Lloyd relaxation using analytical Voronoi diagram in the L_∞ norm and its application to quad optimization.

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Summary. An incremental Delaunay triangulation algorithm to generate Voronoi diagrams within the L_∞ norm is presented. The main qualities are the preservation of the simplicity of the classical L_2 version and its its intrinsic robustness It is then coupled to the well known Lloyd algorithm for computing Centroidal Voronoi Tesselations of point sets. This algorithm is then used to generate well shaped quadrilateral meshes.

1 Introduction

The Voronoi diagram of a set of vertices in the euclidian 2D space is one of the most studied topic of the computational geometry field. Nethertheless, if its L_2 metric version is well known, its extension in different L_p metrics is less known. As a consequence, the resulting applications have not been extensively explored, due to the lack of a practical algorithm to build such Voronoi diagrams.

In this research note we will first describe a version of the Bowyer-Watson algorithm used to compute the Delaunay triangulation and its associated Voronoi diagram of a set of points in the L_∞ metrics. We will show that these diagrams can be used to compute the Centroidal Voronoi Tesselation (CVT) of a set of point in the L_∞ metrics. We will use the nice properties of these diagrams to generate and optimize a quadrilateral mesh.

1.1 Construction to the L_∞ Voronoi Diagram

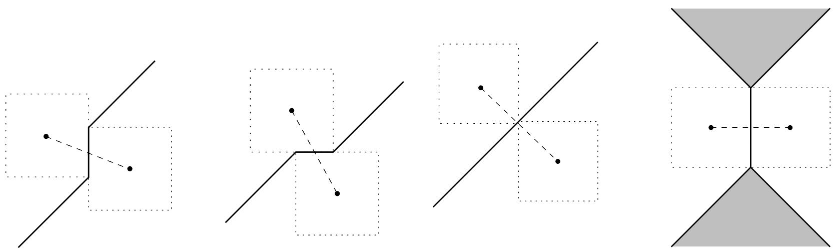
The L_p **norm** of a vector $\mathbf{x}$, noted $\|\mathbf{x}\|_p$, is a function that assigns a positive length to all vectors in a vector space E_d:

$$\|\mathbf{x}\|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}} \tag{1}$$

It can be shown that for $p \to \infty$, $\|\mathbf{x}\|_\infty = \max(|x_i|)$.

The Voronoi diagramm in the L_∞ metric has already been studied by [3], [6] and [1]. At least the two implementations are plane sweeping algorithms. We propose here an incremental version.

The L_∞ **bisector** of two generator points is composed of 3 line segments (see Fig. 1). Because the Voronoi cell of a generator point in the L_∞ norm is starshaped and considering the convex shape of the L_∞ norm, the classical algorithms for Delaunay triangulation can be used under minimal changes.



(a) Regular cases (b) Singular cases

Fig. 1. Bissector of 2 generator points.

Delaunay Triangulation

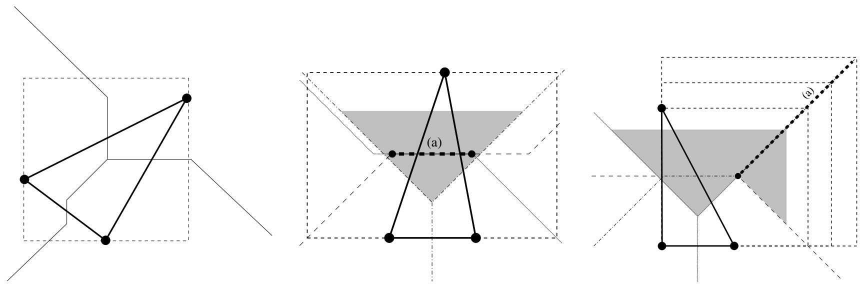
The predicate used when constructing a Delaunay triangulation in the L_2 norm is the *incircle test*. In the L_∞ norm the predicate has obviously to be changed. In 2D, it occurs that, geometrically, the levelset of the L_∞ distance to a point is a square. The L_2 *incircle* test thus becomes an *insquare* test in the L_∞ norm. In practice, the test consists in finding the smallest square that encompasses the 3 points of a triangle (see Fig. 2(a)).

It is worth noting that the center of the circumsquare is not always uniquely defined, as two degenerated cases can be encountered. Every point of the thick dotted line (a) is equidistant to all 3 vertices of the triangles.

1. Figure 2(b): The *insquare test* thus becomes a *inrectangle test* in this case.
2. Figure 2(c): The square whose size is minimal is chosen for the *insquare* test.

Voronoi Diagram

Using the Delaunay triangulation, the Voronoi diagram is built by joining the center of the circumsquares of the simplices of the triangulation. Nevertheless, recall that in the L_∞ norm, the bisector of 2 generator points is a broken line composed of at most 3 segments. Thus, it is necessary to compute the intersection of the bisector and the Voronoi vertices (see Fig. 3).



(a) Square encompassing a triangle. (b) Rectangle encompassing a triangle. (c) Infinite set of encompassing squares

Fig. 2. Singular cases for the insquare test.

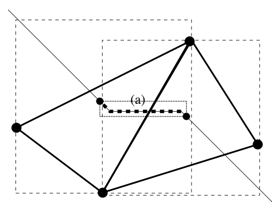


Fig. 3. Bounded bissector of a Voronoi diagram.

2 Application to Quad optimization

A centroidal Voronoi tessellation (CVT) is a special Voronoi tessellation of a given point set such that the generating points are the centroids of their corresponding Voronoi regions with respect to a prescribed density function.

The most used method for computing a CVT is the well known Lloyd algorithm [5]. It consists in generating a new points set from a previous one until certain stopping criterion is met.

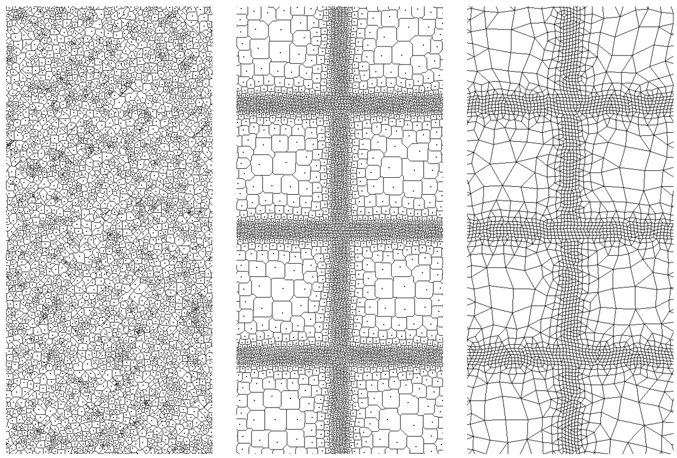
The aim of building a Voronoi diagram generator in the L_∞ norm was to find a way to optimize a quadrilateral mesh. Indeed, it has been shown that a CVT in the case of the L_∞ norm leads to aligned generator points along the frame axis. An important step in the Lloyd algorithm is to compute accurately the centroid of each Voronoi cell. Unlike [4] which compute the CVT energy in the L_∞ norm from a Voronoi cell of a L_2 Voronoi diagram, we have here the opportunity to use an explicit Voronoi diagram in the L_∞ norm, for which we can compute easily the centroid of each cell :

$$\mathbf{z}_i = \frac{\int_{\Omega_i} \mathbf{y}\rho(\mathbf{y})d\mathbf{y}}{\int_{\Omega_i} \rho(\mathbf{y})d\mathbf{y}} \tag{2}$$

Since the Voronoi cells are star shaped as seen from the generator point, they can be decomposed into non overlapping triangular sub-domains for which centroids are easily computed.

3 Example

This test case is obtained in a unit square domain with a prescribed mesh size field which have a value of $h(x, y) = 0.01$ along the crosslines and grow linearly away from the crosslines. The initial set of points is randomly generated inside the domain (see Fig. 4(a)).



(a) Initial Voronoi diagram of a random point distribution

(b) Final Voronoi diagram after 500 iterations

(c) Resulting quad mesh generated from the above Voronoi diagram

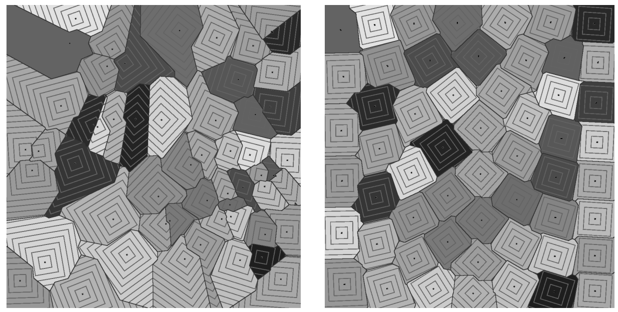
Fig. 4. Example of the Lloyd algorithm applied on a random point set.

The resulting CVT of the initial set of points is presented in Fig. 4(b). The recombination of the Delaunay triangulation then results in a good quality grading quad mesh (see Fig. 4(c)).

4 Conclusion

In this work, an algorithm to generate a quality quad element mesh using the Lloyd iteration algorithm coupled to an analytical Voronoi diagram in the L_∞ norm has been presented.

Nevertheless, to make this method appealing in FEM computations, it should be able to take directional information into account. It is usually expected for quadrilateral elements to be oriented relatively to the boundaries of the domain. The integration of this feature involves to generalize the algorithm in order to build Voronoi diagrams in the L_∞ norm with a specified orientation at each point. Using a discrete definition of anisotropic Voronoi diagrams inspired by [2], Fig. 5 already shows that the Voronoi cells tend to be aligned along the direction field. But in such cases, the complexity is much higher. For three given points and their corresponding directions, there can be more than one equidistant vertex. The resulting Voronoi diagram may thus contain not connected pieces of cells, and the connectivity graphs may not be planar. In order to efficiently address these problems, a currently developped idea is to build an approximated Voronoi diagram.



(a) Initial Voronoi diagram (b) Voronoi diagram after Lloyd iterations

Fig. 5. Voronoi diagram under an orientation field (crossfield)

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