Free Group and Recognizability

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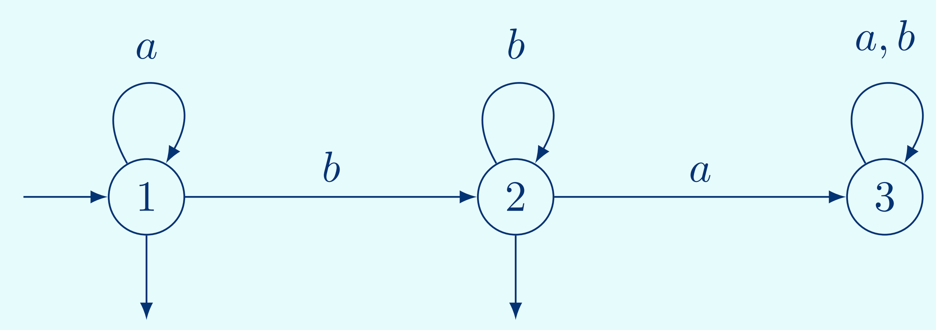
Recognizability

Let Σ be a finite alphabet. A language over Σ is *regular* or *recognizable* if it is accepted by a finite automaton (or, equivalently, if it is described by a regular expression).

a^*b^*



Theorem. A language $L \subset \Sigma^*$ is recognizable if and only if there exists a *finite* monoid M, a subset P of M and a homomorphism $h : \Sigma^* \to M$ such that $L = h^{-1}(P)$.

Generalization. If $(A, (f_i)_{i\in I})$ is an algebra of type $\tau = (n_i)_{i\in I}$ (i.e. $f_i : A^{n_i} \to A$), then $L \subset A$ is *recognizable* if and only if there exists a finite algebra B of type τ, a subset P of B and a homomorphism $h : A \to B$ such that $L = h^{-1}(P)$. This happens if and only if L is saturated by a finite index congruence (first isomorphism theorem).

Free Group

The free group $F(X)$ generated by X is the unique (up to isomorphism) group such that $F(X) = \langle \eta(X) \rangle$ for an injective function $\eta : X \to F(X)$ and such that for all group G and function $f : X \to G$, there exists a unique homomorphism $h : F(X) \to G$ such that $f = h \circ \eta$.

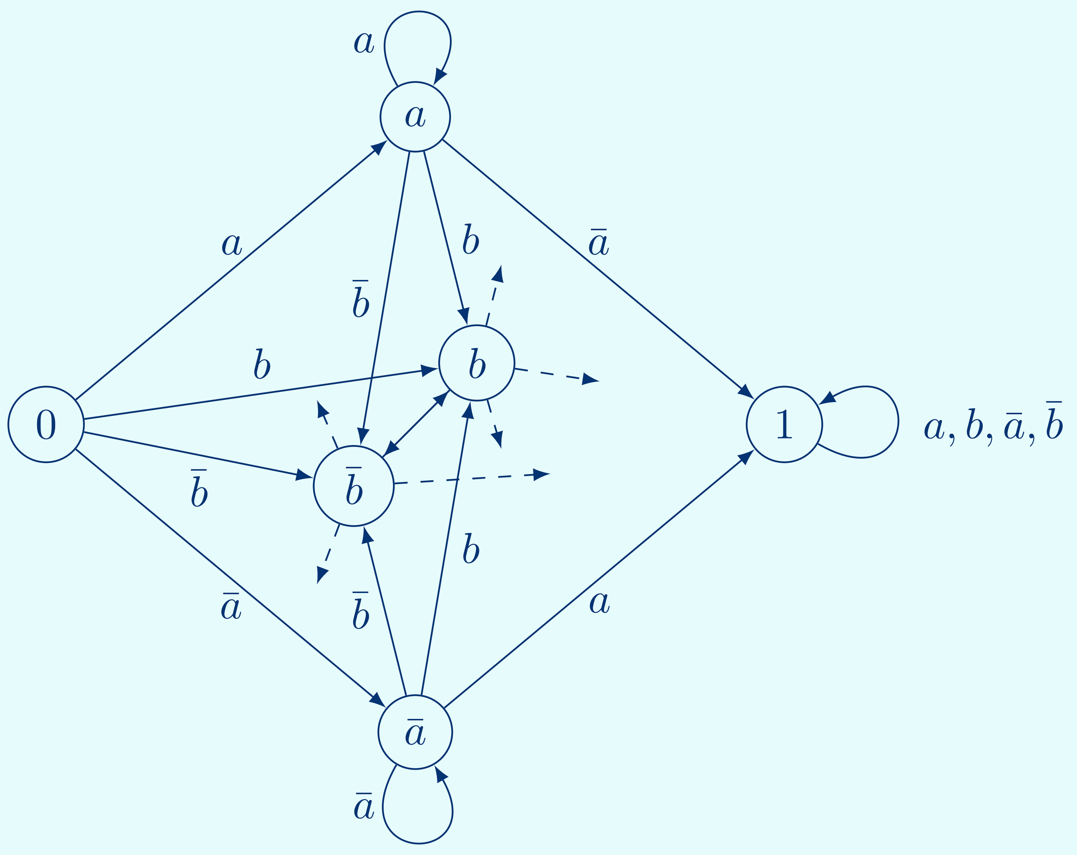
Construction. Let $X = \{x_1, \ldots, x_n\}$ be a finite set, and let $\cdot^{-1}$ be a bijection between X and a set $X' = X^{-1}$ such that $X \cap X' = \emptyset$. By abusing the notation, the inverse bijection is also noted $\cdot^{-1}$, and we note $\bar{X} = X \cup X'$. A word $w \in \bar{X}^*$ is *reduced* if it does not contain any factor of the form xx^{-1}, $x \in \bar{X}$. We note $r(w)$ the reduced word obtained from w by deleting factors of the form xx^{-1} while it is possible.

The free group $F(X)$ is the set of reduced words of $\bar{X}^*$ equipped with the operation

$$\times : F(X)^2 \to F(X) : (u, v) \mapsto r(uv).$$

Its identity element is the empty word ε, and the inverse of $x_1 \cdots x_n$ is $x_n^{-1} \cdots x_1^{-1}$.

As a subset of $\bar{X}^*$, the set $F(X)$ is recognizable.



e-Groups

An *e-group* is a monoid $(E, \cdot, 1)$ endowed with two additional unary operations $\cdot^{-1}$ and r satisfying

- $(xy)^{-1} = y^{-1}x^{-1}$
- $r(xx^{-1}) = r(x^{-1}x) = 1$
- $r(x^{-1}) = r(x)^{-1}$
- $(x^{-1})^{-1} = x$
- $r(xr(y)z) = r(xyz)$

If $(x \equiv y \Leftrightarrow r(x) = r(y))$, the quotient $E/\!\equiv$ is a group. It is isomorphic to the group $r(E)$ with multiplication $\times : (x, y) \mapsto r(xy)$. We call $r(E)$ the *kernel* of E. Note that $F(X)$ is the kernel of $\bar{X}^*$.
If $(G, \cdot, 1, ^{-1})$ is a group, then $(G, \cdot, 1, ^{-1}, \mathrm{id}_G)$ is an e-group, and both have the same congruences and thus the same recognizable subsets.

Recognizability in kernels. If $h : A \to A'$ is a surjective homomorphism between two algebras, then $L \subset A'$ is recognizable if and only if $h^{-1}(L)$ is recognizable. Therefore, if $(E, \cdot, 1, ^{-1}, r)$ is an e-group, then $L \subset r(E)$ is recognizable in the kernel group $r(E)$ if and only if $r^{-1}(L)$ is recognizable in E (as r is a homomorphism between E and the e-group $(r(E), \times, 1, ^{-1}, \mathrm{id}_{r(E)})$). Furthermore, if $M \subset E$ is such that $M = r^{-1}(r(M))$, then M is recognizable in the e-group $(E, \cdot, 1, ^{-1}, r)$ if and only if it is recognizable in the monoid $(E, \cdot, 1)$. This leads us to the following theorem.
Theorem. A subset L of $r(E)$ is recognizable in $(r(E), \times, 1, ^{-1})$ if and only if $r^{-1}(L)$ is recognizable in $(E, \cdot, 1)$.
Corollary. A subset L of $F(X)$ is recognizable if and only if $r^{-1}(L)$ is recognizable in $\bar{X}^*$.

e-Automata

An automaton $\mathcal{A} = (Q, q_0, F, \delta)$ over $\bar{X}$ is an *e-automaton* over X if $Q \backslash \{q_0\}$ can be partitioned into disjoint sets $(Q_x)_{x\in\bar{X}}, P, N$ such that

- $q \notin N \Rightarrow \delta(q, x) \in Q_x \cup P$
- $q \in Q_x \Rightarrow ((\exists q' \notin N\ \delta(q', x) = q)$ and $(\delta(q, x^{-1}) \in P))$
- $P \cap F = \emptyset$ and $p \in P \Rightarrow \delta(p, x) \in P$
- $\delta(q, x) \in P \Rightarrow q \in Q_{x^{-1}} \cup P \cup N$
- The states of N are unreachable
- $q, q' \notin P \cup N, \delta(q, x) = \delta(q', x) \notin P \Rightarrow \forall w \in \bar{X}^* \left(q \xrightarrow[\bar{\mathcal{A}}]{w} F \Leftrightarrow q' \xrightarrow[\bar{\mathcal{A}}]{w} F \right)$

where $\bar{\mathcal{A}}$ is the non-deterministic automaton $(Q, \{q_0\}, F, \Delta)$ with

$$\Delta = \{(q, x, q') | \delta(q, x) = q^{-1} \text{ or } (q \notin P \text{ and } q' \notin N \text{ and } \delta(q, x^{-1}) = q)\}.$$

Theorem. A subset L of $F(X)$ is recognizable if and only if it is accepted by an e-automaton.