A R C H I V E O F M E C H A N I C A L E N G I N E E R I N G

VOL. LXI 2014 Number 2

10.2478/meceng-2014-0018

Key words: dynamic beam, finite element, Lie group, special Euclidean group

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GEOMETRIC INTERPRETATION OF A NON-LINEAR BEAM FINITE ELEMENT ON THE LIE GROUP SE(3)

Recently, the authors proposed a geometrically exact beam finite element formulation on the Lie group SE(3). Some important numerical and theoretical aspects leading to a computationally efficient strategy were obtained. For instance, the formulation leads to invariant equilibrium equations under rigid body motions and a locking free element. In this paper we discuss some important aspects of this formulation. The invariance property of the equilibrium equations under rigid body motions is discussed and brought out in simple analytical examples. The discretization method based on the exponential map is recalled and a geometric interpretation is given. Special attention is also dedicated to the consistent interpolation of the velocities.

1. Introduction

Rotational variables are widely used in structural mechanics as they are particularly convenient to handle the kinematic assumptions involved in structural elements such as rigid bodies, beams or shells [1, 2, 3, 4]. Nevertheless, rotations belong to a non-linear space, $SO(3)$, so that they must be handled carefully. Several methods have been explored to represent rotation variables in multibody systems such as the parameterization of rotation[1], the director vector method [5] or the Lie group methods [3, 6, 7, 8]. This paper is concerned with a beam formulation recently proposed by the authors [9] which is based on the Lie group formalism of the special Euclidean group $SE(3)$ and thus differs from the classical approach on $\mathbb{R}^3 \times SO(3)$.

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The invariance of a formulation with respect to rigid body motions, namely the solution is equivalent whatever the rigid body motion of the reference frame, is of fundamental importance and is satisfied by classical geometrically exact beam formulations, e.g. [3, 5, 10, 11, 12, 13]. Using the special Euclidean group formalism, the translation and rotation variables are inherently coupled and the deformations and the velocities are naturally expressed in a material frame. As a consequence, the formulation is naturally invariant under rigid body motion but in addition the equilibrium equations themselves, i.e. the numerical expression of the internal and inertia forces and of the tangent stiffness and mass matrices, are invariant under rigid body motions. The non-linearities in the equilibrium equations are thus reduced, which is valuable from a computational point of view. In particular, it is shown through simple examples that the solution to geometrically non-linear problems can be obtained by solving the equilibrium equations in terms of deformations and velocities only, without having to refer to the actual position or orientation of the beam. In a second step, the position and orientation can be obtained by solving kinematic compatibility equations. This is similar to the well-known fact that Euler equations for a rigid body moving freely in space can be solved for the velocities expressed in the material frome, without having to consider the actual position and orientation of the body.

In order to solve general problems, a finite element method that preserves the invariance property of the equilibrium equations with respect to rigid body motions shall be considered. A finite element discretization relying on the exponential map of the special Euclidean group was introduced in [9] and a consistent velocity interpolation was derived. In this paper, the discretization method presented in [9] is recalled and a geometric interpretation is given. The consistent interpolation of the velocities is also addressed. It is shown that the Lie bracket relationship is automatically satisfied, which reveals the consistency of the proposed framework. The resulting discrete equilibrium equations are second-order ordinary differential equations on the Lie group and can be solved without introducing any global parameterization of the motion thanks to the use of Lie group integrators, such as the generalized-α scheme presented in [6, 8]. From a computational point of view, solving equilibrium equations which are invariant under rigid body motion reduces drastically the computation costs. In particular, geometrically non-linear problems can be solved using implicit schemes without updating the iteration matrix.

The paper is structured as follows. In Section 2, some fundamentals about the special Euclidean group are given. In Section 3.1., the beam kinematics is described and the continuous equations of motion in the $SE(3)$ context are given. The invariance not only of the results but of the equilibrium equations themselves under rigid body motions is discussed through simple examples. The spatial and time discretization method based on the exponential map of the special Euclidean group is discussed in Section 4. Some computational aspects are then discussed in Section 4.4. and illustrated by an academic numerical example. Finally, some conclusions and perspectives are presented in Section 5.

2. Fundamentals about the special Euclidean group $SE(3)$

The special Euclidean group $SE(3)$ is a matrix Lie group [14, 15, 16] whose elements can be represented by 4×4 matrices

$$\mathbf{H} = \mathcal{H}(\mathbf{R}, \mathbf{x}) = \begin{bmatrix} \mathbf{R} & \mathbf{x} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^3$ and $\mathbf{R}$ is a rotation matrix (i.e. it belongs to the special Orthogonal group $SO(3)$, which is defined as the set of 3×3 matrices such that $\mathbf{R}^T\mathbf{R} = \mathbf{I}_{3\times 3},\ \det(\mathbf{R}) = +1$). Considering that $\mathbf{x}$ is a position or displacement vector and that a rotation matrix defines an orientation, $SE(3)$ elements represent frames. Being a group, there is a composition rule on $SE(3)$, namely the matrix product of two such 4×4 matrices, which reads

$$\mathbf{H}_3 = \mathcal{H}(\mathbf{R}_3, \mathbf{x}_3) = \mathbf{H}_1\mathbf{H}_2 = \mathcal{H}(\mathbf{R}_1\mathbf{R}_2, \mathbf{R}_1\mathbf{x}_2 + \mathbf{x}_1) \tag{2}$$

where it is interesting to notice that $\mathbf{x}_3$ involves $\mathbf{R}_1$ as well as $\mathbf{x}_1$ and $\mathbf{x}_2$. Geometrically speaking, $\mathbf{H}_3$ is thus interpreted as the result of the frame transformation $\mathbf{H}_2$ from the frame defined by $\mathbf{H}_1$. The neutral element for this matrix product is the 4×4 identity matrix $\mathbf{I}_{4\times 4}$ and the inverse of $\mathbf{H} \in SE(3)$, $\mathbf{H}^{-1} \in SE(3)$, is given by

$$\mathbf{H}^{-1} = \mathcal{H}(\mathbf{R}^T, -\mathbf{R}^T\mathbf{x}) = \begin{bmatrix} \mathbf{R}^T & -\mathbf{R}^T\mathbf{x} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \tag{3}$$

Derivatives on $SE(3)$ with respect to any parameter α can be introduced by a left invariant vector field as

$$d_\alpha(\mathbf{H}) = \mathbf{H}\widetilde{\mathbf{h}} \tag{4}$$

where, denoting $T_{\mathbf{H}}SE(3)$ the tangent space at $\mathbf{H}$, $d_\alpha(\mathbf{H}) :\ SE(3) \to T_{\mathbf{H}}SE(3)$. The 4×4 matrix $\widetilde{\mathbf{h}}$ belongs to $\mathfrak{se}(3)$, the Lie algebra of $SE(3)$, which is the tangent space at the identity. $\mathfrak{se}(3)$ is isomorphic to $\mathbb{R}^6$ which means that $\widetilde{\bullet}$ is an invertible linear map from $\mathbb{R}^6$ to $\mathfrak{se}(3)$ such that

$$\mathbf{h} = \begin{bmatrix} \mathbf{h}_U \\ \mathbf{h}_\Omega \end{bmatrix} \in \mathbb{R}^6 \quad \text{and} \quad \widetilde{\mathbf{h}} = \begin{bmatrix} \widetilde{\mathbf{h}}_\Omega & \mathbf{h}_U \\ \mathbf{0}_{1\times 3} & 0 \end{bmatrix} \in \mathfrak{se}(3) \tag{5}$$

where $\mathbf{h}_U \in \mathbb{R}^3$ and $\widetilde{\mathbf{h}}_\Omega \in \mathfrak{so}(3)$, the Lie algebra of the special Orthogonal group $SO(3)$, which is the set of skew-symmetric matrices built upon the three components of $\mathbf{h}_\Omega \in \mathbb{R}^3$ as

$$\widetilde{\mathbf{h}}_\Omega = \begin{bmatrix} 0 & -h_{\Omega 3} & h_{\Omega 2} \\ h_{\Omega 3} & 0 & -h_{\Omega 1} \\ -h_{\Omega 2} & h_{\Omega 1} & 0 \end{bmatrix} \in \mathfrak{so}(3) \tag{6}$$

It is clear from the argument whether the tilde operator denotes the mapping to $\mathfrak{so}(3)$ or $\mathfrak{se}(3)$. From Eq. (4), we have

$$d_\alpha(\mathbf{R}) = \mathbf{R}\widetilde{\mathbf{h}}_\Omega \quad \text{and} \quad d_\alpha(\mathbf{x}) = \mathbf{R}\mathbf{h}_U \tag{7}$$

where $d_\alpha(\mathbf{R}) : \ SO(3) \to T_\mathbf{R}SO(3)$ and $d_\alpha(\mathbf{x}) : \ \mathbb{R}^3 \to \mathbb{R}^3$. It is important to notice that both derivatives involve the rotation matrix of the frame whence the derivatives are interpreted as taking place in the frame described by $\mathbf{H}$. $\mathfrak{se}(3)$ elements have a so-called adjoint representation. Observing that $\widetilde{\mathbf{h}}_2 = \mathbf{H}^{-1}\widetilde{\mathbf{h}}_1\mathbf{H} \in \mathfrak{se}(3)$, the adjoint representation acting on vectors of $\mathbb{R}^6$ is defined as

$$\mathbf{h}_2 = \mathrm{Ad}_\mathbf{H}(\mathbf{h}_1) = \begin{bmatrix} \mathbf{R} & \widetilde{\mathbf{x}}\mathbf{R} \\ \mathbf{0}_{3\times 3} & \mathbf{R} \end{bmatrix} \mathbf{h}_1 \tag{8}$$

where $\mathrm{Ad}_\mathbf{H}(\mathbf{h}) : \ \mathfrak{se}(3) \to \mathfrak{se}(3)$. Considering the commutativity of the derivatives of the variable $\mathbf{H}$ when treated as a matrix in $\mathbb{R}^4 \times \mathbb{R}^4$, $d_\alpha(d_\beta(\mathbf{H})) = d_\beta(d_\alpha(\mathbf{H}))$ together with the representation of the derivatives on $SE(3)$ with $d_\alpha(\mathbf{H}) = \mathbf{H}\widetilde{\mathbf{h}}_\alpha$ and $d_\beta(\mathbf{H}) = \mathbf{H}\widetilde{\mathbf{h}}_\beta$ yields

$$d_\alpha(\widetilde{\mathbf{h}}_\beta) = d_\beta(\widetilde{\mathbf{h}}_\alpha) + [\widetilde{\mathbf{h}}_\beta, \widetilde{\mathbf{h}}_\alpha] \tag{9}$$

where $[\widetilde{\mathbf{h}}_\beta, \widetilde{\mathbf{h}}_\alpha] = \widetilde{\mathbf{h}}_\beta\widetilde{\mathbf{h}}_\alpha - \widetilde{\mathbf{h}}_\alpha\widetilde{\mathbf{h}}_\beta = -[\widetilde{\mathbf{h}}_\alpha, \widetilde{\mathbf{h}}_\beta]$ defines the Lie bracket operator $[\bullet, \bullet]$ which does not vanish in general. Eq. (9) can be expressed in terms of the vectors in $\mathbb{R}^6$ as

$$d_\alpha(\mathbf{h}_\beta) = d_\beta(\mathbf{h}_\alpha) + \widehat{\mathbf{h}}_\beta\mathbf{h}_\alpha \tag{10}$$

where the $\widehat{\bullet}$ operator is defined as

$$\widehat{\mathbf{h}} = \begin{bmatrix} \widetilde{\mathbf{h}}_\Omega & \widetilde{\mathbf{h}}_U \\ \mathbf{0}_{3\times 3} & \widetilde{\mathbf{h}}_\Omega \end{bmatrix} \tag{11}$$

Notice that $\widehat{\mathbf{h}}_\beta\mathbf{h}_\alpha = -\widehat{\mathbf{h}}_\alpha\mathbf{h}_\beta$.

Starting from a given six-dimensional vector $\mathbf{n} = [\mathbf{n}_U^T \ \mathbf{n}_\Omega^T]^T$, a $SE(3)$ element can be built using the so-called exponential map $\exp_{SE(3)}(\widetilde{\mathbf{n}}) : \ \mathfrak{se}(3) \rightarrow SE(3)$ or equivalently $\exp_{SE(3)}(\mathbf{n}) : \ \mathbb{R}^6 \rightarrow SE(3)$ which is defined as

$$\exp_{SE(3)}(\mathbf{n}) = \sum_{i=0}^{\infty} \frac{\widetilde{\mathbf{n}}^i}{i!} \tag{12}$$

The exponential map may be seen as a local parameterization in the sense that the argument of the exponential map belongs to a linear space while $SE(3)$ is a non-linear space. In practice, it means that standard vector calculus applies to the argument of the exponential map, such as the multiplication by a scalar or the addition of another six-dimensional vector and that its effect can be projected onto the group. Exploiting the Lie algebra structure, the exponential map can be expressed in a closed form as

$$\exp_{SE(3)}(\mathbf{n}) = \begin{bmatrix} \exp_{SO(3)}(\mathbf{n}_\Omega) & \mathbf{T}_{SO(3)}^T(\mathbf{n}_\Omega)\mathbf{n}_U \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \tag{13}$$

where, using $a = \sin(||\mathbf{n}_\Omega||)/||\mathbf{n}_\Omega||$ and $b = 2(1 - \cos(||\mathbf{n}_\Omega||))/||\mathbf{n}_\Omega||^2$,

$$\exp_{SO(3)}(\mathbf{n}_\Omega) = \mathbf{I}_{3\times 3} + a\widetilde{\mathbf{n}}_\Omega + \frac{b}{2}\widetilde{\mathbf{n}}_\Omega^2 \tag{14}$$

$$\mathbf{T}_{SO(3)}(\mathbf{n}_\Omega) = \mathbf{I}_{3\times 3} - \frac{b}{2}\widetilde{\mathbf{n}}_\Omega + \frac{1-a}{||\mathbf{n}_\Omega||^2}\widetilde{\mathbf{n}}_\Omega^2 \tag{15}$$

where $\exp_{SO(3)}(\mathbf{n}) : \ \mathbb{R}^3 \rightarrow SO(3)$ is the exponential map of the special Orthogonal group and $\mathbf{T}_{SO(3)}(\mathbf{n})$ is the tangent operator which defines an map from $\mathbb{R}^3$ to $\mathbb{R}^3$. The inverse map of the exponential map is called the logarithmic map, namely $\log_{SE(3)}(\mathbf{H}) : \ SE(3) \rightarrow \mathfrak{se}(3)$. It is explicitly given as

$$\log_{SE(3)}(\mathcal{H}(\mathbf{R}, \mathbf{x})) = \begin{bmatrix} \widetilde{\boldsymbol{\omega}} & \mathbf{T}_{SO(3)}^{-T}(\boldsymbol{\omega})\mathbf{x} \\ \mathbf{0}_{1\times 3} & 0 \end{bmatrix} \tag{16}$$

where $\widetilde{\boldsymbol{\omega}} = \log_{SO(3)}(\mathbf{R})$ with $\log_{SO(3)}(\mathbf{R}) : SO(3) \rightarrow \mathfrak{so}(3)$ is defined as

$$\log_{SO(3)}(\mathbf{R}) = \begin{cases} \dfrac{\theta}{2\sin(\theta)}(\mathbf{R} - \mathbf{R}^T) & \text{if } \theta \neq 0 \\ \widetilde{\mathbf{0}} & \text{otherwise} \end{cases} \tag{17}$$

with $\theta = \operatorname{acos}\left(\frac{1}{2}(\operatorname{trace}(\mathbf{R}) - 1)\right)$, $|\theta| < \pi$. The derivative of the exponential map introduces the so-called tangent operator $\mathbf{T}_{SE(3)}$ which defines a map from $\mathbb{R}^6$ to $\mathbb{R}^6$. Consider the transformation from $\mathbf{H}_0$ to $\mathbf{H}$ as $\mathbf{H} = \mathbf{H}_0 \exp_{SE(3)}(\mathbf{n})$. The derivative of $\mathbf{H}$ with respect to any parameter α leads to $d_\alpha(\mathbf{H}) = \mathbf{H}_0 \mathrm{Dexp}(\mathbf{n}) d_\alpha(\widetilde{\mathbf{n}}) = \mathbf{H} \exp^{-1}_{SE(3)}(\mathbf{n}) \mathrm{Dexp}(\mathbf{n}) d_\alpha(\widetilde{\mathbf{n}})$ where Dexp is the derivative of the exponential map, and this can be written as

$$d_\alpha(\mathbf{H}) = \mathbf{H} \widetilde{\left(\mathbf{T}_{SE(3)}(\mathbf{n})\ d_\alpha(\mathbf{n})\right)} \tag{18}$$

and $\mathbf{T}_{SE(3)}$ and its inverse are given by

$$\mathbf{T}(\mathbf{x}) = \sum_{i=0}^{\infty} (-1)^i \frac{\widehat{\mathbf{x}}^i}{(i+1)!}; \quad \mathbf{T}^{-1}(\mathbf{x}) = \sum_{i=0}^{\infty} (-1)^i B_i \frac{\widehat{\mathbf{x}}^i}{i!} \tag{19}$$

in which B_i are the Bernoulli numbers of first kind. Exploiting the Lie algebra structure, it can be expressed in a closed form as

$$\mathbf{T}_{SE(3)}(\mathbf{n}) = \begin{bmatrix} \mathbf{T}_{SO(3)}(\mathbf{n}_\Omega) & \mathbf{T}_{U\Omega+}(\mathbf{n}_U, \mathbf{n}_\Omega) \\ \mathbf{0}_{3\times3} & \mathbf{T}_{SO(3)}(\mathbf{n}_\Omega) \end{bmatrix} \tag{20}$$

where $\mathbf{T}_{SO(3)}$, the tangent operator of the exponential map on $SO(3)$, was given in Eq. (15) and where $\mathbf{T}_{U\Omega+}(\mathbf{n}_U, \mathbf{n}_\Omega)$ reads

$$\begin{aligned} \mathbf{T}_{U\Omega+}(\mathbf{n}_U, \mathbf{n}_\Omega) = \; & \frac{-b}{2}\widetilde{\mathbf{n}}_U + \frac{1-a}{\|\mathbf{n}_\Omega\|^2}(\widetilde{\mathbf{n}}_U\widetilde{\mathbf{n}}_\Omega + \widetilde{\mathbf{n}}_\Omega\widetilde{\mathbf{n}}_U) \\ & + \frac{\mathbf{n}_\Omega^T\mathbf{n}_U}{\|\mathbf{n}_\Omega\|^2}\Big((b-a)\widetilde{\mathbf{n}}_\Omega + (\frac{b}{2} - \frac{3(1-a)}{\|\mathbf{n}_\Omega\|^2})\widetilde{\mathbf{n}}_\Omega^2\Big) \end{aligned} \tag{21}$$

with a and b as defined for Eq. (15). Notice that $\mathbf{T}_{U\Omega+}(\mathbf{n}_U, \mathbf{0}) = -\widetilde{\mathbf{n}}_U/2$. Using the notation $d_\alpha(\mathbf{H}) = \mathbf{H}\widetilde{\mathbf{h}}$, one has

$$\mathbf{h} = \mathbf{T}_{SE(3)}(\mathbf{n})\ d_\alpha(\mathbf{n}) \tag{22}$$

which relates the derivatives of $\mathbf{H}$, namely $\mathbf{h}$, and the derivative of $\mathbf{n}$ which is used in the representation of $\mathbf{H}$ as $\mathbf{H} = \mathbf{H}_0 \exp_{SE(3)}(\mathbf{n})$.

Based on the formalism of this section, the rest of the paper addresses the formulation of a beam finite element on $SE(3)$. As it can be observed in the composition rule in Eq. (2) or in the expression of the derivatives in Eq. (7), the special Euclidean group $SE(3)$ exhibits a coupling of the position part and the rotation part. This coupling appears as a sound foundation to build a beam formulation since, for example in a cantilever beam, it is expected that the cross-sections rotate under a tip shear load and that a tip bending moment induces a displacement of the neutral axis.

3. Beam formulation

3.1. Beam kinematics

Let us define $s \in [0, L]$ as the longitudinal coordinate along the neutral axis of a beam of length L and t and u as the cross-section coordinates. In the $SE(3)$ formalism, a material frame is attached to any material point and any point of the neutral axis of the beam is described by a mapping $\mathbb{R} \to SE(3) : s \mapsto \mathbf{H}(s) = \mathcal{H}(\mathbf{R}(s), \mathbf{x}(s))$, that is

$$\mathbf{H}(s) = \begin{bmatrix} \mathbf{R}(s) & \mathbf{x}(s) \\ \mathbf{0}_{3\times 1} & 1 \end{bmatrix} \tag{23}$$

where $\mathbf{x}(s)$ is the position vector of the neutral axis and $\mathbf{R}(s)$ characterizes the orientation of the cross-section. Accordingly, the material frame $\mathbf{H}_p(s, t, u) = \mathcal{H}(\mathbf{R}_p(s), \mathbf{x}_p(s, t, u))$ at any beam point p of coordinates (s, t, u) is related to the material frame attached to the neutral axis by the frame transformation

$$\mathbf{H}_p(s, t, u) = \mathbf{H}(s) \begin{bmatrix} \mathbf{I}_{3\times 3} & \mathbf{O}_0 \mathbf{y}(t, u) \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} = \mathbf{H}(s)\mathbf{H}_{\mathbf{y}}(t, u) \tag{24}$$

in which $\mathbf{y}(t, u) = \begin{bmatrix} 0 & t & u \end{bmatrix}^T$ and $\mathbf{O}_0 = \begin{bmatrix} \mathbf{i}_s & \mathbf{i}_t & \mathbf{i}_u \end{bmatrix}$ is a constant frame, built along the neutral and transversal axes, that accounts for the orientation of the beam in the reference configuration with respect to the inertial frame. The assumption that the cross-sections remain straight is made and implies that $\mathbf{R}_p(s) = \mathbf{R}(s)$. From Eq. (24), the position at any point of the beam is thus described as

$$\mathbf{x}_p(s, t, u) = \mathbf{x}(s) + \mathbf{R}(s)\mathbf{O}_0\mathbf{y}(t, u) \tag{25}$$

3.2. Beam equations

According to the formalism presented in Section 2, the spatial derivative, i.e. with respect to parameter s, and time derivative, i.e. with respect to parameter T, of the frames representing the neutral axis of the beam can be introduced as

$$\mathbf{H}' = \mathbf{H}(\widetilde{\mathbf{f}^0 + \boldsymbol{\epsilon}}) \quad \text{and} \quad \dot{\mathbf{H}} = \mathbf{H}\widetilde{\mathbf{v}} \tag{26}$$

where $\mathbf{f}^0$ is interpreted as a deformation gradient in the reference configuration, $\boldsymbol{\epsilon}$ as a deformation and $\mathbf{v}$ as a velocity. As observed in Eq. (7), these quantities are evaluated in the frame described by $\mathbf{H}$, which means that they are naturally expressed in the local frame and that they are invariant under rigid body motions.

The decomposition of the deformation into the position part and the rotation part is introduced as $\boldsymbol{\epsilon} = \begin{bmatrix}\boldsymbol{\gamma}^T & \boldsymbol{\kappa}^T\end{bmatrix}^T$, where γ_1 is interpreted as the axial strain, γ_2 and γ_3 as the shear strains, κ_1 as the torsion and κ_2 and κ_3 as the bending curvatures. For a straight beam, Eq. $(26)_1$ yields

$$\mathbf{R}' = \mathbf{R}\widetilde{\boldsymbol{\kappa}} \quad \text{and} \quad \mathbf{x}' = \mathbf{R}(\mathbf{f}_U^0 + \boldsymbol{\gamma}) \tag{27}$$

Similarly, the decomposition of the velocity into the position part and the rotation part is introduced as $\mathbf{v} = \begin{bmatrix}\mathbf{v}_U^T & \mathbf{v}_\Omega^T\end{bmatrix}^T$ and Eq. $(26)_2$ yields

$$\dot{\mathbf{R}} = \mathbf{R}\widetilde{\mathbf{v}}_\Omega \quad \text{and} \quad \dot{\mathbf{x}} = \mathbf{R}\mathbf{v}_U \tag{28}$$

From these deformation and velocity measures, the strain energy and the kinetic energy for a linear elastic material can be defined as [9]

$$\mathcal{W} = \frac{1}{2}\int_0^L \boldsymbol{\epsilon}^T\mathbf{K}\boldsymbol{\epsilon}\, ds \quad \text{and} \quad \mathcal{K} = \frac{1}{2}\int_0^L \mathbf{v}^T\mathbf{M}_C\mathbf{v}\, ds \tag{29}$$

where $\mathbf{K}$ and $\mathbf{M}_C$ are the stiffness and mass matrices of the cross-sections. For simple cross-section geometries, the stiffness matrix has a classical form $\mathbf{K} = \text{diag}(\mathbf{K}_U, \mathbf{K}_\Omega)$ where $\mathbf{K}_U = \text{diag}(EA, GA_t, GA_u)$ contains the axial and shear stiffnesses whereas $\mathbf{K}_\Omega = \text{diag}(GJ, EI_t, EI_u)$ contains the torsional and bending stiffnesses, and the mass matrix has a classical form $\mathbf{M}_C = \text{diag}(\rho A\mathbf{I}_{3\times3}, \mathbf{J})$ where $\mathbf{J}$ is the second moment of inertia and is diagonal. These matrices are evaluated in the local frame and are therefore invariant under rigid body motions. Furthermore, due to the assumption that the cross-sections do not deform, they do not depend on the deformation state.

Denoting $\delta\mathbf{h} \in \mathbb{R}^6$ as an arbitrary variation such that $\delta(\mathbf{H}) = \mathbf{H}\widetilde{\delta\mathbf{h}}$, and assuming small deformations, Hamilton's principle leads to

$$[\delta\mathbf{h}^T(\mathbf{K}\boldsymbol{\epsilon} - \mathbf{g}_{extBC})]_0^L + \int_0^L \delta\mathbf{h}^T(\mathbf{M}_C\dot{\mathbf{v}} - \widehat{\mathbf{v}}^T\mathbf{M}_C\mathbf{v} - \mathbf{K}\boldsymbol{\epsilon}' + \hat{\mathbf{f}}^T\mathbf{K}\boldsymbol{\epsilon} - \mathbf{g}_{ext})ds = 0 \tag{30}$$

$$\mathbf{v}' - \dot{\mathbf{f}} - \hat{\mathbf{v}}\mathbf{f} = \mathbf{0} \tag{31}$$

where $\mathbf{g}_{ext} = \begin{bmatrix}\mathbf{g}_U^T & \mathbf{g}_\Omega^T\end{bmatrix}^T$ are the distributed external forces and $\mathbf{g}_{extBC} = \begin{bmatrix}\mathbf{g}_{BC,U}^T & \mathbf{g}_{BC,\Omega}^T\end{bmatrix}^T$ are the end external forces, which are both expressed in the local frame. Eq. (31) emanates from the Lie bracket (see Eq. (9)) which ensures that Eq. (27) and Eq. (28) are compatible. Accordingly, writing explicitly the position part and the rotation part, the strong form of the dynamic equilibrium equations for an initially straight beam is given by

$$\rho A \dot{\mathbf{v}}_U + \rho A \widetilde{\mathbf{v}}_\Omega \mathbf{v}_U - \mathbf{K}_U \boldsymbol{\gamma}' - \boldsymbol{\kappa}(\mathbf{K}_U \boldsymbol{\gamma}) - \mathbf{g}_U = \mathbf{0} \quad (32)$$

$$\mathbf{J}\dot{\mathbf{v}}_\Omega + \widetilde{\mathbf{v}}_\Omega \mathbf{J}\mathbf{v}_\Omega - \mathbf{K}_\Omega \boldsymbol{\kappa}' - (\widetilde{\mathbf{f}}_U^0 + \boldsymbol{\gamma})(\mathbf{K}_U \boldsymbol{\gamma}) - \boldsymbol{\kappa}(\mathbf{K}_\Omega \boldsymbol{\kappa}) - \mathbf{g}_\Omega = \mathbf{0} \quad (33)$$

$$\mathbf{v}_U' - \dot{\boldsymbol{\gamma}} - \widetilde{\mathbf{v}}_\Omega(\mathbf{f}_U^0 + \boldsymbol{\gamma}) - \widetilde{\mathbf{v}}_U \boldsymbol{\kappa} = \mathbf{0} \quad (34)$$

$$\mathbf{v}_\Omega' - \dot{\boldsymbol{\kappa}} - \widetilde{\mathbf{v}}_\Omega \boldsymbol{\kappa} = \mathbf{0} \quad (35)$$

and the boundary conditions must satisfy $\delta \mathbf{h}^T(\mathbf{K}\boldsymbol{\epsilon} - \mathbf{g}_{extBC})$ at $s = 0$ and $s = L$. Besides $\mathbf{g}_U$ and $\mathbf{g}_\Omega$, these equations only depend on the local deformations and the local velocities and are therefore invariant under rigid body motion, in the sense that their expression is not modified by any rigid body motion. Furthermore, when the external forces do not depend on the position and the orientation of the beam, Eqs. (32-35) and Eq. (26) are decoupled. Accordingly, Eqs. (32-35) can be solved for the velocities and the deformations and then, in a second step, the position and orientation of the beam can be obtained by integrating Eq. (26). Nevertheless, the external forces and the boundary conditions may in general depend on the position and the orientation of the beam.

In order to emphasize the relevance of this framework, three simple examples are considered in the rest of this section. The analytical solution for the deformations and the velocities is obtained without involving the actual position and orientation of the beam, which are computed in a second step using Eq. (27) and Eq. (28). In each example, as illustrated in Fig. 1, the

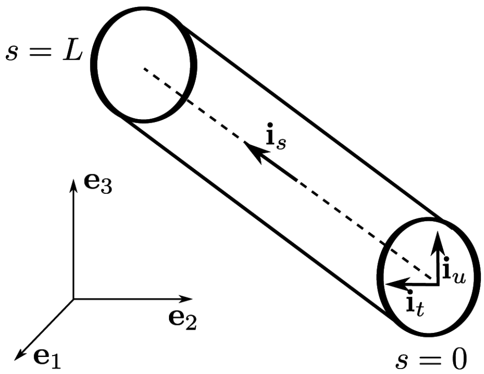

Fig. 1. Beam configuration

initially straight beam is aligned along $\mathbf{i}_s$, and $\mathbf{i}_t$ and $\mathbf{i}_u$ span the cross-section which is constant along the beam. Accordingly, in the local frames, we can set $\mathbf{i}_s = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, $\mathbf{i}_t = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$ and $\mathbf{i}_u = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. The inertial axes are denominated by $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. Without loss of generality, it is assumed that, in the reference configuration, the beam triad and the inertial frame coincide, namely $\mathbf{i}_s = \mathbf{e}_1$, $\mathbf{i}_t = \mathbf{e}_2$ and $\mathbf{i}_u = \mathbf{e}_3$ in order to simplify the integration of the compatibility equations which provides the position and orientation of the beam.

3.3. Example 1: planar rotation of a beam

Let us consider a beam rotating at a constant velocity w_0 about $\mathbf{i}_t$, which is aligned with $\mathbf{e}_2$ in the initial configuration. It is expected that the beam remains straight, but is stretched due to centrifugal forces. Accordingly, we have $\boldsymbol{\gamma}(s,T) = \begin{bmatrix}\gamma_1(s,T) & 0 & 0\end{bmatrix}^T$, $\boldsymbol{\kappa}(s,T) = \mathbf{0}$, $\mathbf{v}_U(s,T) = \begin{bmatrix}0 & 0 & v_3(s,T)\end{bmatrix}^T$ and $\mathbf{v}_\Omega(s,T) = \begin{bmatrix}0 & w(s,T) & 0\end{bmatrix}^T$. Furthermore, we set the boundary conditions such that there is no deformation at the extremities of the beam and no vertical velocity at the midspan. However, this does not determine the position or orientation at this point. From Eqs. (32-35), the equilibrium equations are thus given by

$$\rho A v_3 w - EA\gamma_1' = 0 \quad (36)$$
$$\rho A \dot{v}_3 = 0 \quad (37)$$
$$J_t \dot{w} = 0 \quad (38)$$
$$\dot{\gamma}_1 = 0 \quad (39)$$
$$v_3' + w(1+\gamma_1) = 0 \quad (40)$$
$$w' = 0 \quad (41)$$

since $\mathbf{g}_{ext} = \mathbf{0}$. Eq. (38) and Eq. (41) lead directly to

$$w = w_0 \quad (42)$$

Furthermore, Eq. (37) and Eq. (39) indicate that $\gamma_1 = \gamma_1(s)$ and $v_3 = v_3(s)$. Deriving Eq. (40) with respect to s and replacing γ_1' from Eq. (36) yields

$$v_3'' + \frac{\rho w_0^2}{E} v_3 = 0 \quad (43)$$

whose solution, posing $\alpha = \sqrt{\frac{\rho}{E}}$, reads

$$v_3(s) = a \sin\left(\alpha w_0 (s - \frac{L}{2})\right) \quad (44)$$

where a is not yet determined and the boundary condition $v_3(L/2) = 0$ has been taken into account. Notice that this expression is compatible with Eq. (37). Substituting the solution for v_3 into Eq. (40) leads to

$$\gamma_1(s) = -a\alpha \cos\left(\alpha w_0(s - \frac{L}{2})\right) - 1 \tag{45}$$

Notice that this expression is compatible with Eq. (39). Considering that there is no deformation at the beam extremities allows the determination of a as

$$a = \frac{-1}{\alpha \cos(\alpha w_0 \frac{L}{2})} \tag{46}$$

and finally we have

$$v_3(s) = \frac{\sin\left(\alpha w_0(\frac{L}{2} - s)\right)}{\alpha \cos(\alpha w_0 \frac{L}{2})} \text{ and } \gamma_1(s) = \frac{\cos\left(\alpha w_0(s - \frac{L}{2})\right)}{\cos(\alpha w_0 \frac{L}{2})} - 1 \tag{47}$$

So far, the deformation state and the velocity state of the beam could be determined without referring to the actual position and orientation of the beam, which can now be recovered from Eq. (27) and Eq. (28). For the rotation, Eq. $(27)_1$ indicates that $\mathbf{R}$ does not depend on s since $\boldsymbol{\kappa} = \mathbf{0}$ and Eq. $(28)_1$ yields

$$\mathbf{R}(T) = \exp(\mathbf{v}_\Omega T) = \begin{bmatrix} \cos(w_0 T) & 0 & \sin(w_0 T) \\ 0 & 1 & 0 \\ -\sin(w_0 T) & 0 & \cos(w_0 T) \end{bmatrix} \tag{48}$$

where we imposed $\mathbf{R}(0) = \mathbf{I}_{3\times 3}$. For the position part, either Eq. $(27)_2$ or Eq. $(28)_2$ gives

$$\mathbf{x}(s,T) = \begin{bmatrix} s + \frac{\cos(w_0 T)\sin\left(\alpha w_0(s - \frac{L}{2})\right)}{\alpha w_0 \cos(\alpha w_0 \frac{L}{2})} & 0 & -\frac{\sin(w_0 T)\sin\left(\alpha w_0(s - \frac{L}{2})\right)}{\alpha w_0 \cos(\alpha w_0 \frac{L}{2})} \end{bmatrix}^T \tag{49}$$

where we imposed $\mathbf{x}(s,0) = [s \quad 0 \quad 0]^T$.

This example shows that the problem can be solved in two steps: first the equilibrium equations for the deformations and the velocities, then the kinematic relationships for the position and orientation. Without this clear separation between local deformation and velocities on one hand and large amplitude motion on the other hand, as in classical beam theories, the position field and the rotation field should have been solved together with the equilibrium equations which would have brought significant non-linearities.

3.4. Example 2: static cantilever beam under a tip load

Let us consider a cantilever beam aligned along $\mathbf{i}_s$ and submitted to a follower tip load F along $\mathbf{i}_u$. Furthermore, let us assume that the displacements and the deformations remain small, such that quadratic and higher order terms of the deformation are neglected. From Eqs. (32-33) (Eqs. (34-35) are not considered for a static problem), the static equilibrium equations read

$$\mathbf{K}_U \boldsymbol{\gamma}' = \mathbf{0} \quad (50)$$

$$\mathbf{K}_\Omega \boldsymbol{\kappa}' + \widetilde{\mathbf{i}}_s \mathbf{K}_U \boldsymbol{\gamma} = \mathbf{0} \quad (51)$$

since $\mathbf{f}_U^0 = \mathbf{i}_s$ and $\mathbf{g}_{ext} = \mathbf{0}$. Considering a follower tip load $\mathbf{F} = \begin{bmatrix} 0 & 0 & F \end{bmatrix}^T$, we have $\mathbf{g}_{BC,\Omega} = \mathbf{0}_{3\times 1}$ and $\mathbf{g}_{BC,U} = \mathbf{F}$, and the boundary conditions at $s = L$ read

$$\mathbf{K}_U \boldsymbol{\gamma}(L) = \mathbf{F} \quad (52)$$

$$\mathbf{K}_\Omega \boldsymbol{\kappa}(L) = \mathbf{0} \quad (53)$$

The solution to Eqs. (50-51) takes the form

$$\boldsymbol{\gamma} = \mathbf{a} \quad (54)$$

$$\boldsymbol{\kappa}(s) = (\mathbf{K}_\Omega^* \mathbf{a}) s + \mathbf{b} \quad (55)$$

where $\mathbf{K}_\Omega^* = -\mathbf{K}_\Omega^{-1} \widetilde{\mathbf{i}}_s \mathbf{K}_U$ and $\mathbf{a}$ and $\mathbf{b}$ are integration constants determined from the end loading condition Eqs. (52-53)

$$\mathbf{K}_U \mathbf{a} = \mathbf{F} \quad (56)$$

$$\mathbf{K}_\Omega ((\mathbf{K}_\Omega^* \mathbf{a}) L + \mathbf{b}) = \mathbf{0} \quad (57)$$

The solution is straightforwardly obtained as

$$\mathbf{a} = \mathbf{K}_U^{-1} \mathbf{F} = \begin{bmatrix} 0 & 0 & \frac{F}{GA_u} \end{bmatrix}^T \quad \text{and} \quad \mathbf{b} = -L \mathbf{K}_\Omega^* \mathbf{a} = \begin{bmatrix} 0 & -\frac{LF}{EI_t} & 0 \end{bmatrix}^T \quad (58)$$

such that, using Eq. (54-55),

$$\boldsymbol{\gamma} = \begin{bmatrix} 0 & 0 & \frac{F}{GA_u} \end{bmatrix}^T \quad \text{and} \quad \boldsymbol{\kappa}(s) = \begin{bmatrix} 0 & \frac{LF}{EI_t}(\frac{s}{L} - 1) & 0 \end{bmatrix}^T \quad (59)$$

The solution to the static equilibrium equations yields the deformation state. Notice that it can be obtained without having to consider the actual position or the rotation since the equilibrium equations only depend on the deformations. Now that the static equilibrium is solved and the deformation state is known, the orientation and position of the beam can be deduced by integrating Eq. (27). According to the small displacement assumption, the rotation matrix and its derivative read

$$\mathbf{R}(s) = \mathbf{I}_{3\times3} + \widetilde{\boldsymbol{\theta}}(s) \quad \text{and} \quad \mathbf{R}'(s) = \widetilde{\boldsymbol{\theta}}'(s) \tag{60}$$

where, the motion being planar, $\boldsymbol{\theta}(s) = \begin{bmatrix} 0 & \theta_t(s) & 0 \end{bmatrix}^T$. Hence, using Eq. (27), we have

$$\boldsymbol{\theta}'(s) = \boldsymbol{\kappa}(s) \quad \text{and} \quad \mathbf{x}' = \mathbf{i}_s + \mathbf{a} - \widetilde{\mathbf{i}}_s\boldsymbol{\theta}(s) \tag{61}$$

whose solution is straightforwardly given by

$$\boldsymbol{\theta}(s) = (\mathbf{K}^*_\Omega\mathbf{a})\frac{s^2}{2} + \mathbf{b}s + \mathbf{c} \quad \text{and} \quad \mathbf{x}(s) = (\mathbf{i}_s + \mathbf{a})s - \widetilde{\mathbf{i}}_s((\mathbf{K}^*_\Omega\mathbf{a})\frac{s^3}{6} + \mathbf{b}\frac{s^2}{2}) + \mathbf{d} \tag{62}$$

where the clamped boundary conditions $\boldsymbol{\theta}(0) = \mathbf{0}$ and $\mathbf{x}(0) = \mathbf{0}$ yield $\mathbf{c} = \mathbf{0}$ and $\mathbf{d} = \mathbf{0}$. Using Eq. (58), we have

$$\boldsymbol{\theta}(s) = \begin{bmatrix} 0 & -\frac{FL^2}{EI_t}\left(\frac{s}{L} - \frac{s^2}{2L^2}\right) & 0 \end{bmatrix}^T \text{ and } \mathbf{x}(s) = \begin{bmatrix} s & 0 & s\frac{F}{GA_u} + \frac{FL^3}{6EI_t}(\frac{3s^2}{L^2} - \frac{s^3}{L^3}) \end{bmatrix}^T \tag{63}$$

We observe that the solution matches the exact solution expected from the classical beam theory. The equilibrium equations are first solved for the deformations and it appears that there is no need to know the position and the orientation of the beam to obtain the deformation state. In a second step, the rotation and the position are computed by integrating the deformations, and boundary conditions on rotation and position can be applied.

3.5. Example 3: static cantilever beam under a tip bending moment

Let us consider a cantilever beam aligned along $\mathbf{i}_s$ and submitted to a tip bending moment $\mathbf{M}$ about $\mathbf{i}_t$, namely $\mathbf{M} = \begin{bmatrix} 0 & M & 0 \end{bmatrix}^T$. It is expected that there is no axial nor shear deformation, whence, considering Eq. $(27)_2$, $\boldsymbol{\gamma} = \mathbf{0}$. From Eq. (32-33) (Eq. (34-35) are not considered for a static problem), the static equilibrium equation reads

$$\mathbf{0} = \mathbf{0} \tag{64}$$

$$\mathbf{K}_\Omega\boldsymbol{\kappa}' + \widetilde{\boldsymbol{\kappa}}(\mathbf{K}_\Omega\boldsymbol{\kappa}) = \mathbf{0} \tag{65}$$

since $\mathbf{g}_{ext} = \mathbf{0}$. Since the problem is planar, $\boldsymbol{\kappa} = \begin{bmatrix} 0 & \kappa_t & 0 \end{bmatrix}^T$, which means $\widetilde{\boldsymbol{\kappa}}(\mathbf{K}_\Omega\boldsymbol{\kappa}) = \mathbf{0}$. The solution is thus given as

$$EI_t\kappa_t' = 0 \Leftrightarrow \kappa_t = a \tag{66}$$

where a is an integration constant. Satisfying the boundary condition at $s = L$ yields

$$EI_t\kappa_t(L) = M \Leftrightarrow a = \frac{M}{EI_t} \tag{67}$$

The neutral axis of the beam is thus a planar curve whose constant curvature is given by $\kappa_t = M/EI_t$. From Eq. (27), the position and rotation field can be recovered

$$\begin{array}{rclcrcl} \mathbf{R}' & = & \mathbf{R}\widetilde{\boldsymbol{\kappa}} & \Leftrightarrow & \mathbf{R}(s) & = & \exp_{SO(3)}(s\boldsymbol{\kappa}) \\ \mathbf{x}' & = & \mathbf{R}(s)\mathbf{i}_s & \Leftrightarrow & \mathbf{x}(s) & = & s\mathbf{T}_{SO(3)}(-s\boldsymbol{\kappa})\mathbf{i}_s \end{array} \tag{68}$$

where the clamped boundary conditions $\mathbf{R}(0) = \mathbf{I}_{3\times3}$ and $\mathbf{x}(0) = \mathbf{0}$ have been taken into account. More explicitly, the solution reads

$$\mathbf{R}(s) = \begin{bmatrix} \cos(s\kappa_t) & 0 & \sin(s\kappa_t) \\ 0 & 1 & 0 \\ -\sin(s\kappa_t) & 0 & \cos(s\kappa_t) \end{bmatrix} \text{ and } \mathbf{x}(s) = \left[\frac{1}{\kappa_t}\sin(s\kappa_t)\ 0\ \frac{1}{\kappa_t}(1-\cos(s\kappa_t))\right]^T \tag{69}$$

which is the exact solution expected from classical beam theory. This example shows that the solution to a large displacement problem, which leads to strongly non-linear position and rotation fields, can be solved exactly in a simple way. The solution to the differential equation governing the static equilibrium of the beam could be obtained without having to refer to the actual position and orientation, which are recovered afterwards.

4. Discretization method

The equilibrium equations Eqs. (32-35) are not easy to solve in general and the use of an approximation method is required, such as a finite element method. Although they can be solved in some cases for the velocities and the deformations only, it is convenient, from a practical point of view, to solve simultaneously the position and orientation of the beam, namely include Eqs. (27) and Eq. (28) in the solution process. Indeed, this information is needed to connect elements of a finite element mesh, apply general boundary conditions or express external forces that are not naturally expressed in the local frame. Nevertheless, as pointed out in Section 3, the homogeneous equilibrium equations are not affected by this information which leads to a framework with reduced non-linearities compared to classical formulations.

It is thus of fundamental interest to preserve the invariance property of the equilibrium equations through the discretization process. To that purpose, a consistent discretization method is discussed in this section. It relies on two steps. First, a spatial interpolation of nodal frames is introduced. Then, an interpolation formula for the nodal velocities consistent with the spatial interpolation is derived.

4.1. Spatial discretization

4.1.1. Nodal frame interpolation formula

The spatial discretization along the neutral axis of the beam is introduced by an interpolation with the variable $s \in [0, L]$ between two end nodes A at $s = 0$ and B at $s = L$, where the nodal frames $\mathbf{H}_A$ and $\mathbf{H}_B$ are located (Fig. 2).

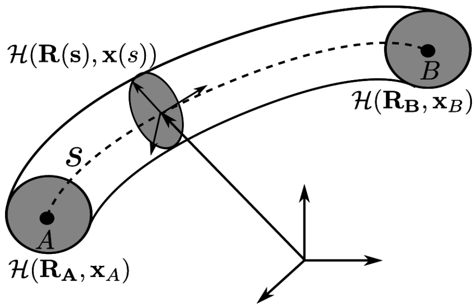

Fig. 2. Beam model

The proposed interpolation formula reads

$$\mathbf{H}(s) = \mathbf{H}_A \exp_{SE(3)}(\frac{s}{L}\mathbf{d}) \tag{70}$$

where $\mathbf{d} = \begin{bmatrix} \mathbf{d}_U^T & \mathbf{d}_\Omega^T \end{bmatrix}^T$ is called the relative configuration vector and is defined as

$$\mathbf{d} = \log_{SE(3)}(\mathbf{H}_A^{-1}\mathbf{H}_B) \tag{71}$$

Eq. (70) can be interpreted as, starting from the nodal frame $\mathbf{H}_A$, the nodal frame $\mathbf{H}_B$ is reached by moving along the frame transformation implied by the projection on the group of relative configuration vector $\mathbf{d}$. As a consequence, $\exp_{SE(3)}(s\mathbf{d}/L)$ is a frame transformation which takes place in frame $\mathbf{H}_A$. Notice that $\mathbf{d}$ is invariant under rigid body motions since, for any $\mathbf{H}^* \in SE(3)$, $\mathbf{H}^*\mathbf{H}_A$ and $\mathbf{H}^*\mathbf{H}_B$ correspond to a rigid motion of the beam and lead to the same value of $\mathbf{d}$. Therefore, the interpolation formula automatically satisfies the frame invariance requirement and $\mathbf{d}$ is a suitable measure to preserve the invariance of the equilibrium equations. In the reference configuration, $\mathbf{H}_A = \mathcal{H}(\mathbf{R}^0, \mathbf{x}_A^0)$ and $\mathbf{H}_B = \mathcal{H}(\mathbf{R}^0, \mathbf{x}_B^0)$ so that $\mathbf{d}_U^0 = \mathbf{x}_B^0 - \mathbf{x}_A^0$ and $\mathbf{d}_\Omega^0 = \mathbf{0}_{3\times1}$. Hence, it can be observed that $\|\mathbf{d}_U^0\| = L$. Due to the validity of the logarithmic map, the parametrization holds as long as the relative rotation between the two end nodes is in $]-\pi, +\pi[$.

Eq. (70) can be seen as a straightforward extension of classical interpolation method for linear spaces to non-linear spaces. Indeed, considering two elements in $\mathbb{R}^k$, $\mathbf{P}_A$ at $s = 0$ and $\mathbf{P}_B$ at $s = L$, the classical linear interpolation formula reads

$$\mathbf{P}(s) = (1 - \frac{s}{L})\mathbf{P}_A + \frac{s}{L}\mathbf{P}_B = \mathbf{P}_A + \frac{s}{L}\mathbf{p} \tag{72}$$

where the free vector $\mathbf{p}$ is defined as

$$\mathbf{p} = \mathbf{P}_B - \mathbf{P}_A \tag{73}$$

Eq. (72) can be interpreted as, starting from $\mathbf{P}_A$, $\mathbf{P}_B$ is reached by moving along the free vector $\mathbf{p}$. The expression is straightforward since free vectors such as $\mathbf{p}$ and attached vectors such as $\mathbf{P}_A$ and $\mathbf{P}_B$ belong to the same space. However, for non-linear spaces, additional care is required. Indeed, the tangent spaces are not the same as the space itself and projections are involved through the exponential and logarithmic maps. Hence, Eq. (71) and Eq. (70) are the $SE(3)$ equivalent to Eq. (73) and Eq. (72) for linear spaces.

The proposed interpolation method in Eq. (70) also complies with the weighted average interpolation discussed in [17] which provides an implicit definition of the interpolated field. In [17], the weighted average interpolation formula is given for a relative configuration vector that is expressed in the global frame rather than in the local frame. The formula used here is adapted to the present local frame definition and, for a two node interpolation using one spatial parameter, reads

$$(1 - \frac{s}{L}) \log_{SE(3)}(\mathbf{H}(s)^{-1}\mathbf{H}_A) + \frac{s}{L} \log_{SE(3)}(\mathbf{H}(s)^{-1}\mathbf{H}_B) = \mathbf{0} \tag{74}$$

It is straightforward to verify that Eq. (70) satisfies this definition:

$$\begin{aligned}(1 - \frac{s}{L}) \log_{SE(3)}(\exp_{SE(3)}(-\frac{s}{L}\mathbf{d})\mathbf{H}_A^{-1}\mathbf{H}_A) + \frac{s}{L} \log_{SE(3)}(\exp_{SE(3)}(-\frac{s}{L}\mathbf{d})\mathbf{H}_A^{-1}\mathbf{H}_B) &= \mathbf{0}\\ (1 - \frac{s}{L})(-\frac{s}{L}\mathbf{d}) + \frac{s}{L} \log_{SE(3)}(\exp_{SE(3)}(-\frac{s}{L}\mathbf{d}) \exp_{SE(3)}(\mathbf{d})) &= \mathbf{0}\\ (1 - \frac{s}{L})(-\frac{s}{L}\mathbf{d}) + \frac{s}{L}(1 - \frac{s}{L})\mathbf{d} &= \mathbf{0}\end{aligned}$$

4.1.2. Deformation measure

The derivative of Eq. (70) with respect to s also allows us to gain some insight. Using Eq. (18) and Eq. (20), we have

$$\mathbf{H}'(s) = \mathbf{H}(s)\widetilde{\left(\mathbf{T}_{SE(3)}(\frac{s}{L}\mathbf{d})\frac{\mathbf{d}}{L}\right)} = \mathbf{H}(s)\frac{\widetilde{\mathbf{d}}}{L} \tag{75}$$

where we observe that the $\mathfrak{se}(3)$ element associated with the derivatives along s is constant. Comparing this expression with Eq. (26) gives the definition of the discretized deformation. Denoting the value of the relative configuration vector in the reference configuration by $\mathbf{d}^0$, we have the simple expression

$$\boldsymbol{\epsilon} = \frac{\mathbf{d} - \mathbf{d}^0}{L} \tag{76}$$

Hence, the deformation measure is constant over the element, which appears as an important numerical advantage. Indeed, the strain energy per unit length of the beam is also a constant over the element and the integration over the beam length in Eq. $(29)_1$ can be performed exactly and no numerical integration must be considered. As an important consequence of the framework, the deformation measure depends on the relative configuration vector $\mathbf{d}$ only, which means that it ensures that the invariance of the equations of motion under rigid body motions is preserved.

4.1.3. Geometric interpretation

A meaningful geometric interpretation of the interpolation formula in Eq. (70) can be obtained by constructing the local Frenet triad along the beam axis. From Eq. (75), we have

$$\frac{d\mathbf{x}(s)}{ds} = \mathbf{R}(s)\frac{\mathbf{d}_U}{L} \quad \text{and} \quad \frac{d\mathbf{R}(s)}{ds} = \mathbf{R}(s)\frac{\widetilde{\mathbf{d}}_\Omega}{L} \tag{77}$$

The unit tangent to the neutral axis is thus given by

$$\mathbf{t}(s) = \frac{d\mathbf{x}}{ds}\|\frac{d\mathbf{x}}{ds}\|^{-1} = \mathbf{R}(s)\frac{\mathbf{d}_U}{\|\mathbf{d}_U\|} \tag{78}$$

In order to compute the unit normal vector, let us derive the tangent unit vector with respect to s. Using Eq. $(77)_2$, we have

$$\frac{d}{ds}(\mathbf{t}(s)) = \frac{1}{L\|\mathbf{d}_U\|}\mathbf{R}(s)\widetilde{\mathbf{d}}_\Omega\mathbf{d}_U \tag{79}$$

Hence, the unit normal vector is given by

$$\mathbf{n}(s) = \frac{1}{\kappa}\frac{d}{ds}\Big(\mathbf{t}(s)\Big) = \mathbf{R}(s)\frac{\widetilde{\mathbf{d}}_\Omega\mathbf{d}_U}{\|\widetilde{\mathbf{d}}_\Omega\mathbf{d}_U\|} \tag{80}$$

where the curvature is defined as

$$\kappa = \|\frac{d}{ds}\Big(\mathbf{t}(s)\Big)\| = \frac{\|\widetilde{\mathbf{d}}_\Omega\mathbf{d}_U\|}{L\|\mathbf{d}_U\|} \tag{81}$$

It appears that the geometric description of the neutral axis using the proposed interpolation formula in Eq. (70) leads to a curve that has a constant curvature since $\mathbf{d}$, which results from the interpolated nodal frames, is constant over the element.

The last vector of the Frenet triad is the unit binormal vector $\mathbf{b}(s)$ defined as

$$\mathbf{b}(s) = \widetilde{\mathbf{t}}\mathbf{n} = \frac{1}{||\mathbf{d}_U|| ||\widetilde{\mathbf{d}}_\Omega \mathbf{d}_U||} \widetilde{\mathbf{R}\mathbf{d}_U} \mathbf{R}\widetilde{\mathbf{d}}_\Omega \mathbf{d}_U = \mathbf{R}(s) \frac{\widetilde{\mathbf{d}}_U \widetilde{\mathbf{d}}_\Omega \mathbf{d}_U}{||\mathbf{d}_U|| ||\widetilde{\mathbf{d}}_\Omega \mathbf{d}_U||} \tag{82}$$

Deriving it with respect to s leads to the definition of the torsion τ as

$$\frac{d\mathbf{b}}{ds} = -\tau \mathbf{n}(s) \tag{83}$$

Going through the computations, we have

$$\begin{aligned}
\frac{d\mathbf{b}}{ds} &= \frac{1}{L||\mathbf{d}_U|| ||\widetilde{\mathbf{d}}_\Omega \mathbf{d}_U||} (\underbrace{\widetilde{\mathbf{R}\widetilde{\mathbf{d}}_\Omega \mathbf{d}_U} \mathbf{R}\widetilde{\mathbf{d}}_\Omega \mathbf{d}_U}_{=\mathbf{0}} + \widetilde{\mathbf{R}\mathbf{d}_U} \mathbf{R}\widetilde{\mathbf{d}}_\Omega \widetilde{\mathbf{d}}_\Omega \mathbf{d}_U) \\
&= \frac{1}{L||\mathbf{d}_U|| ||\widetilde{\mathbf{d}}_\Omega \mathbf{d}_U||} (\widetilde{\mathbf{R}\mathbf{d}_U} \mathbf{R}(\mathbf{d}_\Omega^T \mathbf{d}_U)\mathbf{d}_\Omega - \underbrace{\widetilde{\mathbf{R}\mathbf{d}_U} \mathbf{R}(\mathbf{d}_\Omega^T \mathbf{d}_\Omega)\mathbf{d}_U}_{=\mathbf{0}}) \\
&= \frac{(\mathbf{d}_\Omega^T \mathbf{d}_U)}{L||\mathbf{d}_U|| ||\widetilde{\mathbf{d}}_\Omega \mathbf{d}_U||} \mathbf{R}\widetilde{\mathbf{d}}_U \mathbf{d}_\Omega \\
&= -\frac{(\mathbf{d}_\Omega^T \mathbf{d}_U)}{L||\mathbf{d}_U||} \mathbf{R} \frac{\widetilde{\mathbf{d}}_\Omega \mathbf{d}_U}{||\widetilde{\mathbf{d}}_\Omega \mathbf{d}_U||}
\end{aligned} \tag{84}$$

which leads to

$$\tau = \frac{(\mathbf{d}_\Omega^T \mathbf{d}_U)}{L||\mathbf{d}_U||} \tag{85}$$

It is observed that the torsion of the curve describing the neutral axis of the beam using Eq. (70) is constant along the axis. Developing Eq. (81) and Eq. (85), we have

$$\kappa = \frac{||\mathbf{d}_\Omega|| \sin(\mathbf{d}_\Omega, \mathbf{d}_U)}{L} \quad \text{and} \quad \tau = \frac{||\mathbf{d}_\Omega|| \cos(\mathbf{d}_\Omega, \mathbf{d}_U)}{L} \tag{86}$$

and the Gaussian curvature κ_g is obtained as

$$\kappa_g = \sqrt{\kappa^2 + \tau^2} = \frac{||\mathbf{d}_\Omega||}{L} \tag{87}$$

which gives a geometric interpretation to $\mathbf{d}_\Omega$. κ_g is, as expected, also constant over the beam length.

As pointed out in [9], the ability to represent a curve that has a constant curvature is important with respect to the shear-locking phenomenon. In practice, shear-locking shows up when spurious shear deformations are involved in pure bending. However, in this case, a constant curvature neutral axis and a no shear deformation state are expected. Since the proposed interpolation formula systematically represents a constant curvature neutral axis, the element turns out to be naturally shear-locking free.

4.2. Consistent velocity interpolation

In Section 4.1., a formula to interpolate nodal frames in space has been discussed. Due to the non-commutativity of the special Euclidean group, characterized by the fact that the Lie bracket in Eq. (9) does not vanish in general, the time derivative of the deformation is not equal to the spatial derivative of the velocity. In order to cope with this situation, we proposed to represent the velocity of the beam as an interpolation of nodal velocities which is derived from the spatial discretization formula. The idea is illustrated in Fig. 3.

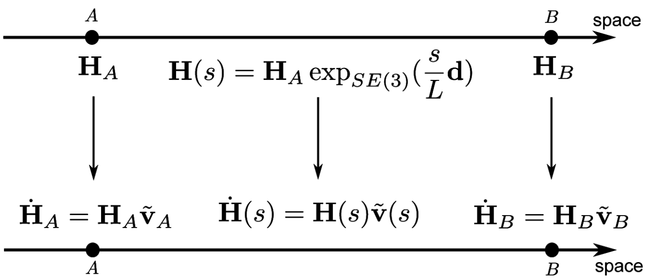

Fig. 3. Consistent velocity interpolation

Let us define the nodal velocities of the beam element as $\dot{\mathbf{H}}_A = \mathbf{H}_A\widetilde{\mathbf{v}}_A$ and $\dot{\mathbf{H}}_B = \mathbf{H}_B\widetilde{\mathbf{v}}_B$. Deriving with respect to time the spatial interpolation formula in Eq. (70), we have

$$\begin{aligned} \dot{\mathbf{H}} &= \mathbf{H}_A\widetilde{\mathbf{v}}_A \exp_{SE(3)}(\frac{s}{L}\mathbf{d}) + \mathbf{H}(\mathbf{T}_{SE(3)}(\frac{s}{L}\mathbf{d})\frac{s}{L}\dot{\mathbf{d}})^{\sim} \\ &= \mathbf{H}\ (\mathrm{Ad}_{\exp_{SE(3)}(-\frac{s}{L}\mathbf{d})}\mathbf{v}_A + \mathbf{T}_{SE(3)}(\frac{s}{L}\mathbf{d})\frac{s}{L}\dot{\mathbf{d}})^{\sim} \end{aligned} \tag{88}$$

Considering the definition of $\mathbf{d}$ in Eq. (71), we have

$$\begin{aligned} \mathbf{H}_B &= \mathbf{H}_A \exp(\mathbf{d}) \quad \Leftrightarrow \quad \mathbf{v}_B = \mathrm{Ad}_{\exp(-\mathbf{d})}(\mathbf{v}_A) + \mathbf{T}_{SE(3)}(\mathbf{d})\ \dot{\mathbf{d}} \\ \mathbf{H}_A &= \mathbf{H}_B \exp(-\mathbf{d}) \quad \Leftrightarrow \quad \mathbf{v}_A = \mathrm{Ad}_{\exp(\mathbf{d})}(\mathbf{v}_B) - \mathbf{T}_{SE(3)}(-\mathbf{d})\ \dot{\mathbf{d}} \end{aligned} \tag{89}$$

which leads to

$$\dot{\mathbf{d}} = \mathbf{P}(\mathbf{d})\mathbf{v}_{AB} \tag{90}$$

where $\mathbf{P}(\mathbf{d}) = \left[-\mathbf{T}_{SE(3)}^{-1}(-\mathbf{d}) \quad \mathbf{T}_{SE(3)}^{-1}(\mathbf{d})\right]$ and $\mathbf{v}_{AB} = \left[\mathbf{v}_A^T \quad \mathbf{v}_B^T\right]^T$. Comparing Eq. (26) and Eq. (88), the velocity of the neutral axis of the beam is interpolated as

$$\mathbf{v}(s,\mathbf{d}) = \left[\mathrm{Ad}_{\exp_{SE(3)}(-\frac{s}{L}\mathbf{d})} - \frac{s}{L}\mathbf{T}_{SE(3)}(\frac{s}{L}\mathbf{d})\mathbf{T}_{SE(3)}^{-1}(-\mathbf{d}) \quad \frac{s}{L}\mathbf{T}_{SE(3)}(\frac{s}{L}\mathbf{d})\mathbf{T}_{SE(3)}^{-1}(\mathbf{d})\right]\mathbf{v}_{AB} \tag{91}$$

As announced previously, the interpolation formula for the velocities is related to the spatial interpolation. It only depends on s and on the relative configuration vector $\mathbf{d}$, which means that it is invariant under rigid body motion and changes only when the beam deforms. Accordingly, this velocity field preserve the invariance under rigid body motion of the equilibrium equations. Some further manipulations of the series expression of the different operators involved yields the following identity

$$\mathrm{Ad}_{\exp_{SE(3)}(-\frac{s}{L}\mathbf{d})} - \frac{s}{L}\mathbf{T}_{SE(3)}(\frac{s}{L}\mathbf{d})\mathbf{T}_{SE(3)}^{-1}(-\mathbf{d}) = \mathbf{I}_{6\times6} - \frac{s}{L}\mathbf{T}_{SE(3)}(\frac{s}{L}\mathbf{d})\mathbf{T}_{SE(3)}^{-1}(\mathbf{d}) \tag{92}$$

which simplifies the interpolation formula as

$$\mathbf{v} = \mathbf{Q}(s,\mathbf{d})\mathbf{v}_{AB} \tag{93}$$

where

$$\mathbf{Q}(s,\mathbf{d}) = \left[\mathbf{I}_{6\times6} - \mathbf{T}^*(s,\mathbf{d}) \quad \mathbf{T}^*(s,\mathbf{d})\right] \tag{94}$$

with $\mathbf{T}^* = (s/L)\ \mathbf{T}_{SE(3)}(s\mathbf{d}/L)\ \mathbf{T}_{SE(3)}^{-1}(\mathbf{d})$. In this form, it appears directly that if $\mathbf{v}_A = \mathbf{v}_B = \mathbf{v}^*$, the interpolated velocity is $\mathbf{v}(s,\mathbf{d}) = \mathbf{v}^*$, namely the velocity in each local frame along the beam is constant.

Let us now consider the Lie bracket relationship: $\mathbf{v}' - \dot{\mathbf{f}} - \widehat{\mathbf{v}}\mathbf{f} = \mathbf{0}$. Using $\mathbf{f} = \mathbf{d}/L$ and Eq. (93), the Lie bracket relationship reads

$$\left(\mathbf{T}^{*\prime} + \frac{1}{L}\mathbf{T}_{SE(3)}^{-1}(-\mathbf{d}) + \frac{\widehat{\mathbf{d}}}{L}(\mathbf{I}_{6\times6} - \mathbf{T}^*)\right)\mathbf{v}_A + \left(\mathbf{T}^{*\prime} - \frac{1}{L}\mathbf{T}_{SE(3)}^{-1}(\mathbf{d}) + \frac{\widehat{\mathbf{d}}}{L}\mathbf{T}^*\right)\mathbf{v}_B = \mathbf{0} \tag{95}$$

This formula should be valid for any $\mathbf{v}_A$ and any $\mathbf{v}_B$, which imposes that

$$\mathbf{T}^{*\prime} + \frac{1}{L}\mathbf{T}_{SE(3)}^{-1}(-\mathbf{d}) + \frac{\widehat{\mathbf{d}}}{L}(\mathbf{I}_{6\times6} - \mathbf{T}^*) = \mathbf{0} \tag{96}$$

$$\mathbf{T}^{*\prime} - \frac{1}{L}\mathbf{T}_{SE(3)}^{-1}(\mathbf{d}) + \frac{\widehat{\mathbf{d}}}{L}\mathbf{T}^* = \mathbf{0} \tag{97}$$

Going through the series developments of the operators involved, it can be shown that these equations are satisfied for any s and any $\mathbf{d}$. Hence, the discretization method discussed here automatically satisfies the Lie bracket relationship.

4.3. Discretized beam equations

Using the interpolation fields discussed in Section 4.1. and 4.2., it can be shown that dynamic equilibrium equations for a beam element take the form of ordinary differential equations on the Lie group [9]

$$\dot{\mathbf{H}}_A = \mathbf{H}_A \widetilde{\mathbf{v}}_A \tag{98}$$

$$\dot{\mathbf{H}}_B = \mathbf{H}_B \widetilde{\mathbf{v}}_B \tag{99}$$

$$\mathbf{M}(\mathbf{d})\dot{\mathbf{v}}_{AB} + \mathbf{C}(\mathbf{d}, \mathbf{v}_{AB})\mathbf{v}_{AB} + \mathbf{P}(\mathbf{d})^T \mathbf{K}\boldsymbol{\epsilon} = \int_0^L \mathbf{Q}(s, \mathbf{d})^T \mathbf{g}_{ext}\, ds \tag{100}$$

where $\mathbf{P}$ was defined in Eq. (90) and, denoting $\mathbf{Q} = \mathbf{Q}(s, \mathbf{d})$ (Eq. (94)), the 12×12 mass matrix is defined as

$$\mathbf{M}(\mathbf{d}) = \int_0^L \mathbf{Q}^T \mathbf{M}_C \mathbf{Q}\, ds \tag{101}$$

and $\mathbf{C}(\mathbf{d}, \mathbf{v}_{AB})\mathbf{v}_{AB}$ are the gyroscopic forces with the 12×12 matrix

$$\mathbf{C}(\mathbf{d}, \mathbf{v}_{AB}) = \int_0^L \mathbf{Q}^T (\mathbf{M}_C \dot{\mathbf{Q}} - \widehat{\mathbf{Q}\mathbf{v}_{AB}}^T \mathbf{M}_C \mathbf{Q})\, ds \tag{102}$$

Eq. (100) are the discretized version of Eq. (32-33). It appears that the discretization process preserves the local frame formulation of the equations since the homogeneous Eq. (100) only depend on $\mathbf{d}$ and $\mathbf{v}_{AB}$, which are evaluated in local frames. Eq. (100) are thus invariant under rigid body motions. The external forces $\mathbf{g}_{ext}$ may however depend on the actual position and orientation of the beam element. As discussed in Section 4.2., the Lie bracket relationship in Eqs. (34-35) is automatically satisfied thanks to the interpolation method and thus does not appear here in the discretized version. According to the treatment of the velocities, Eqs. (98-99) are the discretized version of the compatibility equations in time from Eq. $(26)_2$.

This problem is still continuous in time. These equations can be discretized in time and solved according, e.g., to the generalized-α Lie group time integrator presented in [6]. This algorithm is based on a solution process which does not introduce any parameterization of the global motion and is thus well suited for the present framework. Due to the non-linearity in $\mathbf{d}$ and $\mathbf{v}_{AB}$, an iterative process is required and involves the tangent matrices (see [9]), which inherits naturally the invariance property under rigid body motions from the framework.

4.4. Computational aspects

Based on the special Euclidean group framework, the discretization method discussed in Section 4 exhibits important advantages from a computational point of view.

Besides the importance of the fundamental coupling between the rotation and position variables inherent to the framework, the proposed beam formulation turns out to be particularly suitable for flexible multibody system applications where large amplitude motions involving small deformations are usually expected. Indeed, the equations of motion are naturally expressed in the local frame and, accordingly, the whole formulation is invariant under rigid body motions and only varies due to the deformations. Therefore, assuming small deformations, the non-linearity of the equations of motion is reduced and problems could be solved without updating the iteration matrix which let us expect a significant reduction in computation time. In order to illustrate the potential of the method, a simple numerical example is considered. Following [18], a constant force of 4 N and a constant torque of 80 Nm are applied during 2.5 s at node A of the beam depicted in Fig. 4. Then, the free motion is observed during 50 s. The beam properties are $EA = GA = 1e4$ N, $GJ = EI = 1e3$ Nm^2, $L = 10$ m, $m/L = 1$ kg/m and $\mathbf{J} = \text{diag}(20, 10, 10)$ kgm. The beam is discretized with 10 elements of equal length and the Lie group time integrator proposed in [6, 8] is used with 1e−1 s time step and a 0.9 spectral radius. The displacements of the free end, namely node B, are plotted in Fig. 5. They are in good agreement with [18].

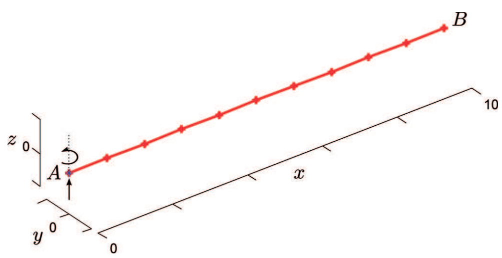

Fig. 4. Force driven flexible beam in helicoidal motion: initial configuration. A force and a torque are exerted at node A during the first 2.5 seconds. Node B is free and Node A is constrained to move along the vertical axes

The motion is clearly three dimensional and has large amplitude. But since the deformations are small, the problem can be solved without updating the iteration matrix related to the beam elements during the entire simulation (that is using the iteration matrix related to the beam elements computed in the reference (undeformed) configuration for the whole simulation). Doing so increases slightly the number of iterations: 4.088 for the updated iteration matrix scheme and 6.252 for the frozen iteration matrix scheme. However,

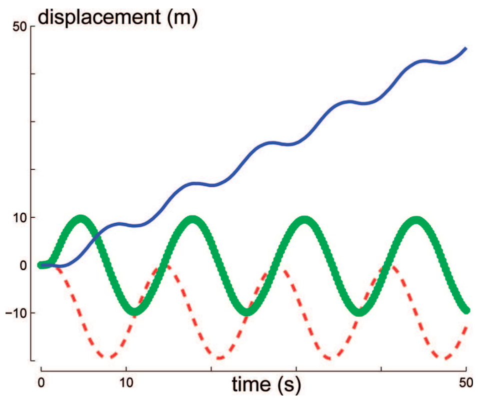

Fig. 5. Force driven flexible beam in helicoidal motion. Displacement of node B. Dashed line (- -): x, Doted line (⋆): y, Solid line (–): z

computation time is saved since the iteration matrix must be inverted only once.

5. Conclusions

The paper discusses a geometric interpretation of a non-linear beam finite element formulated on the special Euclidean group $SE(3)$. The equilibrium equations obtained in this framework are naturally expressed in terms of deformations and velocities evaluated in the material frame. Accordingly, the equilibrium equations are invariant under rigid body motions and free from geometric non-linearities. As it is shown through simple examples, these equations can be often solved without referring to the actual position or orientation of the beam.

However, the equilibrium equations are in general not easy to solve so that approximation methods must be used. Furthermore the knowledge of the position and orientation of the beam is needed in some cases, for instance to apply complex boundary conditions or orientation dependent forces. To this purpose, a discretization method based on position and orientation nodal variables is introduced and is shown to preserve the invariance property of the equilibrium equations. Furthermore, a consistent interpolation formula for the velocity is derived. Thanks to the reduced non-linearities of the equilibrium equations, geometrically non-linear problems can be solved without updating the iteration matrix which is computed once in the reference configuration. This fact leads to significant savings in computation time for flexible multibody system applications.

Acknowledgment

The first author would like to acknowledge the Belgian National Fund for Scientific Research for its financial support (FRIA fellowship).

Manuscript received by Editorial Board, December 22, 2013;
final version, May 30, 2014.

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Interpretacja geometryczna nieliniowego belkowego elementu skończonego w formalizmie grupy Liego SE(3)

S t r e s z c z e n i e

W ostatnim czasie autorzy zaproponowali geometrycznie dokładne sformułowanie dla belkowego elementu skończonego w oparciu o formalizm grupy Liego SE(3). Otrzymano szereg istotnych wyników numerycznych i teoretycznych prowadzących do efektywnej strategii obliczeniowej. Dla przykładu, formalizm ten pozwala uzyskać niezmiennicze równania równowagi przy ruchach ciała sztywnego i elemencie wolnym od blokowania siłami ścinającymi. W obecnym artykule autorzy zajmują się kilkoma istotnymi aspektami tego formalizmu. Właściwość niezmienniczości równań równowagi w warunkach ruchu ciała sztywnego przedyskutowano i zilustrowano prostymi przykładami analitycznymi. Przypomniano metodę dyskretyzacji opartą na mapowaniu wykładniczym i pokazano jej interpretację geometryczną. Specjalną uwagę poświęcono zgodnej interpolacji prędkości.