

ON THE EXTENSIONS OF BARLOW-PROSCHAN IMPORTANCE INDEX AND SYSTEM SIGNATURE TO DEPENDENT LIFETIMES

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ABSTRACT. For a coherent system the Barlow-Proschan importance index, defined when the component lifetimes are independent, measures the probability that the failure of a given component causes the system to fail. Iyer (1992) extended this concept to the more general case when the component lifetimes are jointly absolutely continuous but not necessarily independent. Assuming only that the joint distribution of component lifetimes has no ties, we give an explicit expression for this extended index in terms of the discrete derivatives of the structure function and provide an interpretation of it as a probabilistic value, a concept introduced in game theory. This enables us to interpret Iyer's formula in this more general setting. We also discuss the analogy between this concept and that of system signature and show how it can be used to define a symmetry index for systems.

1. INTRODUCTION

Consider an n -component system $S = (n, \phi, F)$, where ϕ denotes the associated structure function $\phi: \{0, 1\}^n \rightarrow \{0, 1\}$ (which expresses the state of the system in terms of the states of its components) and F denotes the joint c.d.f. of the component lifetimes X_1, \dots, X_n , that is,

$$F(t_1, \dots, t_n) = \Pr(X_1 \leq t_1, \dots, X_n \leq t_n), \quad t_1, \dots, t_n \geq 0.$$

We assume that the system is *semicoherent*, i.e., the structure function ϕ is nondecreasing in each variable and satisfies the conditions $\phi(0, \dots, 0) = 0$ and $\phi(1, \dots, 1) = 1$.

To avoid cumbersome notation, we identify Boolean n -tuples $\mathbf{x} \in \{0, 1\}^n$ and subsets $A \subseteq [n] = \{1, \dots, n\}$ by setting $x_i = 1$ if and only if $i \in A$. We thus use the same symbol to denote both a function $f: \{0, 1\}^n \rightarrow \mathbb{R}$ and its corresponding set function $f: 2^{[n]} \rightarrow \mathbb{R}$, interchangeably. For instance we have $\phi(1, \dots, 1) = \phi([n])$.

An *importance index* was introduced in 1975 by Barlow and Proschan [1] for systems whose components have continuous and independent lifetimes as the n -tuple \mathbf{I}_{BP} whose j th coordinate $I_{\text{BP}}^{(j)}$ is the probability that the failure of component j causes the system to fail. In mathematical terms,

$$(1) \quad I_{\text{BP}}^{(j)} = \Pr(T = X_j), \quad j \in [n],$$

where T denotes the system lifetime.

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When the components have i.i.d. lifetimes, this index reduces to the n -tuple $\mathbf{b} = (b_1, \dots, b_n)$, where

$$(2) \quad b_j = \sum_{A \in [n] \setminus \{j\}} \frac{1}{n \binom{n-1}{|A|}} \Delta_j \phi(A) = \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{\binom{n-1}{k}} \sum_{\substack{A \in [n] \setminus \{j\} \\ |A|=k}} \Delta_j \phi(A)$$

and $\Delta_j \phi(A) = \phi(A \cup \{j\}) - \phi(A \setminus \{j\})$. Thus, in the i.i.d. case the probability (1) does not depend of the c.d.f. F . Due to this feature it is sometimes referred to as a *structural importance*. Note that the expression on the right-hand side of (2) was actually defined in 1953 in cooperative game theory, where it is known as the Shapley-Shubik value [9, 10].

The concept of *signature*, which reveals a strong analogy with that of Barlow-Proschan importance index (1), was introduced in 1985 by Samaniego [7] (see also [8]) for systems whose components have continuous and i.i.d. lifetimes as the n -tuple $\mathbf{s} = (s_1, \dots, s_n)$, where s_k is the probability that the k th component failure causes the system to fail. That is,

$$s_k = \Pr(T = X_{k:n}),$$

where $X_{k:n}$ denotes the k th smallest lifetime, i.e., the k th order statistic obtained by rearranging the variables X_1, \dots, X_n in ascending order of magnitude.

Boland [2] showed that s_k can be explicitly written in the form

$$(3) \quad s_k = \sum_{\substack{A \in [n] \\ |A|=n-k+1}} \frac{1}{\binom{n}{|A|}} \phi(A) - \sum_{\substack{A \in [n] \\ |A|=n-k}} \frac{1}{\binom{n}{|A|}} \phi(A).$$

Just as for the probability $\Pr(T = X_j)$, in the i.i.d. case the probability $\Pr(T = X_{k:n})$ does not depend on the c.d.f. F . Thus \mathbf{s} can be regarded as the *structural signature*.

Example 1. For a system made up of three serially connected components with i.i.d. lifetimes, we have $\mathbf{s} = (1, 0, 0)$ and $\mathbf{b} = (1/3, 1/3, 1/3)$.

Iyer [3] extended the Barlow-Proschan index to the general dependent case where the c.d.f. F is absolutely continuous. In this setting the index \mathbf{I}_{BP} may depend not only on the structure function ϕ but also on the c.d.f. F . Specifically, starting from the multilinear form of ϕ ,

$$\phi(\mathbf{x}) = \sum_{A \in [n]} m_\phi(A) \prod_{i \in A} x_i,$$

where $m_\phi: 2^{[n]} \rightarrow \mathbb{R}$ is the Möbius transform of ϕ , defined by

$$m_\phi(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|} \phi(B),$$

Iyer obtained the integral formula

$$(4) \quad I_{\text{BP}}^{(j)} = \sum_{A \in [n] \setminus \{j\}} m_\phi(A \cup \{j\}) \int_0^\infty \frac{d}{dt_j} \Pr\left(X_j \leq t_j \text{ and } t < \min_{i \in A} X_i\right) \Big|_{t_j=t} dt.$$

The concept of signature was also extended to the general case of dependent lifetimes; see [4] (see also [6] for an earlier work). Denoting this “extended” signature by the n -tuple $\mathbf{p} = (p_1, \dots, p_n)$, where $p_k = \Pr(T = X_{k:n})$, the authors [4] proved

that, if F is absolutely continuous (actually the assumption that there are no ties among the component lifetimes is sufficient), then

$$(5) \quad p_k = \sum_{|A|=n-k+1} q(A) \phi(A) - \sum_{|A|=n-k} q(A) \phi(A),$$

where the function $q: 2^{[n]} \rightarrow [0, 1]$, called the *relative quality function* associated with F , is defined by

$$q(A) = \Pr\left(\max_{i \notin A} X_i < \min_{i \in A} X_i\right).^1$$

Thus (5) is the non-i.i.d. extension of (3). Note also that the function q has the immediate property

$$(6) \quad \sum_{|A|=k} q(A) = 1, \quad k \in \{0, \dots, n\}.$$

In this paper, assuming only that F has no ties, we give an alternative expression for $I_{\text{BP}}^{(j)}$ as a weighted arithmetic mean over $A \subseteq [n] \setminus \{j\}$ of $\Delta_j \phi(A)$ and whose coefficients depend only on F (Theorem 3), thus providing the analog of (5) for the Barlow-Proschan index. This enables us to retrieve and interpret Iyer's formula (4) in this more general setting of distributions having no ties (Corollary 6). We give necessary and sufficient conditions on F for $I_{\text{BP}}^{(j)}$ to always reduce to (2) regardless of the structure function considered (Proposition 9). We also provide explicit expressions for the coefficient of $\Delta_j \phi(A)$ in the general continuous and independent continuous cases (Propositions 10 and 11) and examine the special case of independent Weibull lifetimes, which includes the exponential model (Corollary 12). Finally, we show how the Barlow-Proschan index can be used to measure a symmetry degree of any system (Section 4).

2. EXPLICIT EXPRESSIONS

Throughout we assume that the joint c.d.f. F of the lifetimes has *no ties*, i.e., we have $\Pr(X_i = X_j) = 0$ for every $i \neq j$. For every $j \in [n]$, we define the function $q_j: 2^{[n] \setminus \{j\}} \rightarrow [0, 1]$ as

$$(7) \quad q_j(A) = \Pr\left(\max_{i \notin A \cup \{j\}} X_i < X_j < \min_{i \in A} X_i\right).$$

For instance, when $n = 4$ we have

$$\begin{aligned} q_2(\{1, 3\}) &= \Pr(X_4 < X_2 < \min\{X_1, X_3\}) \\ &= \Pr(X_4 < X_2 < X_1 < X_3) + \Pr(X_4 < X_2 < X_3 < X_1). \end{aligned}$$

From this example we immediately see that (7) can be rewritten as

$$(8) \quad q_j(A) = \sum_{\substack{\sigma \in \mathfrak{S}_n : \{\sigma(n-|A|+1), \dots, \sigma(n)\} = A \\ \sigma(n-|A|) = j}} \Pr(X_{\sigma(1)} < \dots < X_{\sigma(n)}),$$

where \mathfrak{S}_n denotes the set of permutations on $[n]$.

¹Thus $q(A)$ is the probability that the best $|A|$ components are precisely those in A .

Thus defined, $q_j(A)$ is the probability that the components that are better than component j are precisely those in A .² It then follows immediately that

$$(9) \quad \sum_{A \subseteq [n] \setminus \{j\}} q_j(A) = 1, \quad j \in [n].$$

We also observe that

$$(10) \quad q(A) = \sum_{j \notin A} q_j(A), \quad A \neq [n],$$

$$(11) \quad q(A) = \sum_{j \in A} q_j(A \setminus \{j\}), \quad A \neq \emptyset.$$

Moreover, $q_j(\emptyset) = q(\{j\})$ is the probability that component j is the best component, while $q_j([n] \setminus \{j\}) = q([n] \setminus \{j\})$ is the probability that component j is the worst component.

Proposition 2. *If the variables X_1, \dots, X_n are exchangeable, then*

$$(12) \quad q_j(A) = \frac{1}{(n-|A|)\binom{n}{|A|}} = \frac{1}{n\binom{n-1}{|A|}}$$

for every $j \in [n]$ and every $A \subseteq [n] \setminus \{j\}$.

Proof. Since the variables X_1, \dots, X_n are exchangeable, by (8) we immediately obtain

$$q_j(A) = \frac{1}{\binom{n}{n-|A|-1, 1, |A|}},$$

where the denominator is a multinomial coefficient. \square

We now give an expression for $I_{\text{BP}}^{(j)}$ in terms of the functions q_j and $\Delta_j \phi$. This expression, given in (13) below, clearly extends (2) just as formula (5) extends (3).

Theorem 3. *For every $j \in [n]$, we have*

$$(13) \quad I_{\text{BP}}^{(j)} = \sum_{A \subseteq [n] \setminus \{j\}} q_j(A) \Delta_j \phi(A) = \sum_{A \subseteq [n]} (-1)^{|\{j\} \setminus A|} q_j(A \setminus \{j\}) \phi(A).$$

Proof. For every fixed $\sigma \in \mathfrak{S}_n$ we must have

$$(14) \quad \Pr(T = X_j \mid X_{\sigma(1)} < \dots < X_{\sigma(n)}) = \phi(\{\sigma(i), \dots, \sigma(n)\}) - \phi(\{\sigma(i+1), \dots, \sigma(n)\}),$$

where $i = \sigma^{-1}(j)$. Indeed, the left-hand expression of (14) takes its values in $\{0, 1\}$ and is exactly 1 if and only if $\{\sigma(1), \dots, \sigma(i-1)\}$ is not a cut set and $\{\sigma(1), \dots, \sigma(i)\}$ is a cut set.³ That is,

$$\phi(\{\sigma(i), \dots, \sigma(n)\}) = 1 \quad \text{and} \quad \phi(\{\sigma(i+1), \dots, \sigma(n)\}) = 0.$$

Now, by combining (14) with the law of total probability, we get

$$I_{\text{BP}}^{(j)} = \sum_{\sigma \in \mathfrak{S}_n} \left(\phi(\{\sigma(\sigma^{-1}(j)), \dots, \sigma(n)\}) - \phi(\{\sigma(\sigma^{-1}(j)+1), \dots, \sigma(n)\}) \right) \Pr(X_{\sigma(1)} < \dots < X_{\sigma(n)}).$$

²By definition the functions q_j ($j \in [n]$) clearly depend only on the distribution function F (and not on the structure function ϕ).

³Recall that a subset $K \subseteq [n]$ of components is a *cut set* for the function ϕ if $\phi([n] \setminus K) = 0$.

Grouping the terms for which $\{\sigma(\sigma^{-1}(j) + 1), \dots, \sigma(n)\}$ is a fixed set A and then summing over A , we obtain

$$I_{\text{BP}}^{(j)} = \sum_{A \subseteq [n] \setminus \{j\}} (\phi(A \cup \{j\}) - \phi(A)) \sum_{\substack{\sigma \in \mathfrak{S}_n : \{\sigma(n-|A|+1), \dots, \sigma(n)\} = A \\ \sigma(n-|A|) = j}} \Pr(X_{\sigma(1)} < \dots < X_{\sigma(n)}).$$

The result then follows from (8). The second expression in (13) follows immediately from the first one. \square

Example 4. Assume that ϕ defines a k -out-of- n structure, that is, $\phi(\mathbf{x}) = x_{k:n}$, where $x_{k:n}$ is the k th order statistic of the variables x_1, \dots, x_n . In this case we have $\phi(A) = 1$ if and only if $|A| \geq n - k + 1$ and hence, for every $j \in [n]$, we have $\Delta_j \phi(A) = 1$ if and only if $|A| = n - k$. By Theorem 3,

$$I_{\text{BP}}^{(j)} = \sum_{\substack{A \subseteq [n] \setminus \{j\} \\ |A| = n - k}} q_j(A)$$

is the probability that component j has the k th smallest lifetime. This result was expected since $I_{\text{BP}}^{(j)} = \Pr(X_j = T) = \Pr(X_j = X_{k:n})$ by definition.

Example 5. Consider a 5-component system whose structure function $\phi: \{0, 1\}^5 \rightarrow \{0, 1\}$ is defined by

$$\phi(x_1, \dots, x_5) = x_1 x_4 \sqcup x_2 x_5 \sqcup x_1 x_3 x_5 \sqcup x_2 x_3 x_4$$

(see Figure 1), where \sqcup is the binary coproduct (Boolean disjunction) operation defined by $x \sqcup y = 1 - (1 - x)(1 - y)$. In this case we have $\Delta_3 \phi(A) = 1$ if and only if $A = \{1, 5\}$ or $A = \{2, 4\}$. By Theorem 3, we then have

$$\begin{aligned} I_{\text{BP}}^{(3)} &= q_3(\{1, 5\}) + q_3(\{2, 4\}) \\ &= \Pr(\max\{X_2, X_4\} < X_3 < \min\{X_1, X_5\}) + \Pr(\max\{X_1, X_5\} < X_3 < \min\{X_2, X_4\}). \end{aligned}$$

Evidently this value depends on the c.d.f. of the component lifetimes and reduces to $1/15$ in the exchangeable case (see Proposition 2).

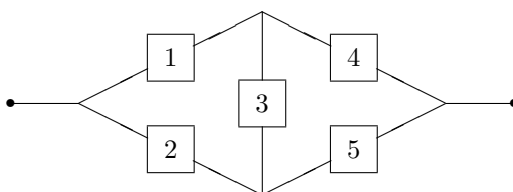


FIGURE 1. Bridge structure

Formula (13) provides an explicit expression for the Barlow-Proshchan index under the sole assumption that F has no ties, which leads to easy interpretations and computations and reveals an interesting analogy with the concept of signature. Combining this formula with (9) also shows that \mathbf{I}_{BP} is a *probabilistic value*, as defined in game theory by Weber [11]. Moreover, \mathbf{I}_{BP} is *efficient* in the sense that $\sum_{j=1}^n I_{\text{BP}}^{(j)} = 1$.

We now use (13) to derive Iyer's formula (4) in an interpretable form and without the absolute continuity assumption. For every $j \in [n]$, we define the function $r_j: 2^{[n] \setminus \{j\}} \rightarrow [0, 1]$ as

$$r_j(A) = \Pr\left(X_j < \min_{i \in A} X_i\right).$$

That is, $r_j(A)$ is the probability that the components in A are better than component j . We then have

$$(15) \quad r_j(A) = \sum_{\substack{B \subseteq [n] \setminus \{j\} \\ B \supseteq A}} q_j(B),$$

which can be inverted into

$$(16) \quad q_j(A) = \sum_{\substack{B \subseteq [n] \setminus \{j\} \\ B \supseteq A}} (-1)^{|B| - |A|} r_j(B).$$

Corollary 6. *For every $j \in [n]$, we have*

$$I_{\text{BP}}^{(j)} = \sum_{A \subseteq [n] \setminus \{j\}} r_j(A) m_\phi(A \cup \{j\}).$$

Proof. Using the inverse Möbius transform $\phi(A) = \sum_{B \subseteq A} m_\phi(B)$ in the right-hand side of (13) and then permuting the resulting sums, we obtain

$$I_{\text{BP}}^{(j)} = \sum_{B \subseteq [n]} m_\phi(B) \sum_{A \supseteq B} (-1)^{|[j] \setminus A|} q_j(A \setminus \{j\}).$$

Noticing that the inner sum vanishes whenever $B \not\supseteq j$ and setting $A' = A \setminus \{j\}$ and $B' = B \setminus \{j\}$, we obtain

$$I_{\text{BP}}^{(j)} = \sum_{B' \subseteq [n] \setminus \{j\}} m_\phi(B' \cup \{j\}) \sum_{\substack{A' \subseteq [n] \setminus \{j\} \\ A' \supseteq B'}} q_j(A').$$

We then conclude by (15). \square

Just as for the Barlow-Proschan index, the signature \mathbf{p} also has an interesting expression in terms of the Möbius transform of ϕ .

Proposition 7. *For every $k \in [n]$, we have*

$$p_k = \sum_{A \subseteq [n]} m_\phi(A) \Pr\left(X_{k:n} = \min_{i \in A} X_i\right),$$

where $\Pr(X_{k:n} = \min_{i \in A} X_i)$ is the probability that the k th failure is that of the worst component in A .

Proof. Using the inverse Möbius transform, for every $k \in [n]$ we have

$$\sum_{|B|=k} q(B) \phi(B) = \sum_{|B|=k} q(B) \sum_{A \subseteq B} m_\phi(A) = \sum_{A \subseteq [n]} m_\phi(A) \sum_{\substack{B \supseteq A \\ |B|=k}} q(B),$$

where the inner sum is the probability $\Pr(X_{n-k:n} < \min_{i \in A} X_i)$ that the components in A be among the best k components. We then conclude by (5). \square

Remark 1. The proof of Proposition 7 shows that

$$\Pr\left(X_{k:n} = \min_{i \in A} X_i\right) = \sum_{\substack{B \supseteq A \\ |B|=n-k+1}} q(B) - \sum_{\substack{B \supseteq A \\ |B|=n-k}} q(B)$$

is exactly the k th coordinate p_k of the signature of the semicoherent system obtained from the current system by transforming the structure function into $\phi(\mathbf{x}) = \prod_{i \in A} x_i$.⁴ It also shows that the *tail signature* [4], defined by $\Pr(T > X_{k:n}) = \sum_{i=k+1}^n p_i$, has the Möbius representation

$$\Pr(T > X_{k:n}) = \sum_{|A|=n-k} q(A) \phi(A) = \sum_{A \subseteq [n]} m_\phi(A) \Pr\left(X_{k:n} < \min_{i \in A} X_i\right).$$

The next corollary, which follows immediately from Proposition 2 and Theorem 3, gives a sufficient condition on F for the equality $\mathbf{I}_{\text{BP}} = \mathbf{b}$ to hold regardless of the structure function considered.

Corollary 8. *If the variables X_1, \dots, X_n are exchangeable, then $\mathbf{I}_{\text{BP}} = \mathbf{b}$.*

We now give necessary and sufficient conditions on F for the equality $\mathbf{I}_{\text{BP}} = \mathbf{b}$ to hold for every structure function.

A system is said to be *coherent* if it is semicoherent and its structure function ϕ has only *essential* variables, i.e., for every $j \in [n]$, there exists $\mathbf{x} \in \{0, 1\}^n$ such that $\phi(\mathbf{x})|_{x_j=0} \neq \phi(\mathbf{x})|_{x_j=1}$.⁵ Let Φ_n (resp. Φ'_n) denote the family of n -variable structure functions corresponding to coherent (resp. semicoherent) systems.

Proposition 9. *The equality $\mathbf{I}_{\text{BP}} = \mathbf{b}$ holds for every $\phi \in \Phi_n$ (or equivalently, for every $\phi \in \Phi'_n$) if and only if (12) holds for every $j \in [n]$ and every $A \subseteq [n] \setminus \{j\}$.*

Proof. We can assume that $n \geq 3$ (the cases $n = 1$ and $n = 2$ can be checked easily). Using (13) and (2), we see that the identity $\mathbf{I}_{\text{BP}} = \mathbf{b}$ can be written as

$$\sum_{A \subseteq [n]} (-1)^{|\{j\} \setminus A|} q_j(A \setminus \{j\}) \phi(A) = \sum_{A \subseteq [n]} (-1)^{|\{j\} \setminus A|} \frac{1}{n \binom{n-1}{|A \setminus \{j\}|}} \phi(A).$$

It was shown in [5] that for any function $\lambda: 2^{[n]} \rightarrow \mathbb{R}$ we have

$$\sum_{A \subseteq [n]} \lambda(A) \phi(A) = 0, \quad \text{for every } \phi \in \Phi_n \text{ (or every } \phi \in \Phi'_n)$$

if and only if $\lambda(A) = 0$ for all $A \neq \emptyset$. Therefore we have $\mathbf{I}_{\text{BP}} = \mathbf{b}$ for every $\phi \in \Phi_n$ (or every $\phi \in \Phi'_n$) if and only if $q_j(A \setminus \{j\}) = 1 / (n \binom{n-1}{|A \setminus \{j\}|})$ for every $j \in [n]$ and every $A \neq \emptyset$. This completes the proof. \square

We observe that $q_j(A)$ has the form (12) for every $j \in [n]$ and every $A \subseteq [n] \setminus \{j\}$ if and only if the map $(j, A) \mapsto q_j(A)$ is symmetric in the sense that

$$q_{\sigma(j)}(\sigma(A)) = q_j(A)$$

for every $j \in [n]$, every $A \subseteq [n] \setminus \{j\}$, and every permutation σ on $[n]$. Indeed, by (6) and (10), for any $k \in [n-1]$ we have

$$\sum_{|A|=k} \sum_{j \notin A} q_j(A) = \sum_{|A|=k} q(A) = 1.$$

⁴This fact also follows immediately from the identity $p_k = \Pr(T = X_{k:n})$ since this modified system has lifetime $T = \min_{i \in A} X_i$.

⁵In other words, every component of the system is relevant.

The identity (12) then follows from the symmetry of the map $(j, A) \mapsto q_j(A)$.⁶

The “signature” version of Proposition 9 can be stated as follows (see [5]). For $n \geq 3$ (resp. $n \geq 2$), the equality $\mathbf{p} = \mathbf{s}$ holds for every $\phi \in \Phi_n$ (resp. every $\phi \in \Phi'_n$) if and only if $q(A) = 1/\binom{n}{|A|}$ for every $A \subseteq [n]$ (i.e., q is a symmetric function).

Remark 2. It can immediately be seen that the function q is symmetric (i.e., $q(A) = 1/\binom{n}{|A|}$ for every $A \subseteq [n]$) as soon as condition (12) holds for every $j \in [n]$ and every $A \subseteq [n] \setminus \{j\}$. Let us now show that the converse statement does not hold in general. Suppose $n = 3$ and set $p_{ijk} = \Pr(X_i < X_j < X_k)$. Since F has no ties, we must have

$$(17) \quad p_{123} + p_{132} + p_{213} + p_{231} + p_{312} + p_{321} = 1.$$

It is clear that the function q is symmetric if and only if

$$(18) \quad p_{231} + p_{321} = p_{132} + p_{312} = p_{123} + p_{213} = 1/3,$$

$$(19) \quad p_{312} + p_{321} = p_{213} + p_{231} = p_{123} + p_{132} = 1/3.$$

Combining Eq. (17)–(19) with the fact that $0 \leq p_{ijk} \leq 1$, we see that q is symmetric if and only if there exists $\lambda \in [0, 1]$ such that

$$(p_{123}, p_{132}, p_{213}, p_{231}, p_{312}, p_{321}) = (\lambda, 1 - \lambda, 1 - \lambda, \lambda, \lambda, 1 - \lambda)/3.$$

On the other hand, we have $q_j(A) = 1/(3\binom{2}{|A|})$ for every $j \in [3]$ and every $A \subseteq [3] \setminus \{j\}$ if and only if $p_{ijk} = 1/6$ for every permutation (i, j, k) of $(1, 2, 3)$.

3. CONTINUOUS AND INDEPENDENT CONTINUOUS LIFETIMES

We now assume that the component lifetimes are absolutely continuous with p.d.f. f . This assumption enables us to derive explicit integral formulas for $q_j(A)$ and $r_j(A)$.

Proposition 10. *For absolutely continuous lifetimes, we have*

$$(20) \quad q_j(A) = \int_0^\infty \int_{]0, t_j[[^{[n] \setminus (A \cup \{j\})} \int_{]t_j, \infty[^A} f(\mathbf{t}) dt_A dt_{[n] \setminus (A \cup \{j\})} dt_j$$

and

$$(21) \quad r_j(A) = \int_0^\infty \int_{]0, \infty[^{[n] \setminus (A \cup \{j\})} \int_{]t_j, \infty[^A} f(\mathbf{t}) dt_A dt_{[n] \setminus (A \cup \{j\})} dt_j$$

for every $j \in [n]$ and every $A \subseteq [n] \setminus \{j\}$.

Proof. By definition we have $q_j(A) = \Pr(E)$, where E is the event

$$\max_{i \notin A \cup \{j\}} X_i < X_j < \min_{i \in A} X_i,$$

which can be described by the set

$$\{(t_1, \dots, t_n) \in]0, \infty[^n : t_i < t_j \ \forall i \notin A \cup \{j\}, t_j < t_i \ \forall i \in A\}.$$

Formula (20) then follows by integrating the p.d.f. over this event. Formula (21) can be established similarly. \square

We now consider the special case of independent and absolutely continuous lifetimes X_1, \dots, X_n , each X_i having p.d.f. f_i and c.d.f. F_i , with $F_i(0) = 0$. The following immediate result shows how formulas (20) and (21) can be simplified. Similar formulas for $q(A)$ can be found in [4].

⁶We note that this result provides an alternative proof of Proposition 2.

Proposition 11. *For independent and absolutely continuous lifetimes, we have*

$$(22) \quad q_j(A) = \int_0^\infty f_j(t) \prod_{i \in A} \bar{F}_i(t) \prod_{i \notin A \cup \{j\}} F_i(t) dt$$

and

$$(23) \quad r_j(A) = \int_0^\infty f_j(t) \prod_{i \in A} \bar{F}_i(t) dt$$

for every $j \in [n]$ and every $A \subseteq [n] \setminus \{j\}$, where $\bar{F}_i(t) = 1 - F_i(t)$.

Using (23) and then (16) leads immediately to the following corollary.

Corollary 12. *For independent Weibull lifetimes, with $F_i(t) = 1 - e^{-(\lambda_i t)^\alpha}$, we have*

$$(24) \quad r_j(A) = \frac{\lambda_\alpha(\{j\})}{\lambda_\alpha(A \cup \{j\})}$$

and

$$(25) \quad q_j(A) = \sum_{\substack{B \subseteq [n] \setminus \{j\} \\ B \supseteq A}} (-1)^{|B| - |A|} \frac{\lambda_\alpha(\{j\})}{\lambda_\alpha(B \cup \{j\})}$$

for every $j \in [n]$ and every $A \subseteq [n] \setminus \{j\}$, where $\lambda_\alpha(A) = \sum_{i \in A} \lambda_i^\alpha$.

We observe that, under the assumptions of Corollary 12, by (24) the ratio

$$\frac{\lambda_\alpha(\{j\})}{\lambda_\alpha([n])} = r_j([n] \setminus \{j\}) = q_j([n] \setminus \{j\})$$

is exactly the probability that X_j is the shortest lifetime.

4. A SYMMETRY INDEX FOR SYSTEMS

A natural concept of symmetry for systems can be defined as follows. We say that a semicoherent system $S = (n, \phi, F)$ is *symmetric* if it has a uniform Barlow-Proschan index, i.e., $\mathbf{I}_{\text{BP}} = (1/n, \dots, 1/n)$.

Example 13. For a system made up of n serially connected components, we have $I_{\text{BP}}^{(j)} = q_j([n] \setminus \{j\}) = q([n] \setminus \{j\})$ for every $j \in [n]$. The system is then symmetric if and only if $q([n] \setminus \{j\})$ is independent of j .

Since a system is rarely symmetric in the non-i.i.d. case, it is natural to define an index measuring a ‘‘symmetry degree’’ of the system.

Recall from probability theory that the uniformity of a probability distribution $\mathbf{w} = (w_1, \dots, w_n)$ over $[n]$ can be measured through the concept of *normalized Shannon entropy*

$$H(\mathbf{w}) = -\frac{1}{\ln n} \sum_{i=1}^n w_i \ln(w_i),$$

with the convention that $0 \ln 0 = 0$. It is well known that $H(\mathbf{w})$ is maximum ($H(\mathbf{w}) = 1$) if and only if \mathbf{w} is the uniform distribution $\mathbf{w}^* = (1/n, \dots, 1/n)$ and minimum ($H(\mathbf{w}) = 0$) if and only if $w_j = 1$ for some $j \in [n]$ and $w_i = 0$ for all $i \neq j$ (Dirac measure). Moreover, for any probability distribution $\mathbf{w} \neq \mathbf{w}^*$, the expression $H(\mathbf{w}_\lambda)$, where $\mathbf{w}_\lambda = \mathbf{w} + \lambda(\mathbf{w}^* - \mathbf{w})$, strictly increases as the parameter λ increases from 0 to 1. Thus, the number $H(\mathbf{w})$, which lies in the interval $[0, 1]$, measures a uniformity (evenness) degree of the probability distribution \mathbf{w} .

On the basis of these observations we define a symmetry index as follows.

Definition 14. The *symmetry index* for a semicoherent system $S = (n, \phi, F)$ is the number $H(\mathbf{I}_{\text{BP}})$, that is, the normalized Shannon entropy of \mathbf{I}_{BP} .

It seems intuitive—at least in the exchangeable case—that the lower $H(\mathbf{I}_{\text{BP}})$ (i.e., the more concentrated the distribution \mathbf{I}_{BP}) the more the system components play an asymmetric role. Similarly, the lower $H(\mathbf{p})$ the more the components play a symmetric role. This observation is confirmed by the next two propositions, where we provide conditions under which the entropies $H(\mathbf{I}_{\text{BP}})$ and $H(\mathbf{p})$ reach their extreme values.

Proposition 15. *Let $S = (n, \phi, F)$ be a semicoherent system.*

- (i) *If the functions q_i ($i = 1, \dots, n$) are strictly positive, then $H(\mathbf{I}_{\text{BP}}) = 0$ if and only if $\phi(\mathbf{x}) = x_j$ for some $j \in [n]$ (i.e., exactly one component of S is relevant).*
- (ii) *If the function q is strictly positive, then $H(\mathbf{p}) = 0$ if and only if $\phi(\mathbf{x}) = x_{k:n}$ for some $k \in [n]$ (i.e., S is a k -out-of- n system).*

Proof. Let us prove (i). Suppose first that $H(\mathbf{I}_{\text{BP}}) = 0$. This means that $I_{\text{BP}}^{(j)} = 1$ for some $j \in [n]$. Equivalently, by (13) we have

$$\sum_{A \in [n] \setminus \{i\}} q_i(A) \Delta_i \phi(A) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$$

By (9) and due to the positivity of q_i , this means that $\Delta_i \phi(A) = 1$ if and only if $i = j$. Equivalently, $\phi(A) = 1$ if and only if $j \in A$. Thus, $\phi(\mathbf{x}) = x_j$. The converse implication immediately follows from (9) and (13).

Let us prove (ii). Suppose first that $H(\mathbf{p}) = 0$. This means that $p_k = 1$ for some $k \in [n]$. Equivalently, by (5) we have,

$$\sum_{|A|=i} q(A) \phi(A) = \begin{cases} 1, & \text{if } i \geq n - k + 1 \\ 0, & \text{otherwise.} \end{cases}$$

By (6) and due to the positivity of q , this condition means that $\phi(A) = 1$ if and only if $|A| \geq n - k + 1$. Thus, $\phi(\mathbf{x}) = x_{k:n}$. The converse implication immediately follows from (5) and (6). \square

Proposition 16. *Let $S = (n, \phi, F)$ be a semicoherent system.*

- (i) *If the functions q_j ($j = 1, \dots, n$) have the form (12) and $\phi(\mathbf{x}) = x_{k:n}$ for some $k \in [n]$, then $H(\mathbf{I}_{\text{BP}}) = H(\mathbf{b}) = 1$.*
- (ii) *If the function q is symmetric and $\phi(\mathbf{x}) = x_j$ for some $j \in [n]$, then $H(\mathbf{p}) = H(\mathbf{s}) = 1$.*

Proof. Let us prove (i). Since $\phi(\mathbf{x}) = x_{k:n}$ for some k , we have $\Delta_j \phi(A) = 1$ if and only if $|A| = n - k$. It follows that

$$I_{\text{BP}}^{(j)} = \binom{n-1}{n-k} \frac{1}{n \binom{n-1}{n-k}} = \frac{1}{n}$$

and hence $H(\mathbf{I}_{\text{BP}}) = 1$. We also have $\mathbf{I}_{\text{BP}} = \mathbf{b}$ by Proposition 9.

Let us prove (ii). Since $\phi(\mathbf{x}) = x_j$ for some j , we have $\phi(A) = 1$ if and only if $j \in A$. It follows that

$$p_k = \sum_{|A|=n-k+1} \frac{1}{\binom{n}{|A|}} \phi(A) - \sum_{|A|=n-k} \frac{1}{\binom{n}{|A|}} \phi(A) = \frac{\binom{n-1}{n-k}}{\binom{n}{n-k+1}} - \frac{\binom{n-1}{n-k-1}}{\binom{n}{n-k}} = \frac{1}{n}$$

for $k < n$ (and also for $k = n$) and hence $H(\mathbf{p}) = 1$. We also have $\mathbf{p} = \mathbf{s}$ (see the paragraph before Remark 2). \square

Remark 3. Put in other words, Proposition 15 states that

- (i) Under positiveness of the functions q_j , the mass of \mathbf{I}_{BP} is concentrated on exactly one coordinate (\mathbf{I}_{BP} is a Dirac measure) if and only if exactly one component of the system is relevant.
- (ii) Under positiveness of the function q , the mass of \mathbf{p} is concentrated on exactly one coordinate (\mathbf{p} is a Dirac measure) if and only if the system is of k -out-of- n type for some k .

Similarly, Proposition 16 states that

- (i) For a k -out-of- n system, if the functions q_j ($j = 1, \dots, n$) have the form (12), then \mathbf{I}_{BP} is uniform.
- (ii) For a system with only one relevant component, if the function q is symmetric, then \mathbf{p} is uniform.

Even though the positiveness of the functions q_j and q seems restrictive, it is a rather natural assumption and actually necessary for Proposition 15 to hold.

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