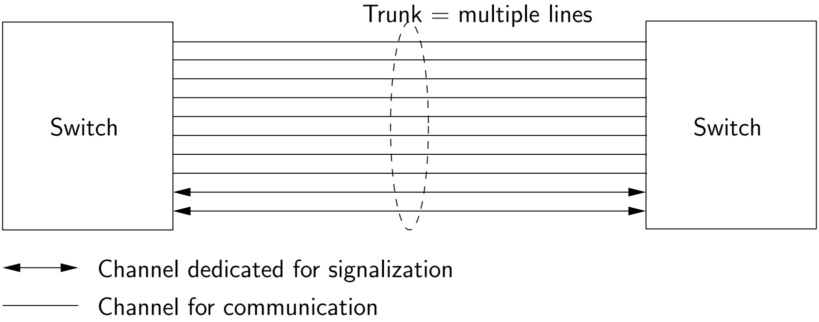


Figure: A trunk between two switches.

Intensity measures I

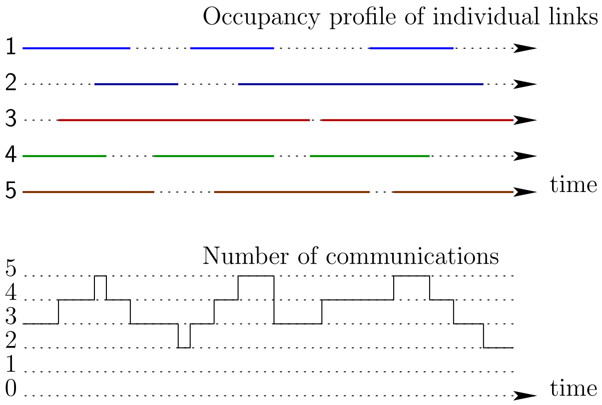


Figure: Activity profile of a trunk of 5 links.

We need a measure for the level of activity!

Intensity measures II

Definition (Intensity of the carried traffic)

The intensity I of the carried traffic is defined as the ratio between the observed traffic volume divided by the observation time T :

$$I = \frac{\int_0^T \sum_{i=1}^N 1_i(t)dt}{T} = \frac{\sum_{i=1}^N \int_0^T 1_i(t)dt}{T} \quad \left[\frac{s}{s}\right] \tag{364}$$

where $1_i(t)$ is the *indicator function* of a link (line i).

The intensity represents the average carried traffic during a given amount of time. It has no unit; however it is given a unit name and is expressed in **Erlang, denoted [E]**.

Intensity measures III

In practice, the traffic is characterized by two important parameters:

1. the *average rate of carried calls* λ_c, measured in [call/s],
2. the *average call duration* t_m, in [s/call].
   If $\#_T$ represents the number of calls done during a time T, then the average duration is given by

$$t_m = \frac{\sum_{i=1}^{N} \int_0^T 1_i(t)dt}{\#_T} \tag{365}$$

Notions of load(s)

Definition (Carried traffic load)

The *carried load*, expressed en Erlang, is the product of the entering call rate (thus carried) by the average duration of a call. It is denoted A_c.

$$A_c = \lambda_c t_m \tag{366}$$

Reference load?

The maximum load on a N-lines trunk is theoretically N.
But in practice, not all the lines are used permanently and we distinguish between two types of loads:

1. the **offered** *load*; it is the load that would be carried if the network could carry all calls, without any limit.
2. the **carried** *load*; it is the load really measured in the network.

By definition, the **carried** *load* is always lower than the **offered** *load*: $A_c \leq A$.

Notions of load(s)

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Offered load

Definition

The offered load A is defined by

$$A = \lambda t_m \tag{367}$$

where λ is the average rate of **call trials**.

How do we determine the load when we have no possibility to measure it? $\Rightarrow$ rules of **good practice**

It is common to choose a *practical reference load* (*capacity*) that represents the load to be allocated to a user during the peak hours. Typically, the network could be sized to allocate a capacity A_0 comprised between 0.02 and 0.1 [E] per user (and per type of activity: internal/in/out)

A good network should be sized such that the carried load is (remember that $A_c \leq A$):

- *as close as possible* to the offered load during the peak hours.
- *equal* to the offered load outside the peak hours.

Measures (statistics) available in a real telephone switch

A telephone switch is usually able to collect the following statistics, for a given period T (typically a quarter of an hour):

- the average call duration t_m. Note that certain calls might have started before the observation time and finish later.
- the number of call trials [attempts] (counter), λ.
- the number of carried calls (counter).
- the number of terminated calls (counter).

To establish our statistical analysis of telephone calls, we will use T and λ.

Counting process

We start by establishing the law of the random process $D(t)$ that counts the number of calls initiated after $t = 0$.

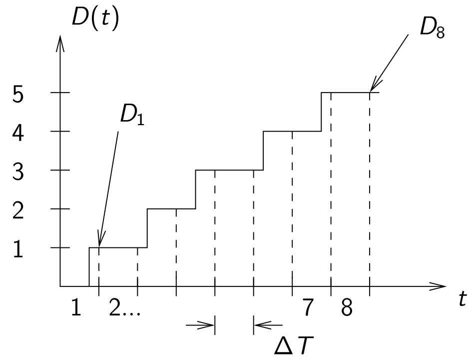


Figure: Integrating the number of calls over time (realization of $D(t)$).

Probabilistic analysis of telephone calls II

Elements of our model:

1. **[Time discretization]** We divide the observation time T into m sub-intervals ΔT such that $T = m\Delta T$.
   Important note: we take ΔT so small that only one occurrence per interval is possible.
2. **[Probability]** Let p be the probability (supposed to be stationary) of a call trial to occur during ΔT.
   Because ΔT is fixed, we can define λ such that $p = \alpha\Delta T$, where α is a constant.
   Later, it appears that in fact $\alpha = \lambda$, so that we take $p = \lambda\Delta T$ directly.

Let consider one random variable per sub-interval ΔT: $D_1, D_2, \ldots$ (there are m of them). Each variable assigns 1 if a trial has occurred, 0 otherwise.

Probabilistic analysis of telephone calls III

By accumulating the values of all the D_1, D_2, ... variables, we get the expression of $D(t)$:

1. the probability of having n call trials during T is (n successes for a binomial law):

$$(\lambda \Delta T)^n = \left(\frac{\lambda T}{m}\right)^n \tag{368}$$

2. the probability of having $m - n$ sub-intervals ΔT with no trial is ($m - n$ failures for a binomial law)

$$(1 - \lambda \Delta T)^{m-n} = \left(1 - \frac{\lambda T}{m}\right)^{m-n} \tag{369}$$

3. we need to consider all the possible permutations: C_m^n

Binomial law

The number of occurrences (call trials) D_m during $T = m\triangle T$ is therefore given by the following binomial probability density function (pdf)

$$f_{D_m}(n) = p(D_m = n) = \begin{cases} C_m^n \left(\frac{\lambda T}{m}\right)^n \left(1 - \frac{\lambda T}{m}\right)^{m-n} & n = 0, 1, \ldots, m \\ 0 & n \neq 0, 1, \ldots, m \end{cases} \tag{370}$$

Its *expectation (average/mean)* is given by the number of trials $\times$ the probability to succeed: $m(\frac{\lambda T}{m}) = \lambda T$ (property of binomial law).

- λ can thus be interpreted as the average number of trials during the time period T.

Poisson distribution: definition and properties I

The previous expression can be rewritten as

$$
\begin{aligned}
f_{D_m}(n) &= C_m^n \left(\frac{\lambda T}{m}\right)^n \left(1-\frac{\lambda T}{m}\right)^{m-n} && (371)\\
&= \frac{m!}{n!(m-n)!} \left(\frac{\lambda T}{m}\right)^n \left(1-\frac{\lambda T}{m}\right)^{m-n} && (372)\\
&= \frac{m(m-1)\ldots(m-n+1)}{n!} \frac{(\lambda T)^n}{m^n} \left(1-\frac{\lambda T}{m}\right)^{m-n} && (373)\\
&= \frac{m(m-1)\ldots(m-n+1)}{m^n} \frac{(\lambda T)^n}{n!} \left(1-\frac{\lambda T}{m}\right)^{m-n} && (374)
\end{aligned}
$$

Then we take the limit $\triangle T \to 0$ ($\equiv m \to +\infty$) of it:

$$
\begin{aligned}
\lim_{m\to+\infty} f_{D_m}(n) &= \lim_{m\to+\infty} \frac{m(m-1)\ldots(m-n+1)}{m^n} \frac{(\lambda T)^n}{n!} \left(1-\frac{\lambda T}{m}\right)^{m-n} && (375)\\
&= \lim_{m\to+\infty} 1 \times \frac{(\lambda T)^n}{n!} \left(1-\frac{\lambda T}{m}\right)^{m-n} && (376)\\
&= \frac{(\lambda T)^n}{n!} \lim_{m\to+\infty} \left(1-\frac{\lambda T}{m}\right)^{m-n} && (377)
\end{aligned}
$$

Poisson distribution: definition and properties II

If x is small, then $e^{-x} = 1 - x + \frac{x^2}{2!} - \ldots$, so that

$$\lim_{m \to +\infty} \left(1 - \frac{\lambda T}{m}\right)^{m-n} = \lim_{m \to +\infty} \frac{\left(1 - \frac{\lambda T}{m}\right)^m}{\left(1 - \frac{\lambda T}{m}\right)^n} \tag{378}$$

$$= \lim_{m \to +\infty} \frac{\left(e^{-\frac{\lambda T}{m}}\right)^m}{1} = e^{-\frac{\lambda T}{m} m} = e^{-\lambda T} \tag{379}$$

Therefore, for $m \to +\infty$,

$$f_D(n) = \begin{cases} \frac{(\lambda T)^n}{n!} e^{-\lambda T} & n = 0, 1, \ldots \\ 0 & n \neq 0, 1, \ldots \end{cases} \tag{380}$$

This is a Poisson probability distribution function of parameter λT.

Poisson distribution: definition and properties III

Theorem

The mean and variance of a Poisson *distribution are respectively*

$$\mu_D = \lambda T \tag{381}$$

$$\sigma_D^2 = \lambda T \tag{382}$$

Poisson distribution: definition and properties IV

Proof.

The expectation (mean) is defined as

$$\mu_D = \sum_{i=0}^{\infty} i f_D(i) = \sum_{i=0}^{\infty} i \frac{(\lambda T)^i}{i!} e^{-\lambda T} \tag{383}$$

$$= 0 \frac{(\lambda T)^0}{0!} e^{-\lambda T} + \sum_{i=1}^{\infty} i \frac{(\lambda T)^i}{i!} e^{-\lambda T} = \sum_{i=1}^{\infty} i \frac{(\lambda T)^i}{i!} e^{-\lambda T} \tag{384}$$

Then we take $j = i - 1$ (variable change):

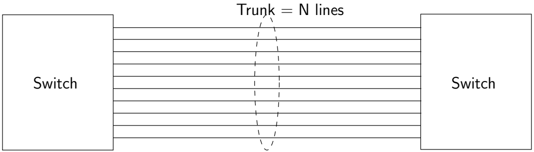
$$\mu_D = \sum_{j=0}^{\infty} (j+1) \frac{(\lambda T)^{j+1}}{(j+1)!} e^{-\lambda T} = \sum_{j=0}^{\infty} \frac{(\lambda T)^{j+1}}{j!} e^{-\lambda T} \tag{385}$$

$$= \lambda T \sum_{j=0}^{\infty} \frac{(\lambda T)^j}{j!} e^{-\lambda T} = \lambda T \times 1 = \lambda T \tag{386}$$

□

Remember that λ is given/measured by the switch and T is chosen!

Sizing a trunk



- We can measure some statistics (λ, t_m, etc.). If they are not available, we take values given by good practice (reference load depending of the users profile).
- We know the law for call trials. A similar law counts the call releases; it is also a Poisson probability distribution function (although with a different parameter value).
- We need to establish the probability P_k to have k busy lines out of N lines of the trunk. Then, P_N is the value we are looking for (for sizing the trunk at *peak* hours).

Sizing a trunk: model based on transition probabilities

$$
\begin{array}{lcl}
p(k+1;\ t) & \searrow^{P_{-1}} & p(k+1;\ t+dt) \\
\boxed{p(k;\ t) \quad \xrightarrow{P_=} \quad p(k;\ t+dt)} & & \\
p(k-1;\ t) & \nearrow_{P_{+1}} & p(k-1;\ t+dt)
\end{array}
$$

Components of our approach:

1. find the expression of transition probabilities from $p(k;\ t)$ to $p(k;\ t+dt)$.
2. add some assumptions to reduce the number of unknown values.
3. derive the expression of P_N.

Towards the Erlang B statistic: a memoryless model for telephone traffic

Number of arriving/entering calls

In a time interval $\triangle t$, the number of *arriving/entering* calls, N_A is a Poisson process given by

$$p(N_A = n) = \frac{(\lambda \triangle t)^n}{n!} e^{-\lambda \triangle t}, \quad n = 0, 1, \ldots \tag{387}$$

where λ is the *average arrival rate* (for the trunk).

Example

The probability of exactly one arrival in $\triangle t$ is $\lambda \triangle t \, e^{-\lambda \triangle t}$.

Number of calls that departure/leaving (releases)

Likewise, the number N_D of calls that *departure/leave* is a Poisson process with *average departure rate* η:

$$p(N_D = n) = \frac{(\eta \triangle t)^n}{n!} e^{-\eta \triangle t}, \quad n = 0, 1, \ldots \tag{388}$$

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Load analysis I

At any time, the load fluctuates between 0 and N *ongoing* calls.
For a very brief time interval dt, considering one line only,

- the probability that exactly one call will arrive ($n = 1$) is
$$\lambda dt e^{-\lambda dt} \simeq \lambda dt \tag{389}$$
- the probability that exactly one call will depart from an occupied line is
$$\eta dt e^{-\eta dt} \simeq \eta dt \tag{390}$$

Thus, given that k lines (of the N lines) are occupied at time t, we need to calculate the following **transition** *probabilities*, from time t to time $t + dt$:

1. P_{+1} the probability that one call will arrive in the interval
2. P_{-1} the probability that one call will depart in the interval
3. $P_{=}$ the probability that neither an arrival nor a departure occurs in the interval

Calculation of P_{+1} , P_{-1} and $P_{=}$:

- P_{+1}: the probability of a call arriving is independent of the number of lines. Therefore:

$$P_{+1} = \lambda dt \tag{391}$$

- P_{-1}: if one of the k lines is released, then the $k-1$ other lines remain occupied.
  Taking all permutations into account:

$$P_{-1} = C_k^1 (\eta dt)^1 (1 - \eta dt)^{k-1} \simeq k\eta dt \tag{392}$$

- $P_{=}$: unchanged situation $\equiv$ no entering call, no leaving call (don't forget that k lines are occupied)

$$P_{=} = (1 - \lambda dt)(1 - k\eta dt) \simeq 1 - \lambda dt - k\eta dt \tag{393}$$

Establishing the transition equations:

Let $p(k;t+dt)$ be the probability to have k busy lines at time $t+dt$. It is then possible to determine $p(k;t+dt)$ based on P_{+1}, P_{-1}, $P_{=}$, and probabilities at time t:

$$\begin{aligned} p(k;t+dt) &= P_{=}\, p(k;t) + P_{+1}\, p(k-1;t) + P_{-1}\, p(k+1;t) && (394)\\ &\simeq (1-\lambda dt - k\eta dt)\, p(k;t) && (395)\\ &+ \lambda dt\, p(k-1;t) && (396)\\ &+ (k+1)\eta dt\, p(k+1;t) && (397)\end{aligned}$$

There are two particular cases:

- $k=0$ (no line occupied)

$$p(0;t+dt) = (1-\lambda dt)\, p(0;t) + \eta dt\, p(1;t) \qquad (398)$$

- $k=N$ (all lines are occupied)

$$p(N;t+dt) = (1-\lambda dt - N\eta dt)\, p(N;t) + \lambda dt\, p(N-1;t) \qquad (399)$$

Summary:

- we have $N+1$ transition equations of type $p(k;t{+}dt) = (1{-}\lambda dt{-}k\eta dt)\,p(k;t){+}\lambda dt\,p(k{-}1;t){+}(k{+}1)\eta dt\,p(k{+}1;t)$
- $p(k;dt)$ and $p(k;t+dt)$ probabilities are unknown: $2(N+1)$ unknown values

$\Rightarrow$ we need to reduce the number of unknown values.

At steady state, it is assumed that the *probabilities are not function of time* (reasonable assumption of [strict!] time stationarity):

$$p(k;t+dt) = p(k;t) = P_k, \quad k = 0, 1, 2, ..., N \tag{400}$$

The transition equations then become

$$\begin{aligned}
P_k &= (1-\lambda dt - k\eta dt)\,P_k + \lambda dt\,P_{k-1} + (k+1)\eta dt\,P_{k+1} && (401)\\
\Rightarrow \quad 0 &= [(-\lambda - k\eta)\,P_k + \lambda\,P_{k-1} + (k+1)\eta\,P_{k+1}]\;dt && (402)\\
\Rightarrow \quad (\lambda + k\eta)\,P_k &= \lambda\,P_{k-1} + (k+1)\eta\,P_{k+1}, \quad 0 < k < N && (403)
\end{aligned}$$

At steady state II

Likewise, we have, for the two particular cases:

$$\lambda P_0 = \eta P_1, \quad k = 0 \tag{404}$$

$$(\lambda + N\eta) P_N = \lambda P_{N-1}, \quad k = N \tag{405}$$

Intermediate conclusion: *we have $N+1$ unknown probabilities and $N+1$ equations. But these are relative equations (all probabilities can be derived up to a scale factor).*

[Scaling condition] Because the number of occupied lines is restricted to $0 \leq k \leq N$, the probabilities P_k also must satisfy

$$P_0 + P_1 + ... + P_N = 1 \tag{406}$$

The form of the probability P_k that satisfies all the conditions is (for all k)

$$P_k = \frac{\frac{(\lambda/\eta)^k}{k!}}{\sum_{i=0}^{N} \frac{(\lambda/\eta)^i}{i!}} \tag{407}$$

This is the probability to have k occupied lines.

Erlang B formula I

Erlang B formula

The state of having all lines occupied is referred to as *congestion*. If calls are rejected when all N lines are occupied, then P_N $(k = N)$ is the probability that a call is *rejected* or *blocked:*

$$B = P_N = \frac{\frac{(\lambda/\eta)^N}{N!}}{\sum_{i=0}^{N} \frac{(\lambda/\eta)^i}{i!}} \tag{408}$$

This is known as the Erlang *B formula.*

If (1) K is a random variable representing the number of occupied lines in a N-lines trunk, and (2) P_k is the probability to have k occupied lines of the trunk, then the carried load A_c is given by the expectation of K:

$$A_c = E\{K\} = \sum_{k=0}^{N} k\, P_k \tag{409}$$

Erlang B formula II

Theorem (Expectation of K?)

The expectation (mean/average) of the variable K provides the average number of occupied lines; this is the carried load A_c. It is equal to

$$A_c = E\{K\} = A(1 - B) \qquad (410)$$

where A is the offered load.

Erlang B formula III

Proof.

Let $\beta = \lambda/\eta$. The expectation (mean) is defined as

$$E\{K\} = \sum_{k=0}^{N} k\,P_k = \frac{\sum_{k=0}^{N} k\,\frac{\beta^k}{k!}}{\sum_{i=0}^{N}\frac{\beta^i}{i!}} = \frac{\sum_{k=1}^{N} k\,\frac{\beta^k}{k!}}{\sum_{i=0}^{N}\frac{\beta^i}{i!}} \tag{411}$$

$$= \frac{\sum_{k=1}^{N} k\,\frac{\beta^k}{k!}}{\sum_{i=0}^{N}\frac{\beta^i}{i!}} = \frac{\sum_{l=0}^{N-1}\frac{\beta^{l+1}}{l!}}{\sum_{i=0}^{N}\frac{\beta^i}{i!}} \tag{412}$$

$$= \beta \sum_{l=0}^{N-1} \frac{\frac{\beta^l}{l!}}{\sum_{i=0}^{N}\frac{\beta^i}{i!}} = \frac{\lambda}{\eta}(1-B) \tag{413}$$

$\square$

Therefore also, $\frac{\lambda}{\eta}$ defines the offered load A:

$$A = \frac{\lambda}{\eta} \quad \text{and} \quad A_c = A(1-B) \tag{414}$$

Engineering

In the expression of the Erlang B statistic,

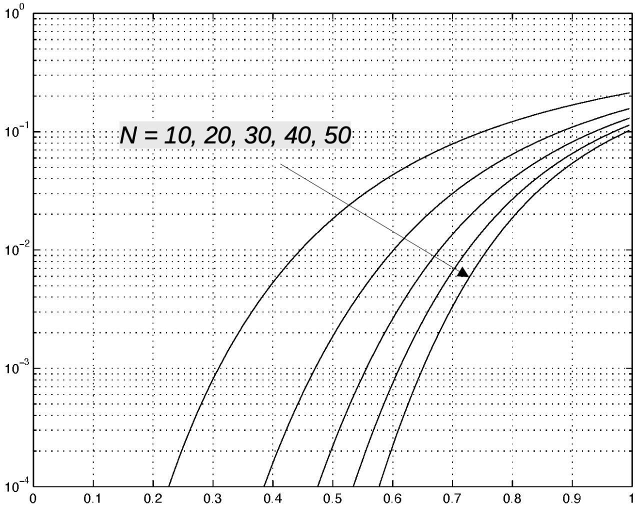
$$B = P_N = \frac{\frac{A^N}{N!}}{\sum_{i=0}^{N} \frac{A^i}{i!}} \tag{415}$$

we have three parameters:

- the *blocking probability* B
- the *size* of the trunk N
- the *offered load* A

$\Rightarrow$ if we set two parameters, we can calculate the third one. But this expression is not easily inverted.
In practice, engineers use tables with values or graphics.

Interpreting Erlang B probability law I



Blocking probability B as a function of the offered load per line (A/N)

Interpreting Erlang B probability law II

N	10	20	30	40	50	60	70	80	90	100
A	4.5	12.0	20.3	29.0	37.9	46.9	56.1	65.4	74.7	84.1
$\frac{A}{N}$	0.45	0.60	0.68	0.73	0.76	0.78	0.80	0.82	0.83	0.84

Table: Example: proportionality of A/N to N for $B = 0.01$.

Sizing the capacity

	B					B			
N	0.01	0.005	0.003	0.001	N	0.01	0.005	0.003	0.001
1	0.01	0.005	0.003	0.001	31	21.2	19.9	19.0	17.4
2	0.153	0.105	0.081	0.046	32	22.0	20.7	19.8	18.2
3	0.46	0.35	0.29	0.19	33	22.9	21.5	20.6	19.0
4	0.87	0.7	0.6	0.44	34	23.8	22.3	21.4	19.7
5	1.4	1.1	1.0	0.8	35	24.6	23.2	22.2	20.5
6	1.9	1.6	1.4	1.1	36	25.5	24.0	23.1	21.3
7	2.5	2.2	1.9	1.6	37	26.4	24.8	23.9	22.1
8	3.1	2.7	2.5	2.1	38	27.3	25.7	24.7	22.9
9	3.8	3.3	3.1	2.6	39	28.1	26.5	25.5	23.7
10	4.5	4.0	3.6	3.1	40	29.0	27.4	26.3	24.4
11	5.2	4.6	4.3	3.7	41	29.9	28.2	27.2	25.2
12	5.9	5.3	4.9	4.2	42	30.8	29.1	28.0	26.0
...	...	...	...	...	...	...	...	...	...

Table: Erlang B: offered load, given B and N.

Alternative models I

Assumptions for building the Erlang B model

- stationarity.
- rejected calls are not re-submitted.
- the blocking probability is given by P_N.

Types of loads:

1. carried load $A_c = E\{k\} = A(1 - B)$
2. offered load $A = \frac{\lambda}{\eta}$
3. what if rejected calls are returning? Then the offered load A' is larger than A, because users return until they are served. Therefore,

$$A' \geq A \geq A_c \tag{416}$$

How do we determine A'?

Trials and calls returning.
Until now, we have ignored what happens when a call has been blocked.

Let A' be the estimated offered load that considers the effects of blocked calls returning and being accepted:

- A is the original load (1st attempt)
- AB is the load that returns to the trunk after calls have been blocked (2nd attempt)
- AB^2 results from 3rd attempts
- ...

Therefore, A' can be expressed as

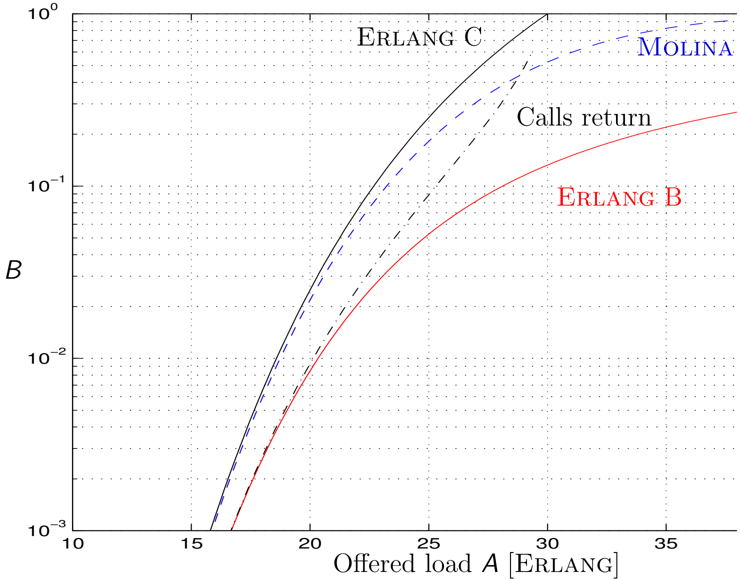
$$A' = A + AB + (AB)B + (AB^2)B + \ldots = \frac{A}{1-B} \tag{417}$$

Note that $A' > A$.

Alternative blocking formulas

Blocking analysis	Treatment of lost calls	Blocking formula
Formulas for a large (infinite) number of traffic sources		
Erlang B	Calls "cleared"; no recur	$B_1 = \frac{P_N}{\sum_{k=0}^{N} P_k}$ with $P_k = \frac{A^k}{k!}$
Lost calls return	Calls reenter until served	Given B, effective load is $A' = A/(1-B)$
Erlang C	Lost calls "held" in infinite queue	$B = \frac{B_1}{\left[1 - \frac{A(1-B_1)}{N}\right]}$
Molina	Same as Erlang C	$B = 1 - e^{-A} \sum_{k=0}^{N-1} P_k = e^{-A} \sum_{k=N}^{\infty} P_k$
Formulas for finite number of traffic sources, M		
Engest	Lost calls are cleared	$B_2(\rho) = \frac{p_N}{\sum_{k=0}^{N} p_k}$, $p_k = \binom{M}{k} \rho^k$ and $A(\rho) \simeq \frac{M\rho}{[1+\rho B_2(\rho)]}$
Bernoulli	Lost calls held	$B = \sum_{k=N}^{M} \binom{M}{k} \left(\frac{A}{M}\right)^k \left(1 - \frac{A}{M}\right)^{M-k}$

Comparison of blocking probability formulas



Outline

- Transmission over copper lines
  - Electrical properties
  - Model
  - Spectral study
  - Examples of lines
- Telephone network structure
- Crosstalk and high bitrate transmission
  - Principles
  - Crosstalk study
  - NEXT, FEXT, signal to noise ratio
- Estimation of the channel capacity
  - Information, uncertainty and entropy
  - Memoryless discrete channel
  - Mutual information
  - Channel capacity

How will we proceed?

1. Understand the principles of transmission over copper lines
2. Establish a model for dealing with disturbers (other users on neighboring copper wires)
3. Estimate the efficiency of transmission over copper lines $\Rightarrow$ notion of *channel capacity*

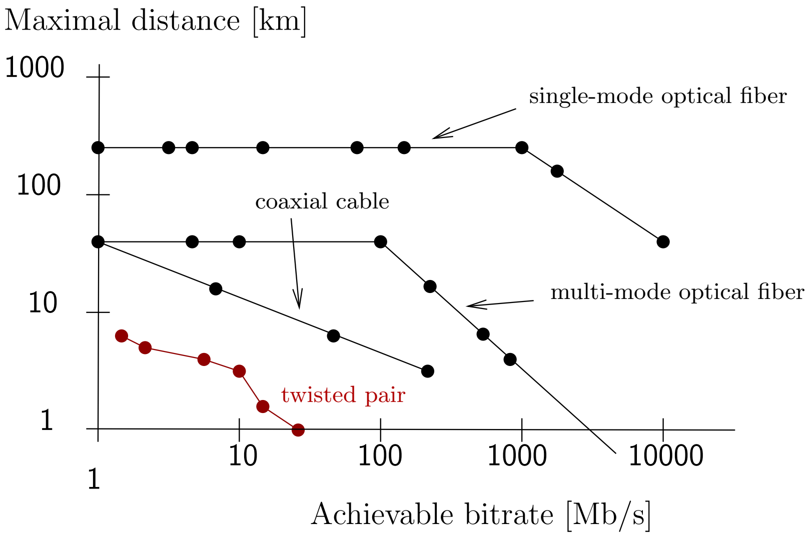


Figure: Comparison of different transmission media.

Factor of limitations

Several phenomenons affect the transmission over copper lines:

Signal [S] attenuation: Attenuation results in a decrease of the power along the copper line. It is commonly expressed $[dB]$ per kilometer. It will be expressed by the α line attenuation constant.

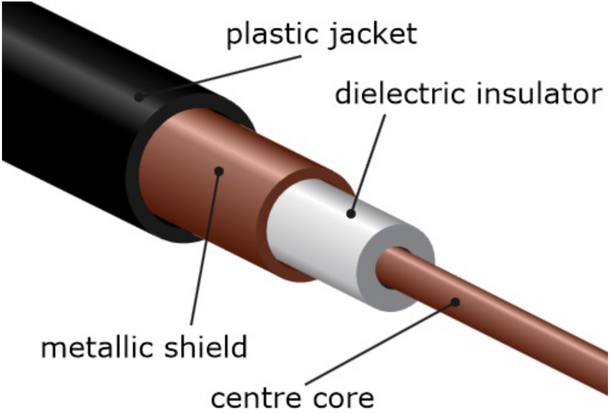
Noise [N]: Noise originates from electromagnetic disturbances: environment + other users (named *disturbers*)

Non-linearities: Here, we will assume that the channel (made of copper wires) is linear to a large extend.

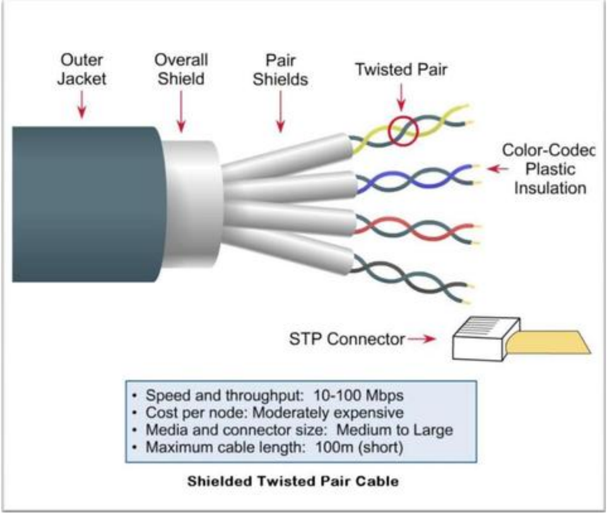
Cables

There are many types of cables, but basically two families:

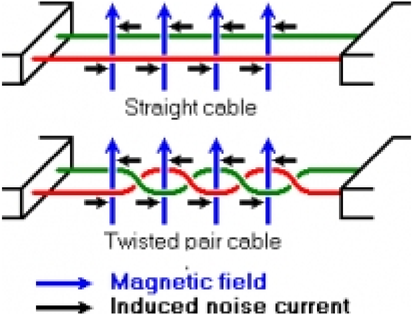
- coaxial cables (mainly used by cable TV operators)
- cables of twisted pairs (used for network and telephone lines)



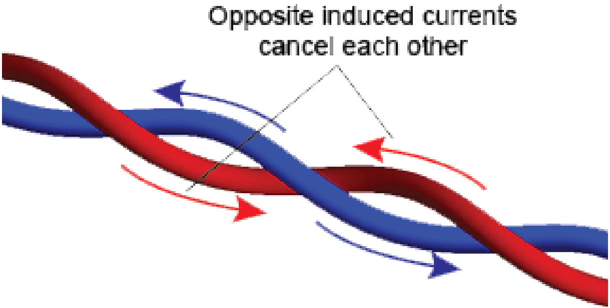
Cables of twisted pairs



A uniform (but time-varying) magnetic field induces a current in a loop of two wires



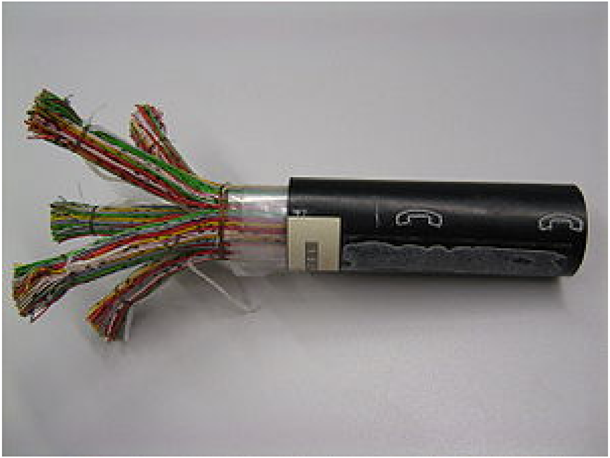
Why do we twist pairs? II



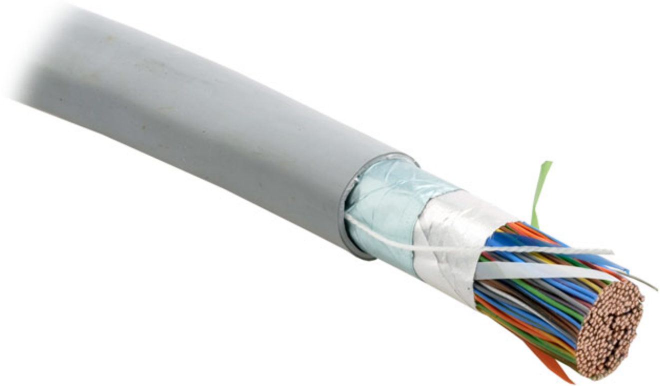
Conclusion: twisting pairs make them more resilient to electro-magnetic noise.

Organization of cables I

Cables might be organized in *binder groups*.



Organization of cables II

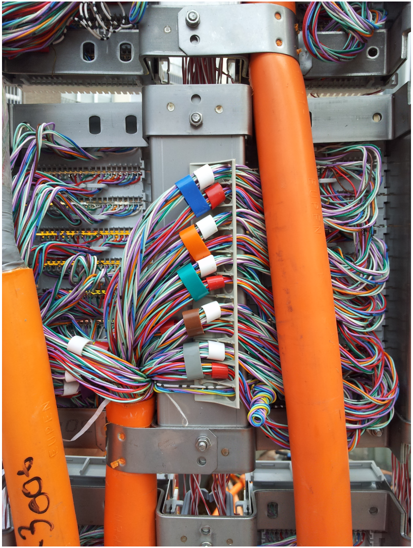


Organization of cables III

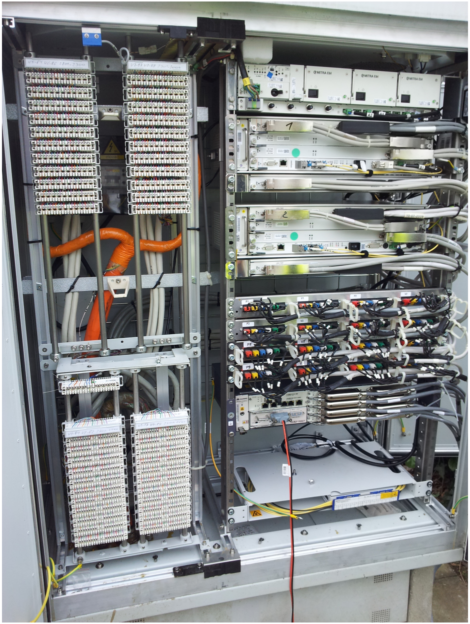
Street cabinets (Belgium) I

Street cabinets (Belgium) II

At the other side:



Local Distribution Center (LDC)




Autres
Directions
N635
Quartier Urbanistes 2
Quartier Urbanistes 1
FASSI

Electrical model of a line and primary line constants (reminder)

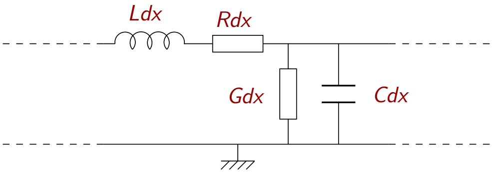
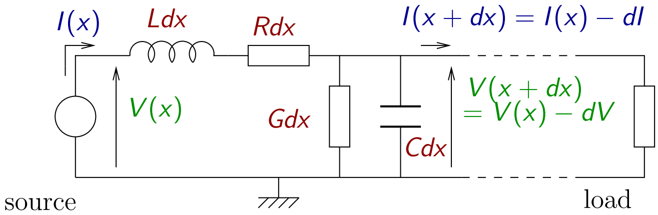


Figure: Infinitesimal section of a copper line.

Name	Symbol	Units	Unit symbol
loop resistance	R	ohms per meter	Ω/m
loop inductance	L	henries per meter	H/m
insulator capacitance	C	farads per meter	F/m
insulator conductance	$G \simeq 0$	siemens per meter	S/m



By taking a small section:

$$\frac{\partial V}{\partial x} = -RI - L\frac{\partial I}{\partial t} \quad (418)$$

$$-\frac{\partial I}{\partial x} = GV + C\frac{\partial V}{\partial t} \quad (419)$$

Solution to these equations

$$\frac{\partial^2 V}{\partial x^2} = RGV + (RC + LG)\frac{\partial V}{\partial t} + LC\frac{\partial V^2}{\partial t^2} \qquad (420)$$

At steady state

At steady state, we express voltages and currents with phasors $V(x, t) = V(x)e^{j\omega t}$. Then, the equation becomes

$$\frac{\partial^2 V}{\partial x^2} = (R + jL\omega)(G + jC\omega)V(x) = \gamma^2 V(x) \qquad (421)$$

with $\gamma = \alpha + j\beta$.
We then obtain

$$V(x) = V_F e^{-\gamma x} + V_B e^{\gamma x} \qquad (422)$$

- $V_F e^{-\gamma x}$: *forwards* propagating wave
- $V_B e^{\gamma x}$: backwards propagating wave

Primary line constants aim at describing the line behavior at the "microscopic" scale. Lines are easier to characterize by their *secondary* line constants:

- the *characteristic impedance* Z_c.
  It is defined as the impedance looking into an infinitely long line. Such a line will never return a reflection since the incident wave will never reach the end to be reflected.
- the *propagation constant*, γ, whose real and imaginary parts are the attenuation constant, α, and phase change constant, β, respectively:

$$\gamma = \alpha + j\beta \tag{423}$$

Relationship between primary and secondary line constants

Primary and secondary line constants are related by the following equations:

$$Z_c = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \tag{424}$$

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \tag{425}$$

If we admit that $G \approx 0$,

$$Z_c \simeq \sqrt{\frac{R + j\omega L}{j\omega C}} \tag{426}$$

$$\gamma \simeq \sqrt{(R + j\omega L)j\omega C} \tag{427}$$

Spectral dependence of parameters

At high frequencies ($\omega = 2\pi f$) and taking $\sqrt{1+\alpha} \approx 1 + \frac{\alpha}{2}$,

$$
\begin{aligned}
\gamma &\simeq \sqrt{(R + j\omega L) j\omega C} && (428)\\
&\simeq \sqrt{j\omega C}\sqrt{j\omega L}\sqrt{1 + \frac{R}{j\omega L}} && (429)\\
&\simeq \sqrt{j\omega C}\sqrt{j\omega L}\left(1 + \frac{R}{j2\omega L}\right) && (430)\\
&\simeq \frac{1}{2} R \sqrt{\frac{C}{L}} + j\omega\sqrt{LC} = \alpha + j\beta && (431)
\end{aligned}
$$

In the following, the take the following approximations:

- C and L are independent of the frequency.
- due to the skin effect,
  - R is proportional to $\sqrt{f}$: $R = R_0\sqrt{f}$
  - and consequently α is also proportional to $\sqrt{f}$: $\alpha = \alpha_0\sqrt{f}$

Attenuation

If $V_{in}(f, 0)$ is the input voltage ($x = 0$), then

$$V(f, L) = V_{in}(f, 0)e^{-\gamma(f)L} \tag{432}$$

is the voltage at location L.

Definition

The attenuation is defined as

$$A = \left|\frac{V_{in}(f, 0)}{V(f, L)}\right| = \left|e^{\gamma(f)L}\right| = e^{\alpha(f)L} \tag{433}$$

In decibels and considering that $\alpha = \alpha_0\sqrt{f}$, we have

$$A\,[\mathrm{dB}] = 20\,\log_{10}\left[e^{\alpha_0\sqrt{f}L}\right] = \frac{20\,\ln\left[e^{\alpha_0\sqrt{f}L}\right]}{\ln 10} = A_0\sqrt{f}L\,[\mathrm{dB}] \tag{434}$$

This means that doubling the length doubles the losses in decibels.

Twisted pair

Frequency	$R\,[\Omega/\text{km}]$	$L\,[\mu H/\text{km}]$	$\lvert Z_c \rvert\ [\Omega]$	$\alpha\,[mNp/\text{km}]$
10 [kHz]	52,3	766	188	151
120 [kHz]	98,7	67,5	156	363

Table: Sample values of lines encountered in a telephone network (note that a neper [Np] is approximately 8.7 [dB])

Structure of a telephone network

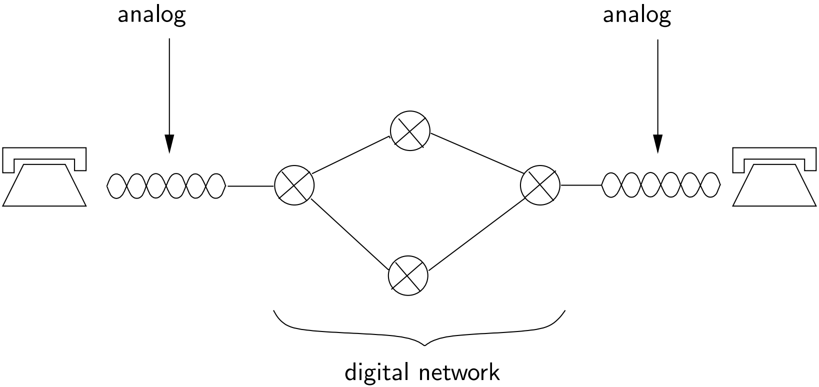


Figure: Transmission of analog voice signals over the telephone network (so called "analog" lines).

Bandwidth for voice communications: [300 Hz, 3400 Hz]

Digital transmission by using modems over a telephone network

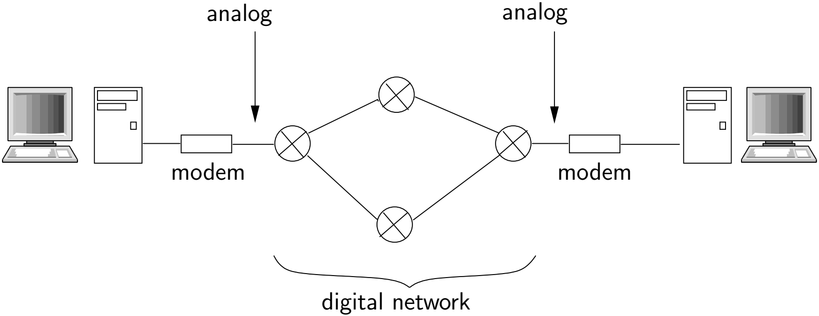


Figure: Digital information is modulated and sent transparently in the [300 Hz, 3400 Hz] bandwidth.

The entry level interface to ISDN is the Basic(s) Rate Interface (BRI), a 2×64 [kb/s] service delivered over a pair of standard telephone copper wires (these channels are identical to the E1 channels).

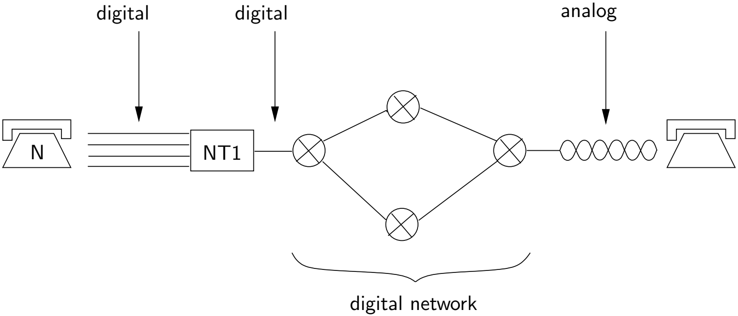


Figure: Integrated Services Digital Network (ISDN).

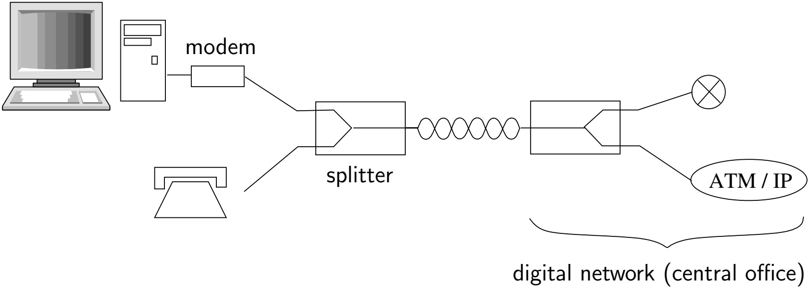
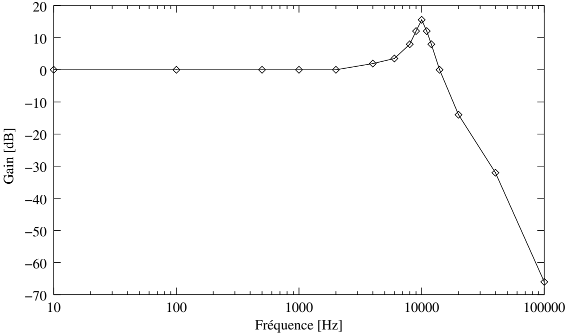


Figure: Configuration of an Asymmetric DSL (ADSL) line.

Cutoff filter (in the splitter)



Spectral occupancy

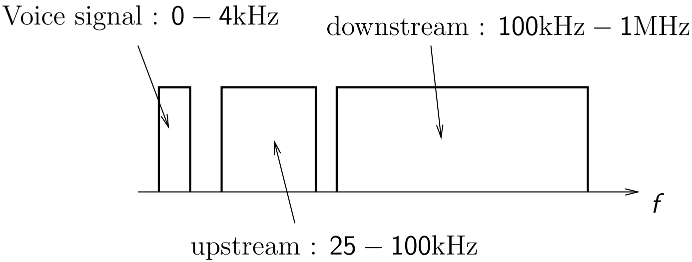


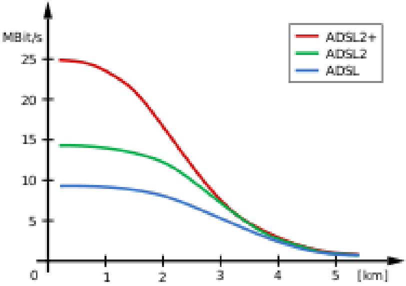
Figure: Spectral occupancy of an ADSL line.

- Lower frequencies are dedicated to the voice and upstream bitrate.

DSL standards I

Technologies for asymmetric bit streams (ADSL):

Version	Standard name	Common name	Downstream	Upstream	Approved in
ADSL	ITU G.992.1	ADSL (G.dmt)	8 [Mb/s]	1.3 [Mb/s]	1999-07
ADSL2	ITU G.992.3	ADSL2	12 [Mb/s]	1.3 [Mb/s]	2002-07
ADSL2+	ITU G.992.5 Annex M	ADSL2+M	24 [Mb/s]	3.3 [Mb/s]	2008



Other DSL technologies:

- High-bit-rate digital subscriber line (HDSL) is a telecommunications protocol standardized in 1994. It was the first digital subscriber line (DSL) technology to use a higher frequency spectrum of copper, twisted pair cables.
- Very-high-bit-rate digital subscriber line (VDSL or VHDSL): up to 52 [Mb/s] downstream and 16 [Mb/s] upstream using the frequency band from 25 [kHz] to 12 [MHz].
  VDSL2 is even faster.

Real cables I

Real cables II

Real cables III

Principle

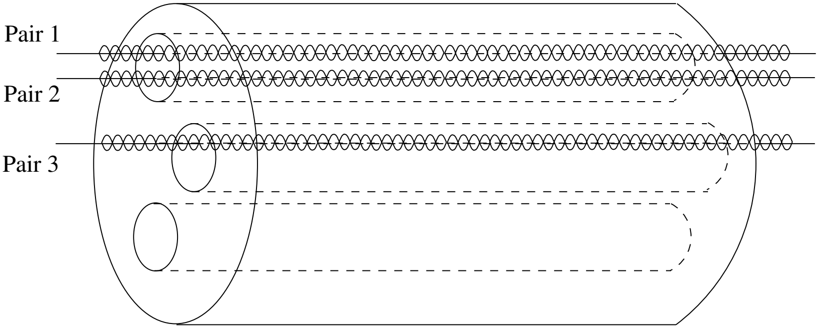
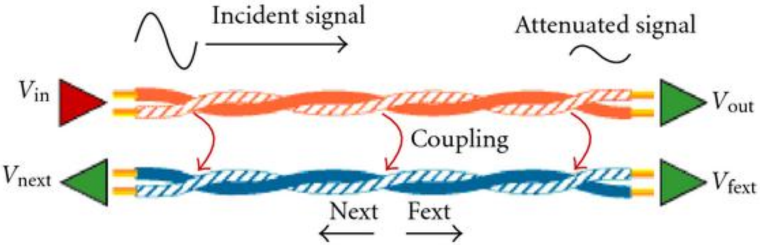


Figure: Crosstalk originates from the proximity of neighboring copper wires.

Each input signal generates **3 output** signals



Types of crosstalk in telephone networks I

1. *NEXT* (*Near-End Crosstalk*): disturbing pair's source is at the same side
2. *FEXT* (*Far-End Crosstalk*)

NEXT is usually more damaging than FEXT.

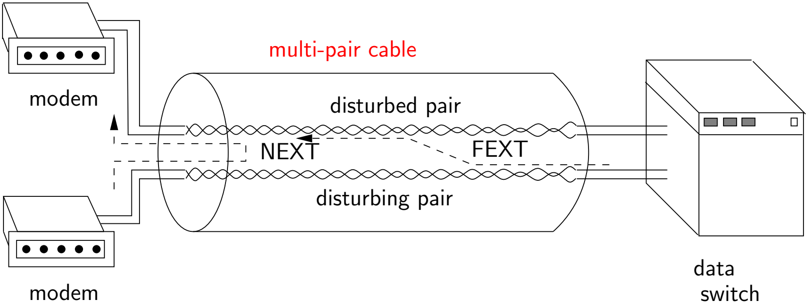


Figure: Two types of crosstalk. We examine NEXT and FEXT at one side.

For two neighboring twisted pairs, capacitive and inductive components exist between each of the four wires and also between each wire and ground. Therefore, we consider two models:

1. capacitive *unbalance* model
2. inductive *unbalance* model

We will see that they produce similar results.

Capacitive *unbalance* model I

Framework:

- we consider 2 pairs, or 4 wires
- there are 8 unknown quantities: V_1, V_2, V_3, V_4, I_1, I_2, I_3, I_4 (voltages are defined with respect to the ground reference)
- how many circuits do we have?
- how many circuits do we use in practice?
  - differential (metallic) modes are easier to use. They are defined by $V_1 - V_2$, $V_3 - V_4$, $I_1 - I_2$, and $I_3 - I_4$.
  - so we use 2 circuits only.

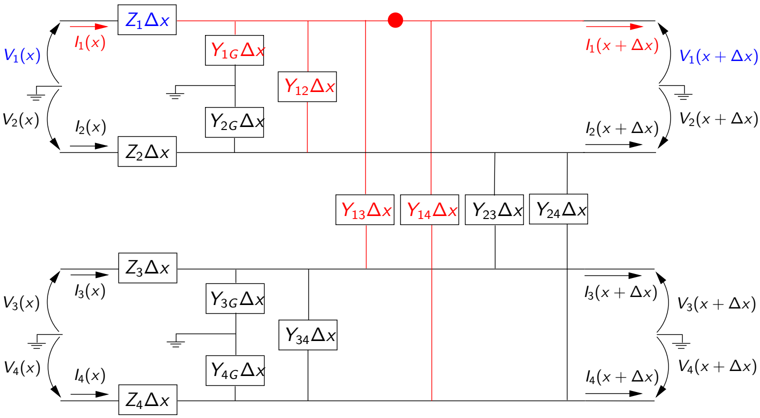


Figure: Capacitive model of a short section of two twisted pairs.

Capacitive *unbalance* model III

For convenience:

- we write all capacitances as admittances: $Y = j\omega C$ (we assume that $G = 0$).
- All voltages are defined with respect to the ground reference.

According to Kirchhoff's mesh rule (on voltages):

$$
\begin{aligned}
V_1(x + \triangle x) &= V_1(x) - I_1(x) Z_1 \triangle x && (435)\\
V_2(x + \triangle x) &= V_2(x) - I_2(x) Z_2 \triangle x && (436)\\
V_3(x + \triangle x) &= V_3(x) - I_3(x) Z_3 \triangle x && (437)\\
V_4(x + \triangle x) &= V_4(x) - I_4(x) Z_4 \triangle x && (438)
\end{aligned}
$$

Capacitive *unbalance* model IV

Likewise, the current nodes law provides:

$$\begin{aligned}
I_1(x+\triangle x) &= I_1(x) - V_1(x+\triangle x)Y_{1G}\triangle x - [V_1(x+\triangle x) - V_2(x+\triangle x)]Y_{12}\triangle x \\
&\quad -[V_1(x+\triangle x) - V_3(x+\triangle x)]Y_{13}\triangle x - [V_1(x+\triangle x) - V_4(x+\triangle x)]Y_{14}\triangle x \\
I_2(x+\triangle x) &= I_2(x) - V_2(x+\triangle x)Y_{2G}\triangle x - [V_2(x+\triangle x) - V_1(x+\triangle x)]Y_{12}\triangle x \\
&\quad -[V_2(x+\triangle x) - V_3(x+\triangle x)]Y_{23}\triangle x - [V_2(x+\triangle x) - V_4(x+\triangle x)]Y_{24}\triangle x \\
I_3(x+\triangle x) &= I_3(x) - V_3(x+\triangle x)Y_{3G}\triangle x - [V_3(x+\triangle x) - V_1(x+\triangle x)]Y_{13}\triangle x \\
&\quad -[V_3(x+\triangle x) - V_2(x+\triangle x)]Y_{23}\triangle x - [V_3(x+\triangle x) - V_4(x+\triangle x)]Y_{34}\triangle x \\
I_4(x+\triangle x) &= I_4(x) - V_4(x+\triangle x)Y_{4G}\triangle x - [V_4(x+\triangle x) - V_1(x+\triangle x)]Y_{13}\triangle x \\
&\quad -[V_4(x+\triangle x) - V_2(x+\triangle x)]Y_{24}\triangle x - [V_4(x+\triangle x) - V_3(x+\triangle x)]Y_{34}\triangle x
\end{aligned}$$

We have:

- 8 equations
- but 16 unknown values?!
  $V_j(x)$, $V_j(x+\triangle x)$, $I_j(x)$, $I_j(x+\triangle x)$, with $j \in \{1, 2, 3, 4\}$.

Therefore, in the equations, we divide both members by $\triangle x$ and take the limit for $\triangle x \to 0$.

For example,

$$V_1(x + \triangle x) = V_1(x) - I_1(x) Z_1 \triangle x \tag{439}$$

becomes

$$\lim_{\triangle x \to 0} \frac{V_1(x + \triangle x) - V_1(x)}{\triangle x} = -Z_1 I_1(x) \tag{440}$$

$$\frac{d}{dx} V_1(x) = -Z_1 I_1(x) \tag{441}$$

Note that $V_1(x)$ depends on a current.
Likewise it appears that currents depend on voltages.
$\Rightarrow$ equations are coupled.

In a matrix form, the 8 equations can be expressed as

$$\frac{d}{dx}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & -Z_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -Z_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -Z_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -Z_4 \\ A_1 & Y_{12} & Y_{13} & Y_{14} & 0 & 0 & 0 & 0 \\ Y_{12} & A_2 & Y_{23} & Y_{24} & 0 & 0 & 0 & 0 \\ Y_{13} & Y_{23} & A_3 & Y_{34} & 0 & 0 & 0 & 0 \\ Y_{14} & Y_{24} & Y_{34} & A_4 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \underline{A}\,\vec{S} \tag{442}$$

with

$$A_1 = -(Y_{1G} + Y_{12} + Y_{13} + Y_{14}) \tag{443}$$
$$A_2 = -(Y_{2G} + Y_{12} + Y_{23} + Y_{24}) \tag{444}$$
$$A_3 = -(Y_{3G} + Y_{13} + Y_{23} + Y_{34}) \tag{445}$$
$$A_4 = -(Y_{4G} + Y_{14} + Y_{24} + Y_{34}) \tag{446}$$

Capacitive *unbalance* model VII

Change of variables (we transform the voltages and currents on the four lines to *metallic* [$\equiv$*differential*] and *longitudinal* voltages and currents): $\overrightarrow{S} \longrightarrow \overrightarrow{S}'$

$$V_{1M} = V_1 - V_2 \tag{447}$$
$$V_{2M} = V_3 - V_4 \tag{448}$$
$$V_{1L} = \frac{V_1 + V_2}{2} \tag{449}$$
$$V_{2L} = \frac{V_3 + V_4}{2} \tag{450}$$
$$I_{1M} = \frac{I_1 - I_2}{2} \tag{451}$$
$$I_{2M} = \frac{I_3 - I_4}{2} \tag{452}$$
$$I_{1L} = I_1 + I_2 \tag{453}$$
$$I_{2L} = I_3 + I_4 \tag{454}$$

In a matrix form:

$$\vec{S} = \underline{T} \begin{bmatrix} V_{1M} \\ V_{1L} \\ V_{2M} \\ V_{2L} \\ I_{1M} \\ I_{1L} \\ I_{2M} \\ I_{2L} \end{bmatrix} = \underline{T}\,\vec{S}\,' \tag{455}$$

where T is a transformation matrix:

$$\underline{T} = \begin{bmatrix} \frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & \frac{1}{2} \end{bmatrix} \tag{456}$$

So that,

$$\vec{S'} = \underline{T}^{-1}\vec{S} \tag{457}$$

Capacitive *unbalance* model IX

This can be written as

$$\frac{d}{dx}\overrightarrow{S} = \underline{A}\,\overrightarrow{S} \tag{458}$$

$$\frac{d}{dx}\underline{T}^{-1}\overrightarrow{S} = \underline{T}^{-1}\underline{A}\,\overrightarrow{S} \tag{459}$$

$$\frac{d}{dx}\overrightarrow{S}' = \underline{T}^{-1}\underline{A\,T}\,\overrightarrow{S'} \tag{460}$$

This results in currents depending only on the metallic and longitudinal voltages.

First solution: the capacitive model I

By replacing Y by $j\omega C$ (only capacitive effects are considered),

$$\frac{d}{dx}\begin{bmatrix} I_{1M} \\ I_{1L} \\ I_{2M} \\ I_{2L} \end{bmatrix} = -\frac{j\omega}{4}\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}\begin{bmatrix} V_{1M} \\ V_{1L} \\ V_{2M} \\ V_{2L} \end{bmatrix} \tag{461}$$

with

$$a_{11} = C_{1G} + C_{2G} + 4C_{12} + C_{13} + C_{14} + C_{23} + C_{24} \tag{462}$$
$$a_{21} = a_{12} = 2C_{1G} - 2C_{2G} + 2C_{13} + 2C_{14} - 2C_{23} - 2C_{24} \tag{463}$$
$$a_{31} = a_{13} = -C_{13} + C_{14} + C_{23} - C_{24} \tag{464}$$
$$a_{41} = a_{14} = -2C_{13} - 2C_{14} + 2C_{23} + 2C_{34} \tag{465}$$
$$a_{22} = 4C_{1G} + 4C_{2G} + 4C_{13} + 4C_{14} + 4C_{23} + 4C_{24} \tag{466}$$
$$a_{23} = a_{32} = -2C_{13} + 2C_{14} - 2C_{23} + 2C_{34} \tag{467}$$
$$a_{24} = a_{42} = -4C_{13} - 4C_{14} - 4C_{23} - 4C_{24} \tag{468}$$
$$a_{33} = C_{3G} + C_{4G} + C_{13} + C_{14} + C_{23} + C_{24} + 4C_{34} \tag{469}$$
$$a_{34} = a_{43} = 2C_{3G} - 2C_{4G} + 2C_{13} - 2C_{14} + 2C_{23} - 2C_{24} \tag{470}$$
$$a_{44} = 4C_{3G} + 4C_{4G} + 4C_{13} + 4C_{14} + 4C_{23} + 4C_{24} \tag{471}$$

First solution: the capacitive model II

The parameter a_{31} defines the coupling between the metallic (differential) voltage in the disturbing pair V_{1M} to the metallic (differential) current in the disturbed pair I_{2M}; it is referred to as capacitance unbalance of a twisted pair:

$$\frac{d}{dx} I_{2M} = -\frac{j\omega}{4} C_{M_1 M_2} V_{1M} \tag{472}$$

where $C_{M_1 M_2}$ is equal to a_{31} (or a_{13}).

Inductive unbalance model I

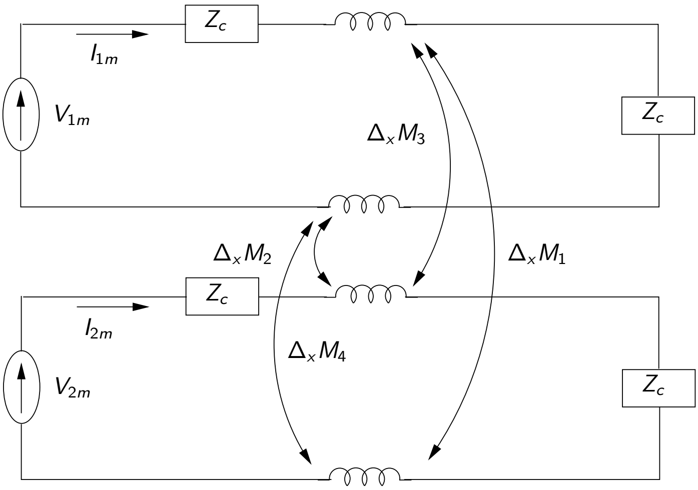


Figure: Mutual inductance model of a short section of two twisted pairs.

Inductive unbalance model II

A detailed analysis leads to

$$\frac{d}{dx} I_{2M} \simeq \frac{j\omega M}{4Z_c^2} V_{1M} \tag{473}$$

where M equals $M_1 + M_2 + M_3 + M_4$.

Both models provide expressions that can be combined to

$$\frac{d}{dx} I_{2M} = \left(\frac{j\omega M}{4Z_c^2} - \frac{j\omega}{4} C_{M_1 M_2} \right) V_{1M} \tag{474}$$

that can be grouped together to form a new unbalanced constant

$$\frac{d}{dx} I_{2M}(x) = j\omega Q_{M_1 M_2}(x) V_{1M}(x) \tag{475}$$

where

- $Q_{M_1 M_2}(x)$ takes into account capacitive and inductive effects.
- (x) emphasizes the dependence with the location along the line.

Near-end crosstalk (NEXT) calculation I

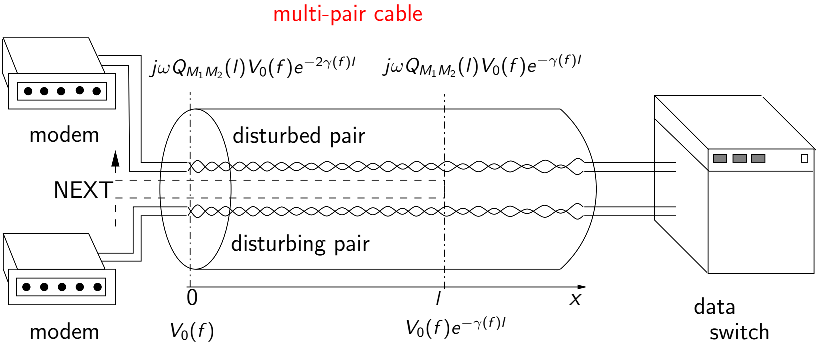


Figure: NEXT at location $x = l$.

Near-end crosstalk (NEXT) calculation II

1. At the input of the disturbing line, the voltage is $V(f, x = 0) = V_0(f)$.
2. At location $x = l$, the voltage along the disturbing pair is equal to ($\rightarrow$ propagation)

$$V_1(f, x = l) = V_0(f)\, e^{-\gamma(f) l} \tag{476}$$

3. At location $x = l$, the induced current on the disturbed pair is

$$\frac{d}{dx} I_2(f, x = l) = j\omega Q_{M_1 M_2}(l)\, V_1(f, x = l) = j\omega Q_{M_1 M_2}(l)\, V_0(f)\, e^{-\gamma(f) l} \tag{477}$$

4. This current propagates back to the input ($x = 0$) of the disturbed line ($\leftarrow$ propagation):

$$I_2(f, x = 0) = I_2(f, x = l)\, e^{\gamma(f)(-l)} = I_2(f, x = l)\, e^{-\gamma(f) l} \tag{478}$$

so that

$$\frac{d}{dx} I_2(f, x = 0) = j\omega Q_{M_1 M_2}(l)\, V_0(f)\, e^{-2\gamma(f) l} \tag{479}$$

Derivation of the NEXT power transfer function: integration along the line l

We have

$$I_2(f) = \int_0^L j\omega Q_{M_1 M_2}(l) V_0(f) e^{-2\gamma(f)l} dl \tag{480}$$

Power?

The power due to the disturbing pair is given by

$$P_2(f) = V_2(f) I_2^*(f) = Z_L I_2(f) I_2^*(f) \tag{481}$$

Z_L is chosen to:

- maximize the transfer of power (conjugate matching: $Z_L^* = Z_c$) and
- to avoid any reflection ($Z_L = Z_c$), where Z_c is the characteristic impedance of a transmission line.

So that, $Z_L = R_L$. Consequently,

$$P_2(f) = R_L I_2(f) I_2^*(f) \tag{482}$$

Derivation of the NEXT power transfer function: integration along the line II

The problem is that $Q_{M_1 M_2}$ is unknown and it depends on the location $\Rightarrow$ we adopt a probabilistic approach for evaluating the power

$$
\begin{aligned}
E\left\{P_2(f)\right\} &= E\left\{R_L V_0^2(f)\int_0^L j\omega Q_{M_1 M_2}(x) e^{-2\gamma(f)x} dx \int_0^L -j\omega Q_{M_1 M_2}^{*}(y) e^{-2\gamma^{*}(f)y} dy\right\} \\
&= R_L \omega^2 V_0^2(f) E\left\{\int_0^L \int_0^L Q_{M_1 M_2}(x) Q_{M_1 M_2}^{*}(y) e^{-2\gamma(f)x} e^{-2\gamma^{*}(f)y} dxdy\right\} \quad (483) \\
&= R_L \omega^2 V_0^2(f) \int_0^L \int_0^L E\left\{Q_{M_1 M_2}(x) Q_{M_1 M_2}^{*}(y)\right\} e^{-2\gamma(f)x} e^{-2\gamma^{*}(f)y} dxdy \quad (484)
\end{aligned}
$$

How do we estimate $E\left\{Q_{M_1 M_2}(x) Q_{M_1 M_2}^{*}(y)\right\}$?

We *assume* that $Q_{M_1 M_2}(x)$ and $Q^*_{M_1 M_2}(y)$ (at two different locations) are *uncorrelated*, which means that

$$E\left\{Q_{M_1 M_2}(x) Q^*_{M_1 M_2}(y)\right\} = k\delta(x-y) \tag{485}$$

According to this assumption:

$$E\left\{P_2(f)\right\} = R_L \omega^2 V_0^2(f) \int_0^L \left[\int_0^L k\delta(x-y) e^{-2\gamma(f)x} e^{-2\gamma^*(f)y} dy\right] dx$$

Knowing that there is no contribution to crosstalk outside of $[0, L]$ and that $\int_{-\infty}^{+\infty} \delta(x-y) f(y) dy = f(x)$:

$$\begin{aligned}
\int_0^L k\delta(x-y) e^{-2\gamma(f)x} e^{-2\gamma^*(f)y} dy &= \int_{-\infty}^{+\infty} k\delta(x-y) e^{-2\gamma(f)x} e^{-2\gamma^*(f)y} dy \\
&= k e^{-2\gamma(f)x} e^{-2\gamma^*(f)x} \\
&= k e^{-2(\alpha(f)+j\beta(f))x} e^{-2(\alpha(f)-j\beta(f))x} \\
&= k e^{-4\alpha(f)x}
\end{aligned}$$

Therefore,

$$
\begin{aligned}
E\{P_2(f)\} &= R_L\omega^2 V_0^2(f)\int_0^L\left[\int_0^L k\delta(x-y)e^{-2\gamma(f)x}e^{-2\gamma^*(f)y}dy\right]dx \\
&= R_L\omega^2 V_0^2(f)k\int_0^L e^{-4\alpha(f)x}dx \qquad (486)\\
&= R_L\omega^2 V_0^2(f)k\left[\frac{e^{-4\alpha(f)x}}{-4\alpha(f)}\right]_0^L \qquad (487)\\
&= \frac{R_L\omega^2 V_0^2(f)k}{-4\alpha(f)}\left(e^{-4\alpha(f)L}-1\right) \qquad (488)
\end{aligned}
$$

For long lines, $e^{-4\alpha(f)L} \ll 1$ and, consequently,

$$
E\{P_2(f)\} \approx \frac{R_L\omega^2 V_0^2(f)k}{-4\alpha(f)}(-1) = \frac{R_L\omega^2 V_0^2(f)k}{4\alpha(f)} \qquad (489)
$$

Derivation of the NEXT power transfer function III

At high *frequencies* (the ones used for ADSL transmission), $\alpha(f)$ is proportional to $\sqrt{f}$, so that

$$E\left\{P_2(f)\right\} = \frac{R_L (2\pi f)^2 V_0^2(f) k}{4\alpha_0 \sqrt{f}} = \frac{R_L V_0^2(f) \pi^2 k}{\alpha_0} f^{\frac{3}{2}} \tag{490}$$

Thus, for a disturber whose power is given by $V_0^2(f)/R_L$, we have the *NEXT power transfer function* between the disturbing transmitter and the disturbed receiver:

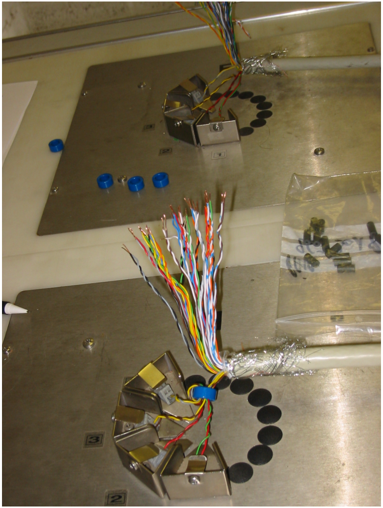
$$H_{NEXT}(f) = \frac{R_L^2 \pi^2 k}{\alpha_0} f^{\frac{3}{2}} = K_{NEXT} f^{\frac{3}{2}} \tag{491}$$

Interpretation:

- the NEXT is proportional to $f^{\frac{3}{2}}$.
- it does not depend on the length. Why? What does it mean?

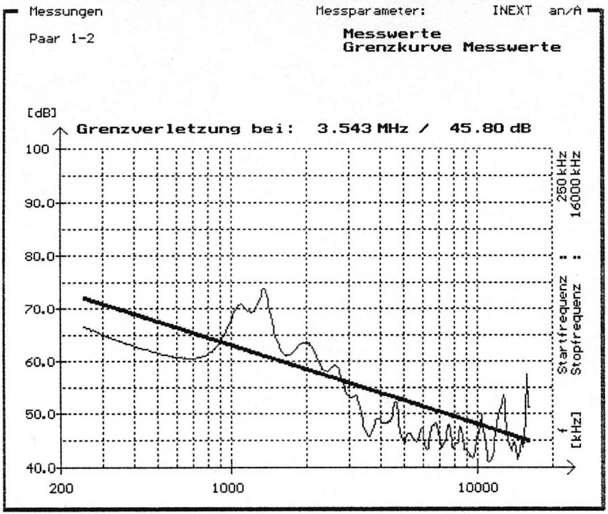
Measured crosstalk values I

Setup:



Measured crosstalk values II

Result: the received power decreases as 15 [dB] per frequency decade



We have:

1. Thanks to the telegrapher's equations, a model for the propagation of waves along an electrical line and a solution:

$$V(x) = V_F e^{-\gamma x} + V_B e^{\gamma x} \quad \text{and} \quad I(x) = I_F e^{-\gamma x} + I_B e^{\gamma x} \tag{492}$$

2. A model and a formula for the calculation of crosstalk (for each infinitesimal section):

$$\frac{d}{dx} I_{2M}(x) = j\omega Q_{M_1 M_2}(x) V_{1M}(x) \tag{493}$$

3. A strategy to derive $E\left\{P_2(f)\right\}$, which includes
   1. Integration of the crosstalk interferences along the line.
   2. An assumption for dealing with $Q_{M_1 M_2}(x)$:

$$E\left\{Q_{M_1 M_2}(x) Q^{*}_{M_1 M_2}(y)\right\} = k\delta(x-y) \tag{494}$$

Calculation of the FEXT I

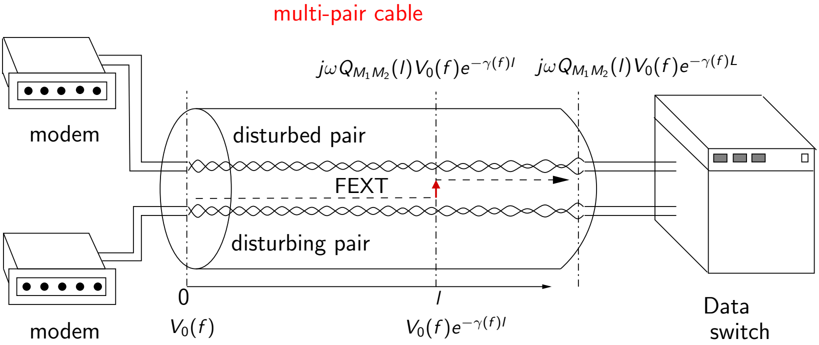


Figure: Far-end crosstalk due to only one section dx (with unbalance) located at $x = l$.

1. At the **input** of the disturbing line, the voltage is $V_0(f)$.
2. At location $x = l$, the voltage on the disturbing pair is equal to ($\rightarrow$ propagation)
$$V_1(f, x = l) = V_0(f)\, e^{-\gamma(f)l} \tag{495}$$
3. At location $x = l$, the induced current on the disturbed pair is
$$\frac{d}{dx} I_2(f, x = l) = j\omega Q_{M_1 M_2}(l)\, V_1(f, x = l) = j\omega Q_{M_1 M_2}(l)\, V_0(f)\, e^{-\gamma(f)l}$$
4. This current propagates to the **output** ($x = L$) of the disturbed line ($\rightarrow$ propagation):
$$I_2(f, x = L) = I_2(f, x = l) e^{-\gamma(f)(L-l)} \tag{496}$$
so that
$$\begin{aligned} \frac{d}{dx} I_2(f, x = L) &= j\omega Q_{M_1 M_2}(l) V_0(f) e^{-\gamma(f)l} e^{-\gamma(f)(L-l)} \\ &= j\omega Q_{M_1 M_2}(l) V_0(f) e^{-\gamma(f)L} \end{aligned} \tag{497}$$

Derivation of the FEXT power transfer function: integration along the line I

We have, at the output of the disturbed line:

$$I_2(f) = \int_0^L j\omega Q_{M_1 M_2}(x) V_0(f) e^{-\gamma(f)L} dx \tag{498}$$

Power?

The power due to the disturbing pair is given by

$$P_2(f) = V_2(f) I_2^*(f) = R_L I_2(f) I_2^*(f) \tag{499}$$

Derivation of the FEXT power transfer function: integration along the line II

By a similar reasoning to that of the NEXT,

$$
\begin{aligned}
E\{P_2(f)\} &= R_L V_0^2(f) E\left\{ \int_0^L j\omega Q_{M_1 M_2}(x) e^{-\gamma(f)L} dx \int_0^L -j\omega Q^*_{M_1 M_2}(y) e^{-\gamma^*(f)L} dy \right\} \\
&= R_L \omega^2 V_0^2(f) e^{-2\alpha(f)L} \int_0^L \int_0^L k\delta(x-y) dx dy \qquad (500) \\
&= R_L \omega^2 V_0^2(f) e^{-2\alpha(f)L} \int_0^L k \, dy \qquad (501) \\
&= R_L (2\pi f)^2 V_0^2(f) e^{-2\alpha(f)L} k L \qquad (502)
\end{aligned}
$$

So, we get the *FEXT power transfer function*

$$
H_{FEXT}(f) = \frac{E\{P_2(f)\}}{V_0^2(f)/R_L} = k_{FEXT} f^2 e^{-2\alpha(f)L} L \qquad (503)
$$

where L is the length of the cable with some FEXT effects.

Derivation of the FEXT power transfer function: integration along the line III

Interpretation:

$$H_{FEXT}(f) = k_{FEXT} f^2 e^{-2\alpha(f)L} L \quad (504)$$

- it depends on the length L.
- it increases with the frequency (similar effect to that of NEXT).
- $e^{-2\alpha(f)L} = \left(e^{-\alpha(f)L}\right)^2$ is the power attenuation for any signal along the line. It is the usual input-output power transfer function or channel power transfer function $\|\mathcal{H}_c(f)\|^2$.

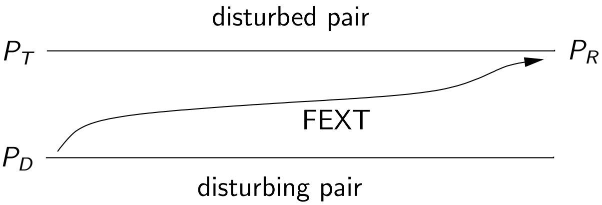


Figure: Calculation of the power spectral density at the receiver with the presence of FEXT.

In the case of FEXT, by applying Wiener-Kintchine's theorem, one gets, at the receiver side R:

$$\gamma_R(f) = \gamma_T(f) \left\| \mathcal{H}_c(f) \right\|^2 + \gamma_D(f) \, H_{FEXT}(f) \tag{505}$$

Calculation of the FEXT Signal to Noise ratio II

This leads to the following signal to noise ratio:

$$\frac{S(f)}{N(f)} = \frac{\gamma_T(f)\,\|\mathcal{H}_c(f)\|^2}{\gamma_D(f)\,H_{FEXT}(f)} \tag{506}$$

$$= \frac{\gamma_T(f)\,\|\mathcal{H}_c(f)\|^2}{\gamma_D(f)Lf^2k_{FEXT}\,\|\mathcal{H}_c(f)\|^2} \tag{507}$$

If we have $\gamma_T(f) \simeq \gamma_D(f)$, which means that both signals are of the same type, then

$$\frac{S(f)}{N(f)} = \frac{\gamma_T(f)\,\|\mathcal{H}_c(f)\|^2}{\gamma_D(f)Lf^2k_{FEXT}\,\|\mathcal{H}_c(f)\|^2} \tag{508}$$

$$= \frac{k'}{Lf^2} \tag{509}$$

Note that the transfer function of the channel does not appear in this expression.

Influence of the number of disturbers

Let consider a cable with N disturbing lines. A simple idea consists to sum up the individual effects of the N disturbing lines:

$$\gamma_{out}(f) = \sum_{i=1}^{N} \gamma_{in_i}(f)\, H_{FEXT/NEXT}(f) \tag{510}$$

If all the signals have the same power spectrum,

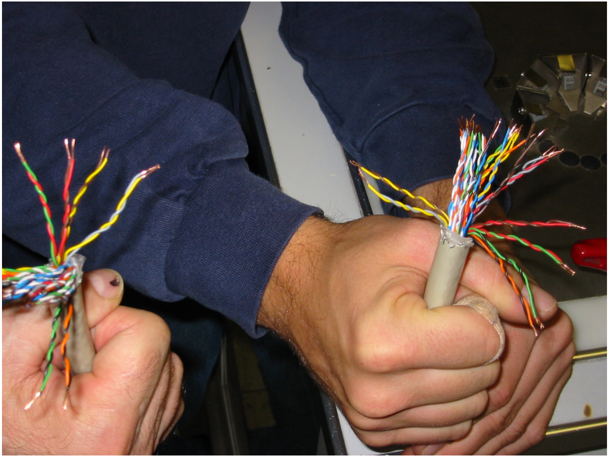
$$\gamma_{out}(f) = N\gamma_{in}(f)\, H_{FEXT/NEXT}(f) \tag{511}$$

However, this formula overestimates the real disturbing power because only neighboring lines interfere with the considered line.

Unger's formula (empirical formula)

$$\gamma_{out}(f) = N^{0.6}\gamma_{in}(f)\, H_{FEXT/NEXT}(f) \tag{512}$$

Cable building considerations I



In the field

$$\text{Bitrate} \quad \leq \quad \text{channel capacity !?}$$

Impact of an error during the transmission: bit/packet error rate (for packets of length N)

Assume a packet of size N and let P_e be the probability error on one bit.
The probability for the packet to be correct is

$$(1 - P_e)^N \tag{513}$$

Therefore the *packet error rate* is

$$P_P = 1 - (1 - P_e)^N. \tag{514}$$

For large packets and small P_e, this becomes

$$P_P \simeq 1 - (1 - NP_e) = N \times P_e. \tag{515}$$

Example

With $N = 10^5$ bits and a bit error rate of $P_e = 10^{-7}$, $P_P \simeq 10^{-2}$.

We thus need to lower $P_e \Rightarrow$ understand how disturbance occurs.

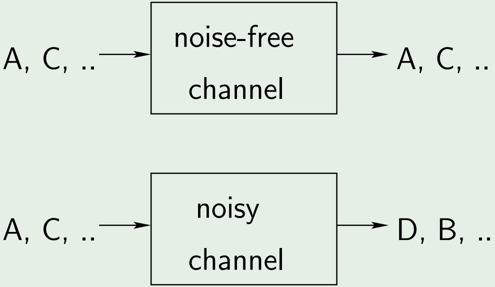
Estimation of the channel capacity I

Approach:

1. Define the notion of *information*
2. Provide a *model* for the channel
3. Establish the *channel capacity*

Example

Two types of channels for *carrying information*:



Estimation of the channel capacity II

Notion of information

Definition (Information of a symbol)

Let s_k be an event with a probability of p_k of a source S, we *quantify* the *information* provided by its observation by

$$i(s_k) = \log_2\left(\frac{1}{p_k}\right) = -\log_2 p_k \qquad (\geq 0) \tag{516}$$

- **Rare event**: small $p_k \rightarrow i(s_k) = -\log_2 p_k$ tends towards $+\infty$ $\rightarrow$ a lot of information
- p_k close to 1 $\rightarrow i(s_k) = -\log_2 p_k$ tends towards 0 $\rightarrow$ no information

The *unit* of information is the *bit of information*.
Note that there is a difference between a *bit* (which measures the number of sent binary symbols) and a *bit of information* (which measures the rate of information).

Definition (Entropy of a source S comprising K symbols)

The *average information*, per symbol, provided by a source is the *entropy*. It is denoted by $H(S)$ and is defined as

$$H(S)=\sum_{k=0}^{K-1} p_k \log_2\left(\frac{1}{p_k}\right)=\sum_{k=0}^{K-1} p_k\, i(s_k) \tag{517}$$

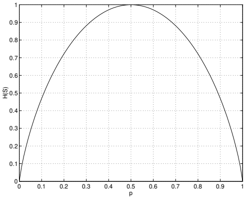


Figure: Entropy of a **binary** source (p denotes the probability of one symbol).
So, $H(S)=-p\log_2 p-(1-p)\log_2(1-p)$.

Binary source with $p(0) = p(1) = \frac{1}{2}$.

In that case, we have

$$H(S) = \frac{1}{2}\log_2 2 + \frac{1}{2}\log_2 2 = \log_2 2 = 1 \text{ [bit of information/symbol]} \tag{518}$$

For all other values of $p(0)$ and $p(1)$, $H(S) < 1$.

Source with 4 symbols, such that $p(0) = p(1) = p(2) = p(3) = \frac{1}{4}$.

In that case, we have

$$H(S) = 4 \times \frac{1}{4}\log_2 4 = \log_2 4 = 2 \text{ [bit of information/symbol]} \tag{519}$$

Question:

Do we want a source with a *minimal* or *maximal* amount of information? Why?

Memoryless discrete channel

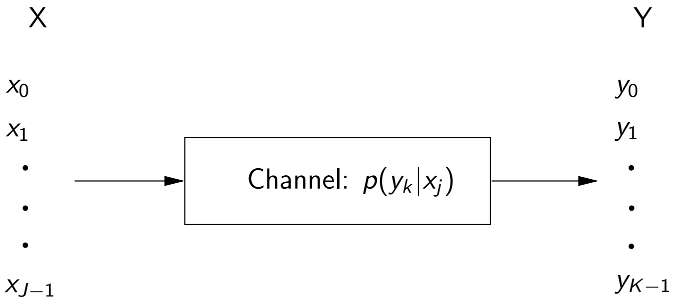


Figure: Model of a discrete memoryless channel.

Ideally,

$$p(y_k|x_j) = 0 \text{ if } \quad k \neq j$$

when there is no noise (and no pre-encoding). In practice however, this is not the case, because *there is always noise on the channel*.

For a noisy channel, the question consists in evaluating the amount of uncertainty relative to the input X when we observe $Y = y_k$ at the output. This leads to the definition of *conditional entropy*.

The uncertainty on X, given $Y = y_k$, is

$$H(X|Y = y_k) = E\{-\log_2 p(X|y_k)\} = \sum_{j=0}^{J-1} p(x_j|y_k)\log_2\left(\frac{1}{p(x_j|y_k)}\right) \tag{520}$$

Definition (Conditional entropy)

Average of this information $\Rightarrow$ *conditional entropy* $H(X|Y)$

$$\begin{aligned} H(X|Y) &= \sum_{k=0}^{K-1} H(X|Y = y_k)p(y_k) && (521) \\ &= \sum_{k=0}^{K-1}\sum_{j=0}^{J-1} p(x_j|y_k)p(y_k)\log_2\left(\frac{1}{p(x_j|y_k)}\right) && (522) \end{aligned}$$

Mutual information I

The uncertainty about X is always larger before observing Y than after doing it. The difference is a measure of the *information passing through the channel*, on average.

Definition (Mutual information)

We define

$$I(X;Y) = H(X) - H(X|Y) \tag{523}$$

as the average *mutual information*.

Theorem

The mutual information is symmetric $I(X;Y) = I(Y;X)$

Therefore:

$$I(X;Y) = H(X) - H(X|Y) \tag{524}$$

$$I(Y;X) = H(Y) - H(Y|X) \tag{525}$$

Mutual information II

Questions:

1. Is it intuitive to have that the mutual information is symmetric?
2. Is it better to have a channel with the *lowest* or *highest* mutual information $I(X, Y) = H(X) - H(X|Y)$? Why?

Definition

The *channel capacity per symbol* is defined as the maximum information conveyed over all possible input probability distributions

$$C_s = \max_{\{p(x_j)\}} I(X;Y) \tag{526}$$

It corresponds to the best possible usage of the channel.

This capacity is expressed in bit/symbol, where a symbol is one sample of X!
If the channel is used s times per second, then the *channel capacity in bits per second is*

$$C = s\,C_s \tag{527}$$

The capacity for one symbol is given by

$$
\begin{aligned}
C_s &= \max_{\{p(x_j)\}} I(X;Y) && (528)\\
&= \max_{\{p(x_j)\}} [H(X) - H(X|Y)] && (529)\\
&= \max_{\{p(x_j)\}} [H(Y) - H(Y|X)] && (530)
\end{aligned}
$$

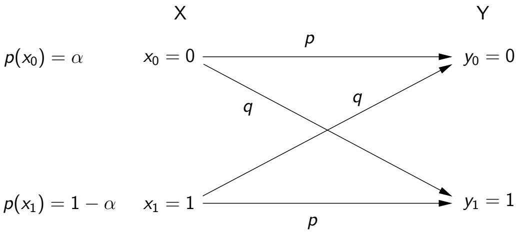


Figure: Calculation of the capacity of a binary symmetric channel.

We have that

$$H(Y|X) = \sum_{j=0}^{J-1} \sum_{k=0}^{K-1} p(y_k|x_j)p(x_j)\log_2\left(\frac{1}{p(y_k|x_j)}\right) \tag{531}$$

So, for the input-output pair (x_0, y_0), the conditional entropy is

$$H(Y|X)_{(x_0,y_0)} = -p(x_0)p(y_0|x_0)\log_2\left(p(y_0|x_0)\right) \tag{532}$$

Given that $p(x_0) = \alpha$ and that $p(y_0|x_0) = p\ (= 1 - q)$, we have

$$H(Y|X)_{(x_0,y_0)} = -\alpha p \log_2 p \tag{533}$$

Likewise:

$$\begin{aligned} H(Y|X)_{(x_0,y_1)} &= -\alpha q \log_2 q \\ H(Y|X)_{(x_1,y_0)} &= -(1-\alpha)q\log_2 q = -q\log_2 q + \alpha q \log_2 q \\ H(Y|X)_{(x_1,y_1)} &= -(1-\alpha)p\log_2 p = -p\log_2 p + \alpha p \log_2 p \end{aligned} \tag{534}$$

Calculation of the capacity of a *binary symmetric* channel III

By summing all these contributions,

$$
\begin{aligned}
H(Y|X) &= \sum_{i=0}^{1}\sum_{j=0}^{1} H(Y|X)_{(x_i,y_j)} && (535)\\
&= -p\log_2 p - q\log_2 q && (536)
\end{aligned}
$$

We observe that $H(Y|X)$ is (≥ 0) and that it only depends on the channel characteristics.
Finally, we get

$$
I(X;Y) = H(Y) + p\log_2 p + q\log_2 q \qquad (537)
$$

that we want to maximize to find the capacity.

Calculation of the capacity of a *binary symmetric* channel IV

Maximum of ?

$$I(X;Y) = H(Y) + p\log_2 p + q\log_2 q \tag{538}$$

Note that $p\log_2 p + q\log_2 q = -H_c \leq 0$ and that it depends on the channel. Therefore, we maximize $I(X;Y)$ by maximizing $H(Y)$, whose maximum is 1 for a binary output alphabet.

In conclusion, (remember that $q = 1 - p$)

$$C_S = 1 + p\log_2 p + q\log_2 q = 1 - H_c < 1 \tag{539}$$

For a transmission with 2 symbols (such as the NRZ or the ASK-2), the bit error rate $q = p_e$ depends on the $\frac{E_b}{N_0}$ ratio and is given by

$$q = p_e = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right) \tag{540}$$

Influence of the type of modulation

For a modulation technique with 4 states, C_s has the same form and is given by

$$C_s = H(Y) - H_c < 2 \qquad (541)$$

with the maximum of $H(Y)$ being equal to 2.

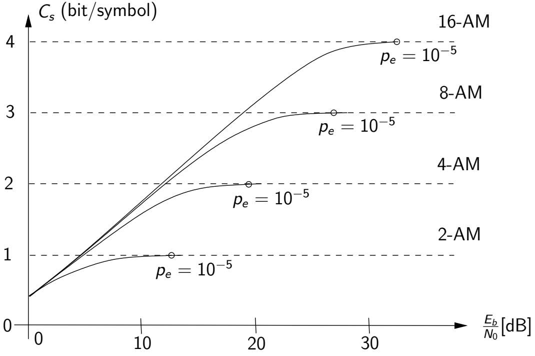


Figure: Channel capacity for different types of modulation.

Discrete inputs and continuous output: Shannon's second law

In an additive noise channel, the output is

$$Y = X + N(0, \sigma_N^2) \tag{542}$$

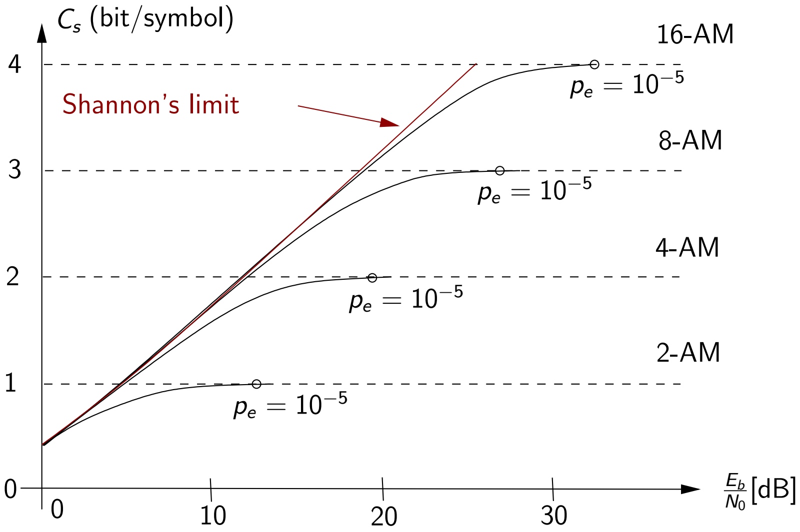
where X is a discrete random input to the channel and $N(0, \sigma_N^2)$ is a continuous noise variable (taken to be a Gaussian here).

Theorem (Shannon's second law)

For an additive Gaussian noise channel (zero-mean, variance σ_N^2), the capacity per symbol is given by

$$C_s = \frac{1}{2}\log_2\left(1 + \frac{\sigma_X^2}{\sigma_N^2}\right) \quad [bit/symbol] \tag{543}$$

where $\frac{\sigma_X^2}{\sigma_N^2}$ represents the signal to noise ratio.



C_s (bit/symbol)
4
3
2
1
0
16-AM
8-AM
4-AM
2-AM
Shannon's limit
$p_e = 10^{-5}$
$p_e = 10^{-5}$
$p_e = 10^{-5}$
$p_e = 10^{-5}$
0
10
20
30
$\frac{E_b}{N_0}$ [dB]

Loop performance for different types of crosstalk

According to Rauschmayer (*ADSL/VDSL principles: a practical and precise study of asymmetric digital subscriber lines and very high speed digital subscriber lines*, page 144, 1999):

	Channel capacity	
Disturbers	**Upstream (kb/s)**	**Downstream (Mb/s)**
AWGN only	2601	16.8
24 ISDN NEXT	1485	14.4
2 HDSL NEXT	1089	12.2
24 T1 NEXT	2338	7.29
24 ADSL NEXT	1126	14.8
24 ADSL FEXT	2072	14.1
Dowstream ADSL NEXT	1109	2.45

Outline

- Introduction
  - Propagation and fading
  - Mobile sensitivity
- Theory: 3 probabilistic propagation models
  - Propagation loss
  - Shadowing
    - Lognormal distribution
  - Fading
    - Rayleigh and Rician fading laws
- Empirical models
  - Types of environment
  - Influence of ground and antenna heights
  - Macrocellular model
  - Indoor propagation

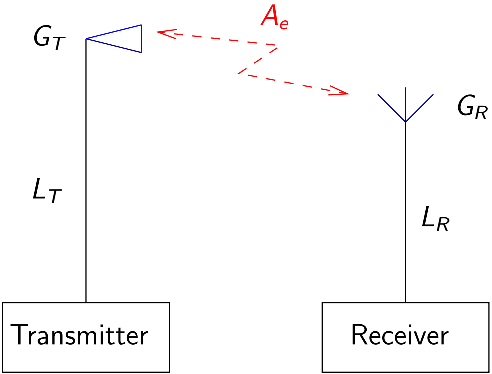


Figure: Elements contributing to the power budget in a radio link.

At the receiver, the power P_R, expressed in [dB], is given by

$$P_R = P_T - L_T + G_T - A_e + G_R - L_R \tag{544}$$

Fading

Definition (Fading)

Fading is a deviation of the attenuation affecting a signal over certain propagation media.

The fading may vary with *time*, *location* or radio *frequency*.
Different classifications/typologies for fading effects exist:

- slow/fast fading. It relates to time and delays considerations.
- flat/frequency selective fading.
- large-scale/small-scale fading.
- ...

Mitigation techniques:

- Rake receiver (estimation of the channel impulse response)
- Convolutional encoding and Viterbi decoding (pre-encoding)
- Diversity (MIMO, ...)
- Margins

Fading channel manifestations

Key manifestations that a fading channel can exhibit:

- large-scale fading, which represents the average signal-power *attenuation* or the path loss due to motion *over large areas* $\Rightarrow$ *"slow"* fading.
- small-scale fading. It refers to the dramatic *changes in signal amplitude* and *phase* that can be experienced as a result of small changes (as small as half a wavelength) in the spatial positioning between a receiver and a transmitter $\Rightarrow$ *"fast"* fading.

We will study examples of both types of fading and develop *probabilistic models* for them.

Mobile sensitivity and quality of service I

For a “good” radio communication, the system needs to fulfill *two conditions*:

1. The level of power received by the mobile device must be larger than the *mobile/device sensitivity level* C, that is the *minimal amount of power* that the device can interpret *for a given level of noise* at the input.
2. The channel should not distort the signal and the level of noise should be acceptable. This relates to E_b/N_0.

Calculation: link the $\frac{E_b}{N_0}$ ratio to the *mobile sensitivity* C (taking a spectral efficiency of $1 \Rightarrow R_b \approx W$)

$$\frac{C}{N} = \frac{E_b R_b}{N_0 W} \approx \frac{E_b W}{N_0 W} = \frac{E_b}{N_0} \tag{545}$$

Therefore

$$C = \left.\frac{E_b}{N_0}\right|_{\mathrm{threshold}} + N \tag{546}$$

Mobile sensitivity and quality of service II

Calculation of the device sensitivity C, at the decoding point in the receiver

For the GSM:

- $\left.\frac{E_b}{N_0}\right|_{\text{threshold}} = 8\,[\text{dB}]$ for a channel with fading ($6\,[\text{dB}]$ for devices with higher power levels)
- The bandwidth of a GSM channel is $W = 271\,[\text{kHz}]$. Therefore, for a temperature of $290\,[\text{K}]$, the amount of noise due to the channel is

$$N = k_B T W = 1.1 \times 10^{-12}\,[mW] \equiv -120\,[\text{dBm}] \qquad (547)$$

- As the typical input amplifier gain is $10\,[\text{dB}]$, the amplified amount of noise at the entrance of the decoder, N', is

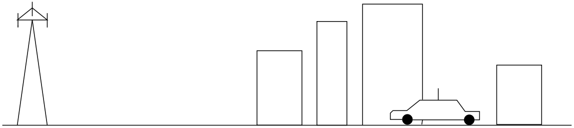
$$N' = -110\,[\text{dBm}] \qquad (548)$$

Mobile sensitivity and quality of service III

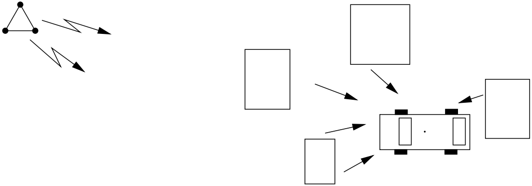
$$C = \left.\frac{E_b}{N_0}\right|_{\mathrm{threshold}} + N' \tag{549}$$

Receiver type	Device sensitivity C in [dBm]
Base station device	-104
8 [W] mobile device	-104
2 [W] mobile device (GSM 900)	-102
Dual-band mobile device (GSM)	-102

General propagation model I



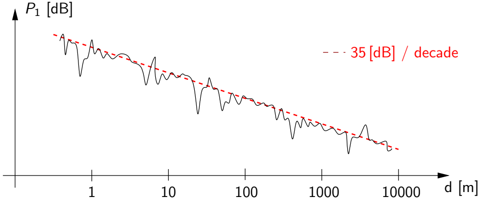
(a) Lateral view



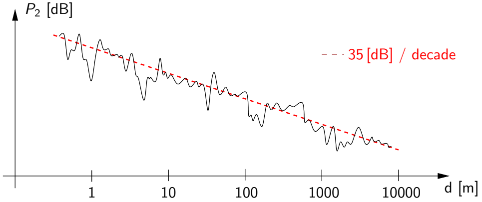
(b) Top view

Figure: Views of a transmission between a mobile and a base station.

General propagation model II



(a) First measurement



(b) Second measurement

Figure: Power measurements as a function of the distance along a same path. There are differences between the two observations.

General propagation model III

Fading is often computed as a combination of three terms:

1. a *median attenuation* due to *distance*,
2. a random component that considers large-scale fading effects. We first consider *shadowing* effects
   $\rightarrow$ *Lognormal* probabilistic law
3. a random component that takes into account small-scale fading effects. Later, we study *multipath* effects
   $\rightarrow$ *Rayleigh* and *Rician* probabilistic laws

Shadowing effects: towards a probabilistic model I

Shadowing is due to obstacles (hills, building, etc.).
Assumption: to model shadowing effects, we assume that the total attenuation is given by (multiplicative model)

$$A = A_1 \times A_2 \times \ldots \times A_N, \qquad \text{with } A_i \geq 1 \tag{550}$$

When there is shadowing, the received power P_R is

$$P_R = P_T / (A_{PL} \times A) = P_T / (A_{PL} \times A_1 \times A_2 \times \ldots \times A_N) \tag{551}$$

where A_{PL} is the propagation loss.
In decibels, the total attenuation loss is the sum of these terms:

$$L_{\text{total}} = L_{PL} + L = L_{PL} + L_1 + L_2 + \ldots + L_N \tag{552}$$

Note: here, L_{PL} is a deterministic value (for example given by Friis's relationship).

Shadowing effects: towards a probabilistic model II

More specifically, the loss due to shadowing L is the sum of N contributions

$$L = L_1 + L_2 + \ldots + L_N \tag{553}$$

If all the contributions are random variables with an identical mean and variance, L is a Gaussian random variable:

$$L\,[\mathrm{dB}] = N(L_{50\%},\, \sigma_s^2)\ [\mathrm{dB}] = L_{50\%}\,[\mathrm{dB}] + \sigma_s\,[\mathrm{dB}] \times N(0,1) \tag{554}$$

where $L_{50\%}$ is the median/(mean) value of the distribution of the propagation loss due to shadowing and σ_s [dB] is its standard deviation (which can typically vary over the range of 8 to 10 [dB]).

If we consider $L_{\mathrm{total}} = L_{PL} + L$, then

$$L_{\mathrm{total}}\,[\mathrm{dB}] = N(L_{PL} + L_{50\%},\, \sigma_s^2)\ [\mathrm{dB}] \tag{555}$$

In absolute numbers (*natural units, instead of [dB]*) and denoting the reduced Gaussian by $N(0,1) = X$, the loss is the variable L_a

$$L_a = 10^{L[\text{dB}]/10} = 10^{(L_{50\%}[\text{dB}] + \sigma_s[\text{dB}] \times X)/10} \quad (556)$$

$$= 10^{L_{50\%}[\text{dB}]/10}\, 10^{\sigma_s[\text{dB}] \times X/10} = L_o V \quad (557)$$

where $L_o \triangleq 10^{L_{50\%}[\text{dB}]/10}$ is a constant, and

$$V = 10^{\sigma_s[\text{dB}] \times X/10} \quad (558)$$

V, and subsequently the loss $L_a = L_o V$, are random variables with a lognormal probability density function.

Let us find the expression of the lognormal probability density function of V

The probability density function (pdf) of the zero-mean, unit-variance Gaussian variable X is, for $x \in [-\infty, +\infty]$,

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{559}$$

The lognormal variable V is given by (changing the base: $b^z = \left(e^{\ln b}\right)^z = \left(e^{z \ln b}\right)$)

$$V = 10^{\frac{\sigma_s X}{10}} = e^{\frac{\sigma_s X}{10} \ln(10)} = e^{\beta \sigma_s X} \tag{560}$$

where

$$\beta = \frac{\ln(10)}{10} = 0.23 \tag{561}$$

Probability density function of a lognormal variable II

The pdf is obtained by taking the formula of variable changes applicable to pdfs:

$$
\begin{aligned}
f_V(v) &= \left.\frac{f_X(x)}{|\partial v/\partial x|}\right|_{x=\frac{\ln v}{\beta\sigma_s}} = \left.\frac{\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}}{\beta\sigma_s e^{\beta\sigma_s x}}\right|_{x=\frac{\ln v}{\beta\sigma_s}} && (562)\\
&= \begin{cases} \dfrac{1}{\beta\sigma_s e^{\beta\sigma_s\frac{\ln v}{\beta\sigma_s}}}\dfrac{1}{\sqrt{2\pi}}e^{-\frac{(\ln v)^2}{2\beta^2\sigma_s^2}} & \text{if } v \geq 0\\ 0 & \text{if } v < 0 \end{cases} && (563)\\
&= \begin{cases} \dfrac{1}{\beta\sigma_s v}\dfrac{1}{\sqrt{2\pi}}e^{-\frac{(\ln v)^2}{2\beta^2\sigma_s^2}} & \text{if } v \geq 0\\ 0 & \text{if } v < 0 \end{cases} && (564)
\end{aligned}
$$

Probability density function of a lognormal variable III

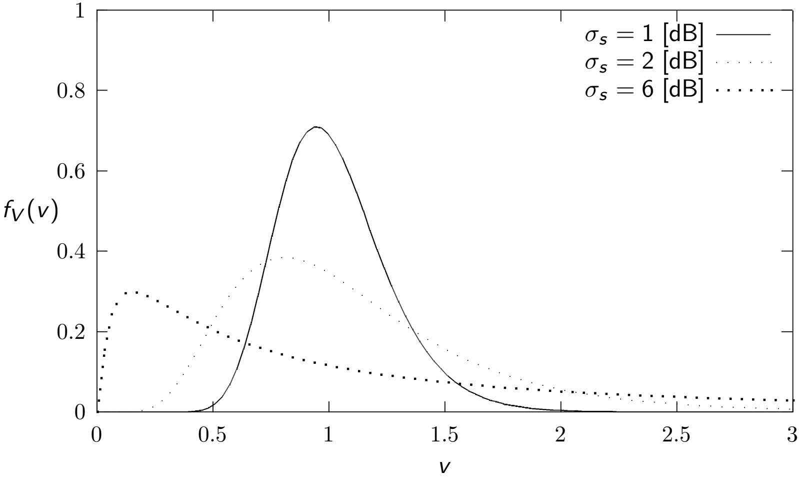


Figure: Probability density function of a lognormal random variable.

Impact of shadowing on the coverage zone at the border of a mobile cell

We have seen that the loss including the shadowing can be modeled as

$$L_{\text{total}}\,[\text{dB}] = L_{50\%}\,[\text{dB}] + \sigma_s\,[\text{dB}] \times N(0,1) = N(L_{50\%},\, \sigma_s^2)\,[\text{dB}] \tag{565}$$

where

- $L_{50\%}$ is the median loss, as given by an empirical model (such models are obtained from measurements and **they include** L_{PL}; see the COST 231-Hata model, later in this document).
- N is a Gaussian variable.

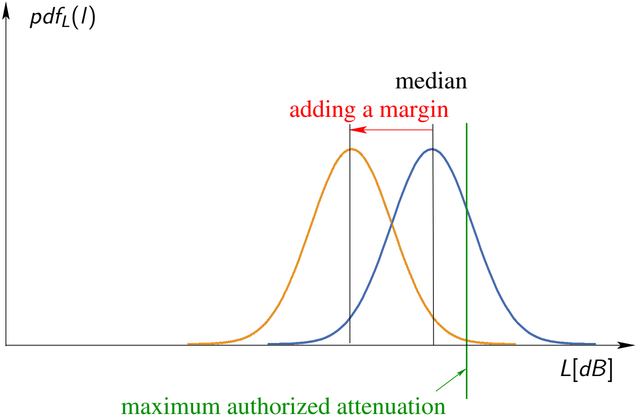
If the observed loss, L_{observed}, is such that

- $L_{\text{observed}} \leq L_{50\%}\,[\text{dB}]$, then we have a favorable case.
- $L_{\text{observed}} \geq L_{50\%}\,[\text{dB}]$, then there is a risk for a **connection failure**.

$\Rightarrow$ the probability to be in the favorable case has to be increased (50% is not enough!).

Lognormal margin I

To decrease the risk (%) of failure and mitigate the effects of shadowing, we add a power margin.



Lognormal margin II

Towards the computation of a power margin m_s [dB]

$$p(l_s < m_s) = \int_{-\infty}^{m_s} \frac{1}{\sigma_s\sqrt{2\pi}} e^{\frac{-l_s^2}{2\sigma_s^2}} dl_s \quad (566)$$

$$= \frac{1}{2} + \int_{0}^{m_s} \frac{1}{\sigma_s\sqrt{2\pi}} e^{\frac{-l_s^2}{2\sigma_s^2}} dl_s \quad (567)$$

$$= \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{m_s}{\sqrt{2}\sigma_s}\right) \quad (568)$$

$\operatorname{erf}(x)$ is the error function defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad (569)$$

Lognormal margin III

Power margin

So, there are two terms:

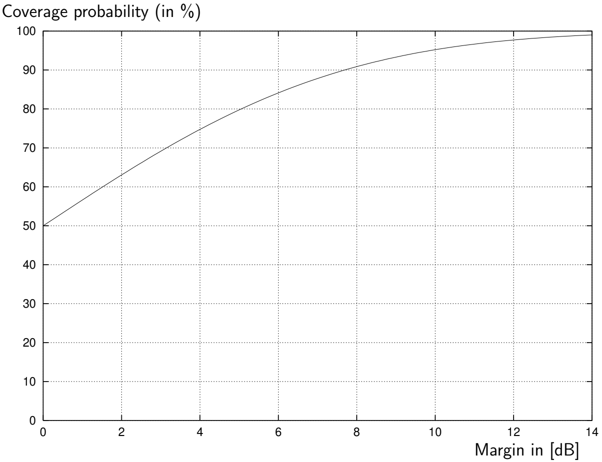
1. $\frac{1}{2} \rightarrow$ level of power to get 50% of favorable cases.
2. $\frac{1}{2}\,\text{erf}\left(\frac{m_s}{\sqrt{2}\sigma_s}\right) \rightarrow$ leads to a power margin m_s if we want more favorable cases (in excess of 50% thus).

Definition (Power margin)

The **power margin** is the **additional amount of power sent by the transmitter** to obtain more favorable cases at the receiver.
The power margin "compensates" for possibly higher fading losses.

Lognormal margin IV

Relationship between a % of coverage and a power margin (at the border of a cell)



Fading channel manifestations (reminder)

Key manifestations that a fading channel can exhibit:

- large-scale fading, which represents the average signal-power *attenuation* or the path loss due to motion *over large areas* $\Rightarrow$ *"slow"* fading.

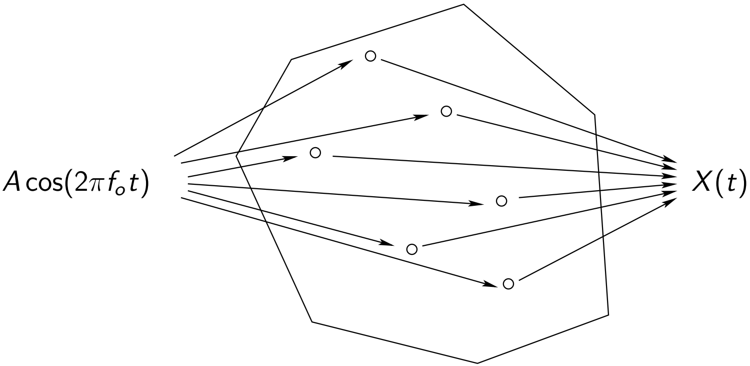
$\rightarrow$ *Lognormal* probabilistic law

- small-scale fading. It refers to the dramatic *changes in signal amplitude* and *phase* that can be experienced as a result of small changes (as small as a half wavelength) in the spatial positioning between a receiver and a transmitter $\Rightarrow$ *"fast"* fading.

We now study the effect of fast fading.

Fast fading I

Consider the transmission over a **scattering** or **multipath channel**



The transmitter emits a clean sinusoidal wave of the form $A\cos(2\pi f_o t)$. At the receiver, we get

$$X(t) = \sum_i C_i \cos(2\pi f_o t + \theta_i) \qquad (570)$$

This expression can be rewritten to express the in-phase and quadrature components ($\cos(a+b) = \cos a \cos b - \sin a \sin b$):

$$\begin{aligned} X(t) &= \sum_i \left[C_i \cos(2\pi f_o t)\cos\theta_i - C_i \sin(2\pi f_o t)\sin\theta_i\right] && (571) \\ &= \sum_i A_i \cos(2\pi f_o t) - \sum_i B_i \sin(2\pi f_o t) && (572) \\ &= X_I \cos(2\pi f_o t) - X_Q \sin(2\pi f_o t) && (573) \end{aligned}$$

where we have defined

$$X_I = \sum_i A_i = \sum_i C_i \cos\theta_i \quad \text{and} \quad X_Q = \sum_i B_i = \sum_i C_i \sin\theta_i \qquad (574)$$

Fast fading III

What are $X_I = \sum_i A_i = \sum_i C_i \cos\theta_i$ **and** $X_Q = \sum_i B_i = \sum_i C_i \sin\theta_i$**?**

- we can consider X_I and X_Q as two random variables.
- they are sums of terms that are identically distributed.

$\Rightarrow$ X_I and X_Q are two Gaussian random variables with:

1. a zero mean and,
2. the same variance σ_X^2.

Calculation of the mean:

$$E\{X_I\} = E\left\{\sum_i C_i \cos\theta_i\right\} = \sum_i E\{C_i \cos\theta_i\} \tag{575}$$

$$= \sum_i E\{C_i\} E\{\cos\theta_i\} = \sum_i E\{C_i\} \times 0 \tag{576}$$

$$= 0 \tag{577}$$

Calculation of the variance:

$$\sigma_{X_I}^2 = E\left\{(X_I - \mu_I)^2\right\} = E\left\{(X_I - 0)^2\right\} \tag{578}$$

$$= E\left\{\left(\sum_i C_i \cos\theta_i\right)^2\right\} \tag{579}$$

$$= E\left\{\sum_i C_i^2 \cos^2\theta_i + \sum_{i,j,\, i\neq j} C_i \cos\theta_i C_j \cos\theta_j\right\} \tag{580}$$

$$= \sum_i E\left\{C_i^2 \cos^2\theta_i\right\} + \sum_{i,j,\, i\neq j} E\left\{C_i \cos\theta_i C_j \cos\theta_j\right\} \tag{581}$$

$$= \sum_i E\left\{C_i^2\right\} E\left\{\cos^2\theta_i\right\} + \sum_{i,j,\, i\neq j} E\left\{C_i\right\} E\left\{\cos\theta_i\right\} E\left\{C_j\right\} E\left\{\cos\theta_j\right\}$$

$$= \sum_i E\left\{C_i^2\right\} \times \frac{1}{2} + \sum_{i,j,\, i\neq j} E\left\{C_i\right\} \times 0 \times E\left\{C_j\right\} \times 0 \tag{582}$$

$$= \frac{1}{2}\alpha = \sigma_X^2 \tag{583}$$

Fast fading V

Probability density functions of X_I and X_Q

In conclusion, we have

$$f_{X_I}(x_I) = \frac{1}{2\pi\sigma_X^2} e^{-\frac{x_I^2}{2\sigma_X^2}} \tag{584}$$

and

$$f_{X_Q}(x_Q) = \frac{1}{2\pi\sigma_X^2} e^{-\frac{x_Q^2}{2\sigma_X^2}} \tag{585}$$

Note also that X_I and X_Q are the Rice components of the stochastic process $X(t)$:

$$X(t) = X_I \cos(2\pi f_o t) - X_Q \sin(2\pi f_o t) \tag{586}$$

To determine the amplitude of the electrical field at the receiver, we need to analyze the amplitude of $X(t)$.
It is given by

$$R = \sqrt{X_I^2 + X_Q^2}, \quad R \geq 0 \tag{587}$$

while the phase is

$$\Phi = \tan^{-1} \frac{X_Q}{X_I}, \quad \Phi \in [0, 2\pi[\tag{588}$$

Both R and Φ are obtained from X_I and X_Q. So, we have to find the joint $f_{R,\Phi}(r,\,\phi)$ from that of $f_{X_I\,X_Q}(x_I,\,x_Q)$.
The joint pdf $f_{X_I\,X_Q}(x_I,\,x_Q)$ is, by definition,

$$f_{X_I\,X_Q}(x_I,\,x_Q) = \frac{1}{2\pi\sigma_X^2}\,e^{-\frac{x_I^2+x_Q^2}{2\sigma_X^2}} \tag{589}$$

From equations (587) and (588), we have that
$x_I = r\cos\phi$, $x_Q = r\sin\phi$. So (formula of change of variables):

$$\begin{aligned} f_{R,\Phi}(r,\,\phi) &= \left|\begin{array}{cc} \cos\phi & \sin\phi \\ -r\sin\phi & r\cos\phi \end{array}\right| f_{X_I\,X_Q}(r\cos\phi,\; r\sin\phi) && (590)\\ &= \frac{r}{2\pi\sigma_X^2}\,e^{-\frac{r^2\cos^2\phi+r^2\sin^2\phi}{2\sigma_X^2}} && (591)\\ &= \begin{cases} \frac{r}{2\pi\sigma_X^2}\,e^{-\frac{r^2}{2\sigma_X^2}}, & r\geq 0,\ \phi\in[0,2\pi[\\ 0 & r<0 \end{cases} && (592) \end{aligned}$$

Marginal probability density function of the envelope $f_R(r)$

By integrating over Φ, we obtain the marginal probability density function of the envelope:

$$f_R(r) = \begin{cases} \frac{r}{\sigma_X^2}\, e^{-\frac{r^2}{2\sigma_X^2}}, & r \geq 0 \\ 0 & r < 0 \end{cases} \tag{593}$$

This is a Rayleigh distribution.

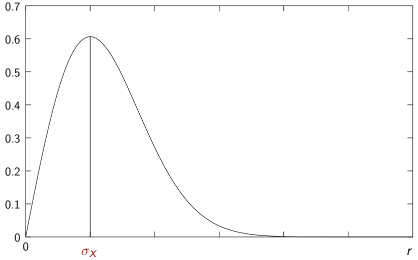


Figure: Rayleigh probability density function.

Probability density function of the phase $f_\Phi(\phi)$

To obtain $f_\Phi(\phi)$, we integrate on the r parameter from 0 to $+\infty$, to get:

$$f_\Phi(\phi) = \frac{1}{2\pi}, \ \ \phi \in [0, 2\pi[\tag{594}$$

Φ is thus the probability density function of a uniform random variable, which is independent of R.

In conclusion,

$$f_{R,\Phi}(r, \phi) = f_R(r) f_\Phi(\phi) \tag{595}$$

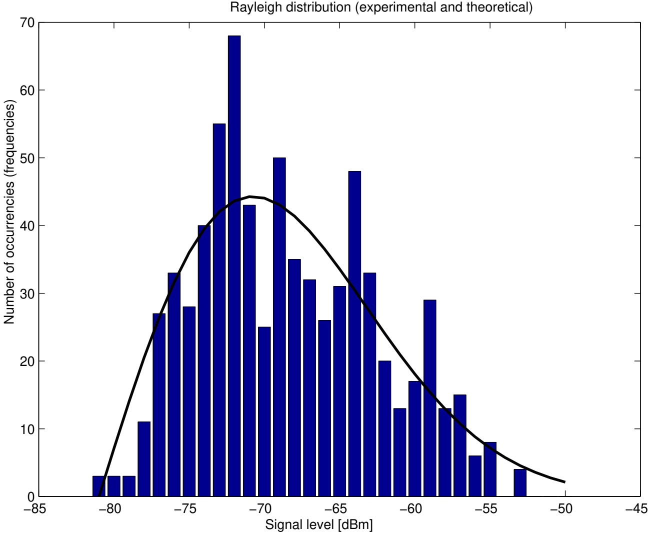


Figure: Histogram of real measured power levels.

If there is a direct path to the receiver, the signal at the receiver is

$$Z(t) = A\cos(2\pi f_o t + \theta) + X(t) \tag{596}$$

Assume that $Z(t)$ is a bandpass signal, we can take its Rice decomposition

$$Z(t) = Z_I \cos(2\pi f_o t) - Z_Q \sin(2\pi f_o t) \tag{597}$$

where

$$Z_I = A\cos\theta + X_I \tag{598}$$
$$Z_Q = A\sin\theta + X_Q \tag{599}$$

In terms of amplitude and phase, this leads to

$$Z = R\cos(2\pi f_o t + \Phi) \tag{600}$$

with

$$R = \sqrt{Z_I^2 + Z_Q^2}, \quad R \geq 0 \tag{601}$$

and

$$\Phi = \tan^{-1} \frac{X_Q}{X_I}, \quad \Phi \in [0, 2\pi[\tag{602}$$

Therefore,

$$\begin{aligned} Z_I &= R \cos \Phi && (603) \\ Z_Q &= R \sin \Phi && (604) \end{aligned}$$

For a given θ, the X_I and X_Q are independent variables. This remains true, so that

$$\begin{aligned} Z_I &= N(A \cos \theta, \sigma_X^2) && (605) \\ Z_Q &= N(A \sin \theta, \sigma_X^2) && (606) \end{aligned}$$

The joint pdf, given θ_o, is then

$$f_{Z_I\, Z_Q}(z_I, z_Q \,|\theta = \theta_o) = \frac{1}{2\pi\sigma_X^2} e^{-\frac{(z_I - A\cos\theta_o)^2 + (z_Q - A\sin\theta_o)^2}{2\sigma_X^2}} \tag{607}$$

$$
\begin{aligned}
f_{R,\Phi}(r,\, \phi\,|\theta_o) &= \left| \begin{array}{cc} \cos\phi & \sin\phi \\ -r\sin\phi & r\cos\phi \end{array} \right| f_{Z_I\, Z_Q}(r\cos\phi,\ r\sin\phi|\theta_o) \\
&= \frac{r}{2\pi\sigma_X^2}\, e^{-\frac{(r\cos\phi - A\cos\theta_o)^2 + (r\sin\phi - A\sin\theta_o)^2}{2\sigma_X^2}} \qquad (608) \\
&= \frac{r}{2\pi\sigma_X^2}\, e^{-\frac{r^2 + A^2 - 2rA\cos(\theta_o - \phi)}{2\sigma_X^2}} \qquad (609)
\end{aligned}
$$

When integrating over Φ, we obtain the Rician pdf

$$f_R(r) = \begin{cases} \frac{r}{\sigma_X^2} e^{-\frac{r^2+A^2}{2\sigma_X^2}} I_0\left(\frac{Ar}{\sigma_X^2}\right), & r \geq 0 \\ 0 & r < 0 \end{cases} \tag{610}$$

where $I_0(x)$ is the modified Bessel function of order 0.

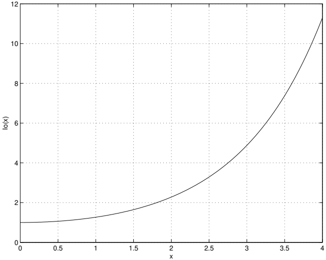


Figure: Modified Bessel function of order 0.

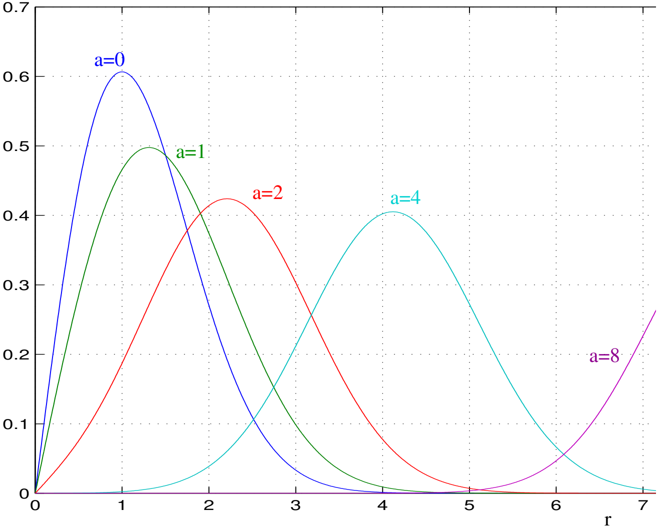


Figure: Rician pdf (for different values of $a = \frac{A}{\sigma_X}$).

Fast fading: conclusions

1. The receiver gets $X(t) = \sum_i C_i \cos(2\pi f_o t + \theta_i)$ (no direct path):
   - the amplitude of $X(t)$ follows a Rayleigh distribution
   - the phase of $X(t)$ follows a uniform distribution
2. The receiver gets $Z(t) = A\cos(2\pi f_o t + \theta) + X(t)$ (there is a direct path):
   - the amplitude of $Z(t)$ follows a Rician distribution
   - the phase of $Z(t)$ is not uniformly distributed

Influence of ground and antenna heights on the radio budget link I

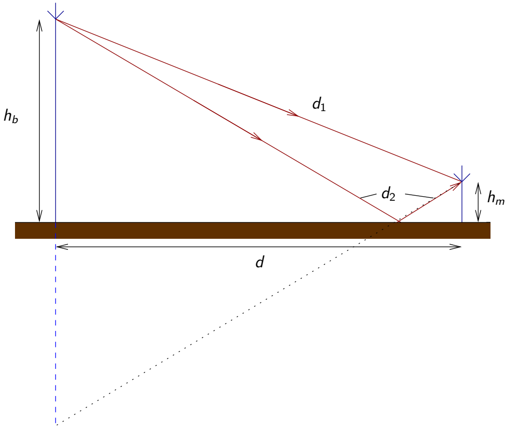


Figure: Influence of a reflection on the ground.

Influence of ground and antenna heights on the radio budget link II

Analysis of the electrical field phasor E:

- direct path $\rightarrow$ $Ae^{-j\beta d_1}$,
- reflected signal $\rightarrow$ $-Ae^{-j\beta d_2}$,
- the received signal is
  $Ae^{-j\beta d_1} - Ae^{-j\beta d_2} = Ae^{-j\beta d_1}\left(1 - e^{j\beta(d_2 - d_1)}\right)$.

So, the reflection adds the following factor:

$$\Gamma = 1 - e^{j\beta(d_2 - d_1)} \tag{611}$$

As (for right-angled triangles)

$$d_1 = \sqrt{d^2 + (h_b - h_m)^2} \tag{612}$$

$$d_2 = \sqrt{d^2 + (h_b + h_m)^2} \tag{613}$$

Influence of ground and antenna heights on the radio budget link III

we have that ($\sqrt{1+\alpha} \simeq 1 + \frac{\alpha}{2}$)

$$
\begin{aligned}
d_2 - d_1 &= \sqrt{d^2 + (h_b + h_m)^2} - \sqrt{d^2 + (h_b - h_m)^2} && (614)\\
&= d\sqrt{1+\left(\frac{h_b + h_m}{d}\right)^2} - d\sqrt{1+\left(\frac{h_b - h_m}{d}\right)^2} && (615)\\
&\simeq d\left(1+\frac{1}{2}\left(\frac{h_b + h_m}{d}\right)^2\right) - d\left(1+\frac{1}{2}\left(\frac{h_b - h_m}{d}\right)^2\right) && (616)\\
&\simeq d\frac{1}{2}\left(\frac{h_b + h_m}{d}\right)^2 - d\frac{1}{2}\left(\frac{h_b - h_m}{d}\right)^2 && (617)\\
&\simeq \frac{1}{2d}\left((h_b + h_m)^2 - (h_b - h_m)^2\right) && (618)\\
&\simeq \frac{1}{2d}\left(\left(h_b^2 + 2h_bh_m + h_m^2\right) - \left(h_b^2 - 2h_bh_m + h_m^2\right)\right) && (619)\\
&\simeq \frac{4h_bh_m}{2d} = \frac{2h_bh_m}{d} && (620)
\end{aligned}
$$

To deal with the power, we take the square of the electrical field modifying factor, that is Γ^2:

$$
\begin{aligned}
\Gamma^2 &= \left|1 - e^{-j\beta(d_2-d_1)}\right|^2 && (621)\\
&= \left|e^{-j\beta(d_2-d_1)/2}\right|^2 \left|e^{j\beta(d_2-d_1)/2} - e^{-j\beta(d_2-d_1)/2}\right|^2 && (622)\\
&= \left|e^{-j\beta(d_2-d_1)/2}\right|^2 \left|2\sin\left(\beta\frac{d_2-d_1}{2}\right)\right|^2 && (623)\\
&= 1 \times \left(2\sin\left(\beta\frac{h_b h_m}{d}\right)\right)^2 && (624)\\
&= 4\sin^2\left(\beta\frac{h_b h_m}{d}\right) && (625)\\
&\simeq 4\left(\beta\frac{h_b h_m}{d}\right)^2 = 4\left(\frac{2\pi}{\lambda}\frac{h_b h_m}{d}\right)^2 = \left(\frac{4\pi}{\lambda}\frac{h_b h_m}{d}\right)^2 && (626)
\end{aligned}
$$

Receiver power P_R

According to Friis's relationship, the received power P_R is given by

$$P_R = P_E G_E G_R \left(\frac{\lambda}{4\pi d} \right)^2 \tag{627}$$

Because of the presence of ground, the received power P_R is given by

$$\begin{aligned} P_R &\simeq P_E G_E G_R \left(\frac{\lambda}{4\pi d} \right)^2 \left(\frac{4\pi}{\lambda} \frac{h_b h_m}{d} \right)^2 & (628) \\ &\simeq P_E G_E G_R \frac{h_b^2 h_m^2}{d^4} & (629) \end{aligned}$$

or, in decibels,

$$P_R \,[\mathrm{dB}] = 10 \log_{10}(P_E G_E G_R) + 20 \log_{10} h_b + 20 \log_{10} h_m - 40 \log_{10} d \tag{630}$$

Different types of environments

- *rural*. Here, the topographic relief is the major factor.
- *suburban* (small towns)
- *urban* (large cities)

Environment	exponent
rural	3.2
suburban	3.5
dense urban	3.8

Table: Exponent for the distance dependency of the attenuation loss.

Model for macrocells

COST 231-Hata model

In a urban environment, the median attenuation L_u is given, in [dB], by

$$L_u = 46.33 + 33.9\log(f) - 13.82\log(h_b) - a(h_m) + [44.9 - 6.55\log(h_b)]\log d + C_m \tag{631}$$

where

- f is the frequency, d the distance, h_b, h_m, heights; all these values are expressed in, respectively, [MHz], [km] and [m].
- $a(h_m) = (1.1\log(f) - 0.7)h_m - (1.56\log(f) - 0.8)$ for middle-sized towns; this correction factor depends on the mobile height but also from the type of environment.
- $C_m = 0$ [dB] middle-sized towns and suburbs, and $C_m = 3$ [dB] for large cities.

Indoor propagation (inside of buildings) I

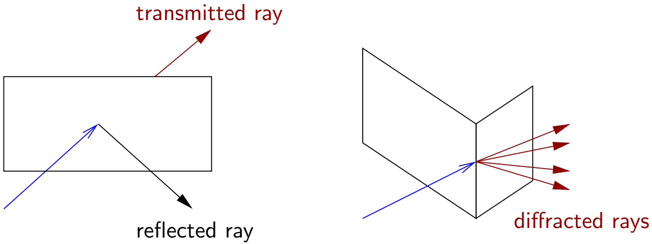


Figure: Transmission, reflection and diffraction.

Indoor propagation (inside of buildings) II

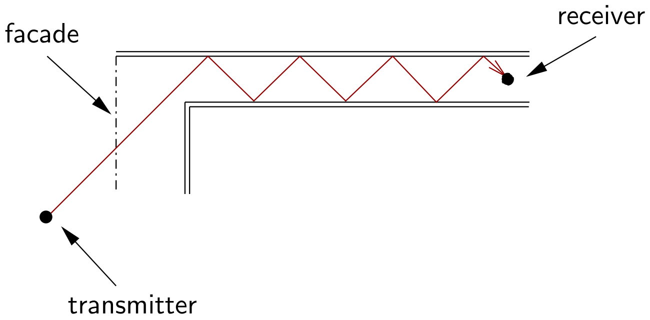


Figure: Wave guide effects.

Indoor propagation (inside of buildings) III

“Outdoor-indoor” propagation

- we can define two types of situations:
  1. *soft indoor*, representing the attenuation close to the front of a building, and
  2. *deep indoor*, representing the attenuation deeper inside the building.

  - Typical values are 10 [dB] *soft indoor* and 20 [dB] for *deep indoor* at 900 [MHz].

“Indoor-indoor” propagation”