Short Prime Quadratizations of Cubic Negative Monomials

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Abstract

Pseudo-Boolean functions naturally model problems in a number of different areas such as computer science, statistics, economics, operations research or computer vision, among others. Pseudo-Boolean optimization (PBO) is $\mathcal{NP}$-hard, even for quadratic polynomial objective functions. However, much progress has been done in finding exact and heuristic algorithms for the quadratic case. Quadratizations are techniques aimed at reducing a general PBO problem to a quadratic polynomial one. Quadratizing single monomials is particularly interesting because it allows quadratizing any pseudo-Boolean function by *termwise quadratization*. A characterization of short quadratizations for negative monomials has been provided. In this report we present a proof of this characterization for the case of cubic monomials, which requires a different analysis than the case of higher degree.

1 Introduction

A pseudo-Boolean function is a mapping $f : \{0,1\}^n \to \mathbb{R}$, i.e., a mapping that assigns a real value to each tuple $(x_1, \ldots, x_n)$ of n binary variables. Every pseudo-Boolean function can be represented by a unique multilinear polynomial, that is, for a function f on $\{0,1\}^n$ there exists a unique mapping $a : 2^{[n]} \to \mathbb{R}$, which assings a real value a_S to every subset S of the n variables, such that

$$f(x_1, x_2, \ldots, x_n) = \sum_{S \in 2^{[n]}} a_S \prod_{i \in S} x_i. \tag{1}$$

Pseudo-Boolean optimization (PBO) problems are of the form

$$\min\{f(x) : x \in \{0,1\}^n\},$$

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where $f(x)$ is a pseudo-Boolean function. Pseudo-Boolean optimization models arise naturally in diverse areas such as computer science, statistics, economics, finance, operations research or computer vision, among others. A detailed list of applications can be found in [2], [3].

Pseudo-Boolean optimization is $\mathcal{NP}$-hard, even if the objective function is quadratic. However the quadratic case is particularly interesting; on one hand, because it encompasses relevant problems such as MAX-2-SAT (satisfiability theory) or MAX-CUT (graph theory), and on the other hand, due to much progress that has been done in finding heuristic and exact algorithms for quadratic pseudo-Boolean optimization (QPBO). Therefore, given a pseudo-Boolean function f, we aim to find an equivalent quadratic function g, for which quadratic binary optimization algorithms are applicable.

Definition 1 *Given a pseudo-Boolean function $f : \{0, 1\}^n \to \mathbb{R}$, $g(x, y)$ is a* quadratization *of f if $g(x, y)$ is a quadratic polynomial depending on x and on m auxiliary binary variables $y_1, y_2, \dots, y_m$, such that*

$$f(x) = \min\{g(x, y) : y \in \{0, 1\}^m\}, \forall x \in \{0, 1\}^n.$$

Using this definition, $\min\{f(x) : x \in \{0, 1\}^n\} = \min\{g(x, y) : (x, y) \in \{0, 1\}^{n+m}\}$, reducing a general PBO problem to the quadratic case.

Anthony, Boros, Crama and Gruber have initiated a systematic study of quadratizations of pseudo-Boolean functions [1]. Among other results, they provide a precise characterization of quadratizations for negative monomials. The aim of this report is to provide a proof of this characterization for cubic negative monomials, which is different from the proof for the case of monomials of degree ≥ 4.

2 Negative Monomials

Finding quadratizations for monomials is particularly interesting; if quadratizations for single monomials are known and well-described, it is possible to use *termwise quadratization procedures*, which are based on the following scheme. For a real number c, let $sign(c) = +1$ (resp., -1) if $c \geq 0$ (resp., $c < 0$). Then, given f as in (1),

1. for each $S \in 2^{[n]}$, let $g_S(x, y_S)$ be a quadratization of the monomial $sign(a_S) \prod_{i \in S} x_i$, where $(y_S, S \in 2^{[n]})$ are disjoint vectors of auxiliary variables, one for each S,

2. let $g(x, y) = \sum_{S \in 2^{[n]}} |a_S|\, g_S(x, y_S)$.

Then $g(x, y)$ is a quadratization of $f(x)$.

Several quadratizations of monomials have been proposed in the literature (see, e.g., [1]). In this report we describe quadratizations for the case where f is a negative monomial. We first introduce the notion of *prime* quadratizations [1], which are interesting because they define ”small” quadratizations, and because our objective is to minimize f. Then, we will prove that there are essentially only two prime quadratizations using a single auxiliary variable for negative cubic monomials.

Definition 2 *A quadratization $g(x, y)$ of f is* prime *if there is no quadratization $h(x, y)$ such that $h(x, y) \leq g(x, y)$ for all $(x, y) \in \{0, 1\}^{n+m}$, and such that $h(x^*, y^*) < g(x^*, y^*)$ for at least one point (x^*, y^*).*

Definition 3 *The* standard quadratization *of a negative monomial $M_n = -\prod_{i=1}^{n} x_i$ is the quadratic function*

$$s_n(x, y) = (n - 1)y - \sum_{i=1}^{n} x_i y. \tag{2}$$

The extended standard quadratization *of M_n is the function*

$$s_n^+(x, y) = (n - 2)x_n y - \sum_{i=1}^{n-1} x_i(y - \bar{x}_n), \tag{3}$$

where $\bar{x}_n = 1 - x_n$.

Anthony et al. [1] state the following theorem:

Theorem 1 *For $n \geq 3$, assume that $g(x, y)$ is a prime quadratization of M_n involving a single auxiliary variable y. Then, up to an appropriate permutation of the x-variables and up to a possible switch of the y-variable, either $g(x, y) = s_n$ or $g(x, y) = s_n^+$.*

The proof in [1] is valid for all $n \geq 4$, but the authors skipped the details of the case $n = 3$, which requires slightly different arguments. We present next the missing details.

Proof. (case $n = 3$). The proof consists in a case study on the coefficients of the general form of a quadratization with a single auxiliary variable for the cubic negative monomial. Until Claim 2, the proof is identical to the case $n \geq 4$ presented in [1].

The general form of a quadratization using a single auxiliary variable is

$$g(x, y) = ay + \sum_{i=1}^{3} b_i x_i y + \sum_{i=1}^{3} c_i x_i + \sum_{1 \leq i < j \leq 3} p_{ij} x_i x_j. \tag{4}$$

Notice that there is no constant term because, since we must have $M_3(x) = \min_{y\in\{0,1\}} g(x,y)$ for all binary vectors x, we can assume $g(0,0) = 0$ after substituting $\bar{y}$ by y if necessary.

For subsets $S \subseteq N = \{1,2,3\}$, we write $b(S) = \sum_{i\in S} b_i$, $c(S) = \sum_{i\in S} c_i$, and $p(S) = \sum_{i,j\in S, i<j} p_{ij}$, and we can write

$$g(S,y) = ay + b(S)y + c(S) + p(S). \tag{5}$$

The fact that g is a quadratization of M_3 can be written as

$$0 = \min_{y\in\{0,1\}} (a + b(S))y + c(S) + p(S), \forall S \subset N, \tag{6}$$

$$-1 = \min_{y\in\{0,1\}} (a + b(N))y + c(N) + p(N). \tag{7}$$

Let us first note that by (6), we have $g(0,1) \geq 0$, and hence

$$a \geq 0. \tag{8}$$

Furthermore, we must have $g(\{i\},0) \geq 0$ for $i = 1,2,3$, implying

$$c_i \geq 0, \text{ for } i = 1,2,3. \tag{9}$$

Based on (9), we can partition the set of indices as $N = N^0 \cup N^+$, where

$$N^0 = \{u \in N \mid c_u = 0\}, \tag{10}$$

$$N^+ = \{i \in N \mid c_i > 0\}. \tag{11}$$

Since $g(\{i\},0) = c_i$, relation (6) implies

$$g(\{i\},1) = a + b_i + c_i = 0, \forall i \in N^+, \text{ and} \tag{12}$$

$$g(\{u\},1) = a + b_u \geq 0, \forall u \in N^0. \tag{13}$$

Let us next write (6) for subsets of size two. Consider first a pair $u,v \in N^0$, $u \neq v$. Sincd $c_u = c_v = 0$, we get $g(\{u,v\},y) = (a + b_u + b_v)y + p_{uv}$, implying

$$\min\{p_{uv}, a + b_u + b_v + p_{uv}\} = 0. \tag{14}$$

Let us consider next $i,j \in N^+$, $i \neq j$. Then, by (12) and by the definitions we get $g(\{i,j\},1) = p_{ij} - a \geq 0$. This, together with (8) implies that $p_{ij} \geq a \geq 0$. Thus, $g(\{i,j\},0) = c_i + c_j + p_{ij} > 0$ implying that $g(\{i,j\},1) = 0$, that is

$$p_{ij} = a \geq 0, \forall i,j \in N^+. \tag{15}$$

This allows us to establish a property of N^0:

Claim 1 $N^0 \neq \emptyset$.

Proof. If $N^0 = \emptyset$, then we have $g(N, y) = (a + b(N^+))y + c(N^+) + \binom{|N^+|}{2}a$ by (15). Since $|N^+|a + b(N^+) + c(N^+) = 0$, by (12), we get $g(N, 1) = \binom{|N^+|-1}{2}a \geq 0$ by (8), and $g(N, 0) = c(N^+) + \binom{|N^+|}{2}a \geq 0$ by (8) and (9). This contradicts (7) and proves the claim. □

The following two claims distinguish two cases: $N^+ = \emptyset$, and $N^+ \neq \emptyset$.

Claim 2 *Theorem 1 holds for $n = 3$ when $N^+ = \emptyset$.*

1. *Case $p_{12}, p_{13}, p_{23} > 0$. All quadratizations are of the form:*

$$\begin{aligned} g(x, y) = {} & (2 + p_{12} + p_{13} + p_{23})y \\ & - (1 + p_{12} + p_{13})x_1y - (1 + p_{12} + p_{23})x_2y - (1 + p_{13} + p_{23})x_3y \\ & + p_{12}x_1x_2 + p_{13}x_1x_3 + p_{23}x_2x_3, \end{aligned}$$

   which is never prime because $g(x, y) \geq s_3(x, y), \forall (x, y) \in \{0, 1\}^{3+1}$.

2. *Case $p_{12} > 0, p_{13}, p_{23} = 0$ (w.l.o.g.). All quadratizations are of the form:*

$$g(x, y) = (-b_1 - b_2 - p_{12})y + b_1x_1y + b_2x_2y - x_3y + p_{12}x_1x_2,$$

   where

   (2.1) $-b_2 - p_{12} \geq 1$,

   (2.2) $-b_1 - p_{12} \geq 1$,

   which is never prime because $g(x, y) \geq s_3(x, y), \forall (x, y) \in \{0, 1\}^{3+1}$.

3. *Case $p_{12}, p_{13} > 0, p_{23} = 0$ (w.l.o.g.). All quadratizations are of the form:*

$$g(x, y) = (1 - b_1)y + b_1x_1y - (1 + p_{12})x_2y - (1 + p_{13})x_3y + p_{12}x_1x_2 + p_{13}x_1x_3$$

   where

   (3.1) $-b_1 - p_{12} \geq 0$,

   (3.2) $-b_1 - p_{13} \geq 0$,

   (3.3) $-1 - b_1 - p_{12} - p_{13} \geq 0$,

   which is never prime because $g(x, y) \geq s_3(x, y), \forall (x, y) \in \{0, 1\}^{3+1}$.

4. *Case $p_{12}, p_{13}, p_{23} = 0$. All quadratizations are of the form:*

$$g(x, y) = (-1 - b_1 - b_2 - b_3)y + b_1x_1y + b_2x_2y + b_3x_3y$$

   where

   (4.1) $-1 - b_1 \geq 0$,

   (4.2) $-1 - b_2 \geq 0$,

   (4.3) $-1 - b_3 \geq 0$,

   which is never prime because $g(x, y) \geq s_3(x, y), \forall (x, y) \in \{0, 1\}^{3+1}$.

Proof.
Since $N^+ = \emptyset$,

$$g(x, y) = ay + \sum_{i=1}^{3} b_i x_i y + \sum_{1 \le i < j \le 3} p_{ij} x_i x_j. \tag{16}$$

By (14), $g(N, 0) = p_{12} + p_{13} + p_{23} \ge 0$ which implies that $g(N, 1) = -1$, or

$$g(N, 1) = a + b_1 + b_2 + b_3 + p_{12} + p_{13} + p_{23} = -1. \tag{17}$$

1. Case $p_{12}, p_{13}, p_{23} > 0$.
   By (14), we have the system of equations

   $$\begin{aligned} a + b_1 + b_2 + p_{12} &= 0, \\ a + b_1 + b_3 + p_{13} &= 0, \\ a + b_2 + b_3 + p_{23} &= 0. \end{aligned}$$

   Considering this system along with equation (17), and solving it as a function of p_{12}, p_{13}, p_{23}, we obtain that the general form (16) of the quadratization in this case is

   $$\begin{aligned} g(x, y) = {} & (2 + p_{12} + p_{13} + p_{23})y \\ & - (1 + p_{12} + p_{13})x_1 y - (1 + p_{12} + p_{23})x_2 y - (1 + p_{13} + p_{23})x_3 y \\ & + p_{12} x_1 x_2 + p_{13} x_1 x_3 + p_{23} x_2 x_3, \end{aligned}$$

   where $p_{12}, p_{13}, p_{23} > 0$.

   It can be easily checked that $g(x, y) - s_3(x, y) \ge 0$, $\forall (x, y) \in \{0, 1\}^{3+1}$, and therefore g is not prime.

2. Case $p_{12} > 0, p_{13} = p_{23} = 0$.
   By (14), we have the equation

   $$a + b_1 + b_2 + p_{12} = 0.$$

   Considering this equation along with equation (17), and solving the system as a function of b_1, b_2, p_{12}, we obtain that the general form (16) of the quadratization in this case is

   $$g(x, y) = (-b_1 - b_2 - p_{12})y + b_1 x_1 y + b_2 x_2 y - x_3 y + p_{12} x_1 x_2.$$

   For g to be a quadratization we also need

   $$g(\{1, 3\}, 1) = -b_2 - p_{12} - 1 \ge 0, \tag{18}$$
   $$g(\{2, 3\}, 1) = -b_1 - p_{12} - 1 \ge 0. \tag{19}$$

   Using conditions (18) and (19), it can be easily checked that $g(x, y) - s_3(x, y) \ge 0$, $\forall (x, y) \in \{0, 1\}^{3+1}$, and therefore g is not prime.

3. Case $p_{12}, p_{13} > 0$, $p_{23} = 0$.
   By (14), we have the system of equations

$$a + b_1 + b_2 + p_{12} = 0,$$
$$a + b_1 + b_3 + p_{13} = 0.$$

   Considering this system along with equation (17), and solving the system as a function of b_1, p_{12}, p_{13}, we obtain that the general form (16) of the quadratization in this case is

$$g(x,y) = (1-b_1)y + b_1x_1y - (1+p_{12})x_2y - (1+p_{13})x_3y + p_{12}x_1x_2 + p_{13}x_1x_3.$$

   For g to be a quadratization we also need

$$g(\{2,3\},1) = -1 - b_1 - p_{12} - p_{13} \geq 0, \tag{20}$$
$$g(\{2\},1) = -b_1 - p_{12} \geq 0, \tag{21}$$
$$g(\{3\},1) = -b_1 - p_{13} \geq 0. \tag{22}$$

   Using conditions (20), (21), (22) and $a = 1 - b_1 \geq 0$, it can be easily checked that $g(x,y) - s_3(x,y) \geq 0$, $\forall (x,y) \in \{0,1\}^{3+1}$, and therefore g is not prime.

4. Case $p_{12} = p_{13} = p_{23} = 0$.
   Equation (17) gives

$$g(N,1) = a + b_1 + b_2 + b_3 = -1.$$

   Using this equation to express a in terms of b_1, b_2 and b_3 in the general form (16) of the quadratization, we obtain

$$g(x,y) = (-1 - b_1 - b_2 - b_3)y + b_1x_1y + b_2x_2y + b_3x_3y.$$

   For g to be a quadratization we also need

$$g(\{1,2\},1) = -1 - b_3 \geq 0, \tag{23}$$
$$g(\{1,3\},1) = -1 - b_2 \geq 0, \tag{24}$$
$$g(\{2,3\},1) = -1 - b_1 \geq 0. \tag{25}$$

   Using conditions (23), (24), (25), it can be easily checked that $g(x,y) - s_3(x,y) \geq 0$, $\forall (x,y) \in \{0,1\}^{3+1}$, and therefore g is not prime. □

Claim 3 *Theorem 1 holds for $n = 3$ when $N^+ \neq \emptyset$. Since $N^0 \neq \emptyset$, there are two cases:*

1. *Case $c_1, c_2 > 0, c_3 = 0$ (w.l.o.g.). All quadratizations are of the form:*

$$g(x,y) = ay - (a+c_1)x_1y - (a+c_2)x_2y - (1+p_{13}+p_{23})x_3y + c_1x_1 + c_2x_2$$
$$ax_1x_2 + p_{13}x_1x_3 + p_{23}x_2x_3$$

where

(5.1) $c_1 + p_{13} \geq 0,$

(5.2) $-1 - p_{13} \geq 0,$

(5.3) $c_2 + p_{23} \geq 0,$

(5.4) $-1 - p_{23} \geq 0,$

which is never prime because $g(x,y) \geq s_3^+(x,y) \forall (x,y) \in \{0,1\}^{3+1}$

2. *Case $c_1 > 0, c_2 = c_3 = 0$ (w.l.o.g.). Then, any quadratization g satisfies $g(x,y) \geq s_3^+(x,\bar{y})$, $\forall (x,y) \in \{0,1\}^{3+1}$.*

Proof.

In this case, the general form of the quadratization is

$$g(x,y) = ay + \sum_{i=1}^{3} b_i x_i y + \sum_{i=1}^{3} c_i x_i + \sum_{1 \leq i < j \leq 3} p_{ij} x_i x_j, \tag{26}$$

where $c_i = 0$ for at least one $i \in \{1,2,3\}$.

1. Case $c_1, c_2 > 0, c_3 = 0$.
   By (12) we obtain equations

$$a + b_1 + c_1 = 0, \tag{27}$$
$$a + b_2 + c_2 = 0. \tag{28}$$

By (15), $p_{12} = a \geq 0$.

For g to be a quadratization we need

$$g(\{1,3\},0) = c_1 + p_{13} \geq 0, \tag{29}$$
$$g(\{2,3\},0) = c_2 + p_{23} \geq 0. \tag{30}$$

Hence,

$$g(N,0) = c_1 + c_2 + a + p_{13} + p_{23} \geq 0. \tag{31}$$

Therefore, for g to be a quadratization we need $g(N,1) = -1$, i.e.,

$$g(N,1) = a + b_1 + b_2 + b_3 + c_1 + c_2 + a + p_{13} + p_{23} = -1. \tag{32}$$

Solving the system given by (27), (28) and (32), as a function of p_{13}, p_{23}, a, c_1 and c_2, the general form (26) of the quadratization becomes

$$\begin{aligned}g(x,y) =&ay-(a+c_1)x_1y-(a+c_2)x_2y-(1+p_{13}+p_{23})x_3y\\&+c_1x_1+c_2x_2\\&+ax_1x_2+p_{13}x_1x_3+p_{23}x_2x_3.\end{aligned}$$

For g to be a quadratization, we also need

$$g(\{1,3\},1) = -1-p_{23} \geq 0, \tag{33}$$
$$g(\{2,3\},1) = -1-p_{13} \geq 0. \tag{34}$$

Using conditions (29), (30), (33) and (34), it can be easily checked that $g(x,y) \geq s_3^+(x,y)$, $\forall (x,y)^{\{3+1\}}$, therefore g is not prime.

2. Case $c_1 > 0, c_2 = c_3 = 0$.
   By (12) we obtain equation

$$a + b_1 + c_1 = 0, \tag{35}$$

and by (13),

$$a + b_2 \geq 0, \tag{36}$$
$$a + b_3 \geq 0. \tag{37}$$

Using (35), we obtain the following conditions for g to be a quadratization,

$$g(\{1,2\},0) = c_1 + p_{12} \geq 0, \tag{38}$$
$$g(\{1,3\},0) = c_1 + p_{13} \geq 0, \tag{39}$$

and

$$g(\{1,2\},1) = b_2 + p_{12} \geq 0, \tag{40}$$
$$g(\{1,3\},1) = b_3 + p_{13} \geq 0. \tag{41}$$

Equations (38) and (40), (39) and (41), respectively imply

$$\min\{c_1 + p_{12}, b_2 + p_{12}\} = 0, \tag{42}$$
$$\min\{c_1 + p_{13}, b_3 + p_{13}\} = 0. \tag{43}$$

For $i \in \{2,3\}$, we say that

- $i \in B$ if $b_i + p_{1i} = 0$, and
- $i \in C$ if $c_1 + p_{1i} = 0$,

in equations (42)-(43).

We will now show that $p_{23} = 0$.
First, note that by (14),

$$\min\{p_{23}, a + b_2 + b_3 + p_{23}\} = 0. \tag{44}$$

Now, (44), (35) and (42)-(43), imply that

$$g(N, 1) = b_2 + b_3 + p_{12} + p_{13} + p_{23} \geq 0. \tag{45}$$

Therefore, for g to be a quadratization we need

$$g(N, 0) = c_1 + p_{12} + p_{13} + p_{23} = -1. \tag{46}$$

Assume now that $p_{23} > 0$. Then, (44) implies that $a + b_2 + b_3 + p_{23} = 0$. Together with (36) and (37), this implies $b_2 < 0$ and $b_3 < 0$. From (42), $p_{12} \geq -b_2 > 0$ and from (43), $p_{13} \geq -b_3 > 0$. Since $p_{12}, p_{13}, p_{23}, c_1 > 0$, we have a contradiction with (46).

Therefore, we can assume from now on that $p_{23} = 0$. Then, (46) reduces to

$$g(N, 0) = c_1 + p_{12} + p_{13} = -1. \tag{47}$$

By (42)-(43), we get $2c_1 + p_{12} + p_{13} \geq 0$ and hence, in view of (47),

$$c_1 \geq 1. \tag{48}$$

We distinguish now among several subcases.

- **Case 1.** If $C = \{2, 3\}$, then (47) implies $c_1 = 1$ and $p_{12} = p_{13} = -1$. With these values, (35) becomes $b_1 = -1 - a$, and (42)-(43) become $b_2 \geq 1, b_3 \geq 1$.

  Moreover, the general form (26) of the quadratization becomes

  $$g(x, y) = ay + (-1 - a)x_1y + b_2x_2y + b_3x_3y + x_1 - x_1x_2 - x_1x_3. \tag{49}$$

  Compare this expression with

  $$s_3^+(x, \bar{y}) = x_1 - x_1y - x_2(x_1 - y) - x_3(x_1 - y),$$

  (where x_1 plays the role of x_3).

  We obtain

  $$g(x, y) - s_3^+(x, \bar{y}) = a\bar{x}_1y + (b_2 - 1)x_2y + (b_3 - 1)x_3y \geq 0,$$

  and g is not prime.

- **Case 2.** If $2 \in B$ and $3 \in C$, then by definition, $p_{12} = -b_2$ and $p_{13} = -c_1$. Then, (47) implies $p_{12} = -b_2 = -1$. Let us substitute the values $p_{12} = -1$, $b_2 = 1$, $p_{13} = -c_1$ and $b_1 = -c_1 - a$ in the general form (26) of the quadratization,

  $$g(x, y) = ay + (-c_1 - a)x_1 y + x_2 y + b_3 x_3 y + c_1 x_1 - x_1 x_2 - c_1 x_1 x_3. \quad (50)$$

  When $y = 0$, this yields (taking (48) into account),

  $$g(x, 0) = c_1 x_1 \bar{x}_3 - x_1 x_2 \geq x_1 \bar{x}_3 - x_1 x_2 = s_3^+(x, 1).$$

  When $y = 1$,

  $$\begin{aligned} g(x, 1) &= a - ax_1 + x_2 + b_3 x_3 - x_1 x_2 - c_1 x_1 x_3 \\ &= a\bar{x}_1 + \bar{x}_1 x_2 + (b_3 - c_1)x_3 + c_1 \bar{x}_1 x_3. \end{aligned}$$

  Note that $a \geq 0$, $b_3 + p_{13} = b_3 - c_1 \geq 0$ by (43), and $c_1 \geq 1$ by (48). So,

  $$g(x, 1) \geq \bar{x}_1 x_2 + \bar{x}_1 x_3 = s_3^+(x, 0).$$

  Obtaining that $g(x, y) \geq s_3^+(x, \bar{y})$, and g is not prime.
- **Case 3.** Assume finally that $B = \{2, 3\}$, meaning that $p_{12} = -b_2$ and $p_{13} = -b_3$. Substituting in (47) yields $c_1 - b_2 - b_3 = -1$, and equations (42)-(43) imply $c_1 - b_2 \geq 0$ and $c_1 - b_3 \geq 0$. From these relations we deduce

  $$b_2 \geq 1, b_3 \geq 1. \quad (51)$$

  With $p_{12} = -b_2$, $p_{13} = -b_3$, $c_1 = b_2 + b_3 - 1$ and $b_1 = -a - c_1 = -a - b_2 - b_3 + 1$, the general form (26) of the quadratization becomes

  $$g(x, y) = ay + (-a - b_2 - b_3 + 1)x_1 y + b_2 x_2 y + b_3 x_3 y + (b_2 + b_3 - 1)x_1 - b_2 x_1 x_2 - b_3 x_1 x_3.$$

  When $y = 0$, and considering (51),

  $$\begin{aligned} g(x, 0) &= (b_2 + b_3 - 1)x_1 - b_2 x_1 x_2 - b_3 x_1 x_3 \\ &= b_2 x_1 \bar{x}_2 + b_3 x_1 \bar{x}_3 - x_1 \\ &\geq x_1 \bar{x}_2 + x_1 \bar{x}_3 - x_1 = s_3^+(x, 1). \end{aligned}$$

  When $y = 1$,

  $$\begin{aligned} g(x, 1) &= a - ax_1 + b_2 x_2 + b_3 x_3 - b_2 x_1 x_2 - b_3 x_1 x_3 \\ &= a\bar{x}_1 + b_2 \bar{x}_1 x_2 + b_3 \bar{x}_1 x_3 \\ &\geq \bar{x}_1 x_2 + \bar{x}_1 x_3 = s_3^+(x, 0). \end{aligned}$$

  Obtaining that $g(x, y) \geq s_3^+(x, \bar{y})$, and g is not prime. □

We have covered all cases for $N^+ = \emptyset$ and for $N^+ \neq \emptyset$. As the theorem states, we have seen that the only possibilities for prime quadratizations using one auxiliary variable of the cubic negative monomial are s_3 or s_3^+. □

3 Conclusion

Quadratization techniques are aimed at transforming a general pseudo-Boolean function expressed as a multilinear polynomial into a quadratic function, in order to apply quadratic pseudo-Boolean optimization algorithms which have been well-studied in both exact and heuristic approaches. Quadratizations of negative monomials are particularly interesting because they allow using techniques such as *termwise quadratization*, which can be applied to any pseudo-Boolean function expressed as a multilinear polynomial.

This technical report presented a proof of the theorem of Anthony, Boros, Crama and Gruber [1], characterizing short prime quadratizations for cubic negative monomials. The proof for the cubic case is based on the proof for the general case $n \geq 4$ of the cited article. However the case study is different for $n = 3$, and requires the exhaustive analysis presented in this report.

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