On the Formulæ of Reduction to Apparent Places of Close Polar Stars. By F. Folie.

(*Communicated by A. M. W. Downing.*)

In the *Bulletin Astronomique* (February 1888 and seq.) I made some criticisms upon the method proposed by Fabritius for the reduction of close polar stars, and on his demonstration of it, and I exhibited the discordance between the form adopted and the formulæ used in the *Berliner Jahrbuch*, the latter being more correct.

M. Fabritius, in tom. iii. of the *Observations of Kiew*, having again contested my criticisms, and given a new demonstration of his formulæ, I have made a new inquiry on the terms of second order, by a process giving the same form as his own, but with some additional terms.

The importance of the question, and the recent investigation of Mr. Downing on the computation of apparent places for *Polaris* (*Monthly Notices*, lii. 5, p. 378), give me hope that the present lines will be read with interest by astronomers.

The investigation of the terms of the second order is threefold:—

1. Terms of the second order of the nutation.
2. Terms of the second order due to the combination of the nutation and aberration.
3. Terms of the second order of the aberration.

1. *Terms of the Second Order of the Nutation.*

The first of these investigations is the most complicated, and the method I have adopted is a new one. I here give a summary of it.

The equations:—

$$(1)\qquad \begin{aligned} \frac{d\alpha}{dt} &= (\cot\epsilon + \sin\alpha\tan\delta)\frac{d\mu}{dt} - \cos\alpha\tan\delta\frac{d\theta}{dt} \\ \frac{d\delta}{dt} &= \cos\alpha\frac{d\mu}{dt} + \sin\alpha\frac{d\theta}{dt}; \end{aligned}$$

where

$$d\mu = \sin\epsilon d\lambda,$$

and, if α, δ are the mean values of the co-ordinates for the beginning of the year, give the terms of first order:—

$$(2)\qquad \begin{aligned} \Delta_1\alpha &= (\cot\epsilon + \sin\alpha\tan\delta)\Delta\mu - \cos\alpha\tan\delta\Delta\theta \\ \Delta_1\delta &= \cos\alpha\Delta\mu + \sin\alpha\Delta\theta. \end{aligned}$$

To these terms we must add $\delta\Delta_1\alpha$, $\delta\Delta_1\delta$, because in the equation (1) are included the true co-ordinates; so that α, δ are the mean co-ordinates, and $\alpha + \Delta_1\alpha$, $\delta + \Delta_1\delta$ the true ones.

From (1), if ϵ is considered constant, we have

$$\frac{d\delta\Delta_1\alpha}{dt} = \tan\delta\Delta_1\alpha\left(\cos\alpha\frac{d\mu}{dt} + \sin\alpha\frac{d\theta}{dt}\right) + \sin^2\delta\Delta_1\delta\left(\sin\alpha\frac{d\mu}{dt} - \cos\alpha\frac{d\theta}{dt}\right)$$

$$(3) \qquad = \tan\delta\Delta_1\alpha\frac{d\delta}{dt} + \frac{1}{\sin\delta\cos\delta}\Delta_1\delta\left(\frac{d\alpha}{dt} - \cot\epsilon\frac{d\mu}{dt}\right)$$

$$= \tan\delta\frac{\Delta_1\alpha d\delta + \Delta_1\delta d\alpha}{dt} + \cot\delta\Delta_1\delta\frac{d\alpha}{dt} - \frac{2}{\sin 2\delta}\cot\epsilon\Delta_1\delta\frac{d\mu}{dt},$$

and

$$\frac{d\delta\Delta_1\delta}{dt} = -\Delta_1\alpha\left(\sin\alpha\frac{d\mu}{dt} - \cos\alpha\frac{d\theta}{dt}\right) = -\cot\delta\Delta_1\alpha\left(\frac{d\alpha}{dt} - \cot\epsilon\frac{d\mu}{dt}\right).$$

Integrating and neglecting the terms in which $\tan\delta$ and $\sec\delta$ are not included:

$$\delta\Delta_1\alpha = \tan\delta\Delta_1\alpha\Delta_1\delta - \frac{2}{\sin 2\delta}\cot\epsilon\int\Delta_1\delta\frac{d\mu}{dt}$$

$$(4) \qquad = \tan\delta\Delta_1\alpha\Delta_1\delta - \frac{2\cot\epsilon}{\sin 2\delta}\left(\tfrac{1}{2}\cos\alpha\,(\Delta\mu)^2 + \sin\alpha\int\Delta\theta d\mu\right).$$

$$\delta\Delta_1\delta = -\tfrac{1}{2}\cot\delta(\Delta_1\alpha)^2.*$$

If

$$F = -\frac{2\cot\epsilon}{\sin 2\delta}\left(\tfrac{1}{2}\cos\alpha(\Delta\mu)^2 + \sin\alpha\int\Delta\theta d\mu\right),$$ the equation (4) becomes

$$(4') \qquad \delta\Delta_1\alpha = \tan\delta\Delta_1\alpha\Delta_1\delta + F$$

$$\delta\Delta_1\delta = -\tfrac{1}{2}\cot\delta(\Delta_1\alpha)^2,$$

forms of Fabritius with the term F in addition.

2. *Combination of the Nutation and the Annual Aberration.*

The terms of the second order arising from the combination of the nutation and the aberration are in the very convenient forms given by Wagner †; if $\Delta_1\alpha$, $\Delta_1\delta$ are the terms of first order of the nutation, and A_α, A_δ those of the aberration,

$$(5) \qquad \delta A_\alpha = \tan\delta A_\alpha\Delta_1\delta + \frac{2}{\sin 2\delta}A_\delta\Delta_1\alpha;$$

$$\delta A_\delta = -\tfrac{1}{2}\sin 2\delta A_\alpha\Delta_1\alpha.$$

3. *Terms of the Second Order of the Aberration.*

The terms of the second order of the annual aberration may be put in a similar form if we omit the terms in which $\tan^2\delta$ in right

* If this form gives $\delta\Delta_1\delta$ infinite for $\delta = 0$, as M. Fabritius thinks (*Obs. de Kiew*, t. iii.), it is only in appearance; $(\Delta_1\alpha)^2$ is, in reality, $\Delta_1\alpha\int\left(\frac{d\alpha}{dt} - \cot\epsilon\frac{d\mu}{dt}\right)dt$, and this is of the form $\tan\delta\int f(t)dt$; so that $\cot\delta$ disappears.

† *Obs. de Poulkova*, t. i. See also my *Traité des Réductions Stellaires*, p. 79.

ascension, tan δ or sec δ in declination, are not included, so that:—

(6) $$\Delta A_\alpha = \frac{2}{\sin 2\delta} A_\alpha A_\delta$$

$$\Delta A_\delta = -\tfrac{1}{4}\sin 2\delta A_\alpha{}^2 - \frac{2}{\sin 2\delta}(A_\delta)^2.$$

4. *Some of the Terms of Second Order of the Nutation and of the Annual Aberration.*

Combining now all the terms of the second order given above, Δ_α, Δ_δ being the terms of the first order of the complete reduction to the apparent place, and $\Delta^2\alpha$, $\Delta^2\delta$ those of the second order, we have, first,

$$\Delta_\alpha = \Delta_1\alpha + A_\alpha$$

$$\Delta\delta = \Delta_1\delta + A_\delta.$$

Taking the sum of the formulæ (4′) (5) and (6) we obtain in right ascension

$$\Delta^2\alpha = \tan\delta\Delta_1\delta(\Delta_1\alpha + A_\alpha) + \frac{2}{\sin 2\delta}A_\delta(\Delta_1\alpha - A_\alpha) + F.$$

But

$$\frac{2}{\sin 2\delta} = \frac{\tan\delta}{\sin^2\delta}, \text{ and, for polar stars, } = \tan\delta + \tfrac{1}{2}\sin 2\delta,$$

and sin 2δ being very small, we may neglect ½ sin 2δ, hence

(7) $$\Delta^2\alpha = \tan\delta\Delta\alpha\Delta\delta + F$$

$$\Delta^2\delta = -\tfrac{1}{2}\cot\delta(\Delta_1\alpha)^2 - \tfrac{1}{2}\sin 2\delta A_\alpha\Delta_1\alpha - \tfrac{1}{4}\sin 2\delta A_\alpha{}^2 - \frac{2}{\sin 2\delta}(A_\delta)^2;$$

also, putting

$$\tfrac{1}{2}\sin 2\delta = \cot\delta$$

$$\Delta^2\delta = -\tfrac{1}{2}\cot\delta(\Delta_1\alpha + A_\alpha)^2 - \tan\delta(A_\delta)^2$$

(7′) $$= -\tfrac{1}{2}\cot\delta(\Delta\alpha)^2 - \frac{2}{\sin 2\delta}(A_\delta)^2,$$

this being the first form of Fabritius* with the additional term

$$-\frac{2}{\sin 2\delta}(A_\delta)^2.$$

If, on the contrary, we put ¼ sin 2δ instead of ½ cot δ, we have

(7″) $$\Delta^2\delta = -\tfrac{1}{4}\sin 2\delta(\Delta\alpha)^2 - \frac{2}{\sin 2\delta}(A_\delta)^2,$$

this differing only by the last term from the second form given

* *Obs. de Kiew*, t. i., and Oppolzer, *Traité de la Détermination des Orbites*, p. 264.

by Fabritius in consequence of our criticism.* We give the preference to this last form for two reasons: first, the calculation of sin 2δ is used in the two terms; second, sin 2δ is factor in two terms in (7), and cot δ only in one. Therefore we adopt for the terms of the second order of the reduction to the apparent place

$$\Delta^2\alpha = \tan\delta\Delta\alpha\Delta\delta + \mathrm{F}$$

$$\Delta^2\delta = -\tfrac{1}{4}\sin 2\delta(\Delta\alpha)^2 - \frac{2}{\sin 2\delta}(\mathrm{A}_\delta)^2,$$

differing from those of Fabritius by the last terms, which should not be neglected.

If we make use, for the reduction of close polar stars, of the very convenient forms of Fabritius, we are then obliged to add: in A R,

$$\mathrm{F} = -\cot.\epsilon\tan\delta\{\tfrac{1}{2}\cos\alpha(\Delta\mu)^2 + \sin\alpha\int\Delta\theta d\mu\}$$

$$= -\tan\delta\,\{\cos\alpha[0''\!\cdot\!0023t^2 - 0''\!\cdot\!0015\sin\Omega t] - 0\!\cdot\!0063\sin\alpha\sin\Omega\};$$

and in δ:

$$-\frac{2}{\sin 2\delta}(\mathrm{A}_\delta)^2.$$

5. *Periodic terms of the systematic aberration.*

I have called attention to these terms in my *Traité des Réductions Stellaires* (1888). I will find these terms in right ascension and in declination, not taking account of other terms of the second order, included in the preceding formulæ, with the exception of very small terms (of the second order) depending on the longitude of the Sun and of the perigee.

The formulæ of the aberration are:

$$\text{(1)}\qquad \mathrm{K}\cos\delta^1\cos\alpha^1 = \cos\delta\cos\alpha + \frac{1}{v}\mathrm{V}_x$$

$$\text{(2)}\qquad \mathrm{K}\cos\delta^1\sin\alpha^1 = \cos\delta\sin\alpha + \frac{1}{v}\mathrm{V}_y$$

$$\text{(3)}\qquad \mathrm{K}\sin\delta^1 = \sin\delta + \frac{1}{v}\mathrm{V}_x;$$

from which we take

$$\text{(4)}\qquad \Delta\alpha = \frac{\sec\delta}{v}(\cos\alpha\mathrm{V}_y - \sin\alpha\mathrm{V}_x) - \frac{\sec^2\delta}{v^2}(\cos\alpha\mathrm{V}_y - \sin\alpha\mathrm{V}_x)(\sin\alpha\mathrm{V}_y + \cos\alpha\mathrm{V}_x).$$

From (1) and (2) we may deduce

$$\text{(5)}\qquad \mathrm{K}\cos\delta^1 = \cos\delta + \frac{\cos\delta}{v}(\sin\alpha\mathrm{V}_y + \cos\alpha\mathrm{V}_x) + \tfrac{1}{2}\frac{\sec\delta}{v^2}(\mathrm{V}_x{}^2 + \mathrm{V}_y{}^2),$$

* *Obs. de Kiew*, t. iii.

and from (3) and (5)

$$\Delta\delta = \frac{\cos\delta}{v}\left\{V_z - \sin\delta(\sin\alpha V_y + \cos\alpha V_x\right\} - \tfrac{1}{2}\frac{\tan\delta}{v^2}(V_x{}^2 + V_y{}^2),$$

if we make abstraction of the terms of second order, in which $\tan\delta$ is not included.

If we take only into account the annual and the systematic aberration, calling m_1 the mean motion of the Sun, σ_1 the systematic velocity, σ' its projection on the equator, A′, D′ the coordinates of the apex, s, c the sin and cos of the obliquity, ⊙ the Sun's longitude, we have:

$$V_x = m_1 \sin\odot + \sigma'\cos A'$$
$$V_y = -m_1 c\cos\odot + \sigma'\sin A'$$
$$V_z = -m_1 s\cos\odot + \sigma'\tan D';$$

and therefore

$$(6)\ \cos\alpha V_y - \sin\alpha V_x = -m_1(c\cos\alpha\cos\odot + \sin\alpha\sin\odot) + \sigma'\sin(A' - \alpha)$$

$$(7)\ \sin\alpha V_y + \cos\alpha V_x = -m_1(c\sin\alpha\cos\odot - \cos\alpha\sin\odot) + \sigma'\cos(A' - \alpha)$$

$$V_x{}^2 + V_y{}^2 = m_1{}^2(\sin^2\odot + c^2\cos^2\odot) + \sigma'^2 - 2m_1\sigma'(c\sin A'\cos\odot - \cos A'\sin\odot);$$

or, neglecting the constant terms,

$$V_x{}^2 + V_y{}^2 = -\tfrac{1}{2}m_1{}^2 s^2\cos 2\odot - 2m_1\sigma^1(c\sin A^1\cos\odot - \cos A^1\sin\odot).$$

Taking only into account in the formulæ (6) and (7) the periodical terms of the systematic aberration:

$$(\cos\alpha V_y - \sin\alpha V_x)(\sin\alpha V_y + \cos\alpha V_x)$$
$$= -m_1\sigma'\{c\cos(A' - 2\alpha)\cos\odot - \sin(A' - 2\alpha)\sin\odot\}.$$

The periodical terms of the systematic aberration will be therefore, if a be the annual constant of aberration, and a' the reduced constant of systematic aberration, viz., projected on the equator:

$$\Delta^2\alpha = -ax'\sec^2\delta\{c\cos(A' - 2\alpha)\cos\odot - \sin(A' - 2\alpha)\sin\odot\}$$
$$\Delta^2\delta = aa'\tan\delta(c\sin A'\cos\odot - \cos A'\sin\odot).$$

If $a^1 = a$, aa^1 (expressed in seconds of arc) = 0″·002, the terms above may not be neglected, in the reduction of close polar stars.

These formulæ offer a very easy means of computing a', a means unknown until to-day.

When the constant of the systematic aberration is known, we may compute the terms arising from the combination of this aberration with the nutation. These are (making abstraction of the terms of first order, A'_α and A'_δ, which are not periodical and cannot be used in the formulæ of reduction):

$$\delta A'_{\alpha} = \tan \delta A'_{\alpha} \Delta_1 \delta - \frac{2}{\sin 2\delta} A'_{\delta} \Delta_1 \alpha$$

$$\delta A'_{\delta} = \tfrac{1}{2} \sin 2\delta A'_{\alpha} \Delta_1 \alpha.$$

If the systematic velocity be of the same order as the velocity of the Earth, these terms will be of the same magnitude as the corresponding terms in the annual aberration.

Uccle: May 5.

On a Parallel-Plate Double-Image Micrometer. By J. H. Poynting.

(*Communicated by Sir R. S. Ball.*)

If a ray of light passes through a plate of glass with parallel faces it emerges parallel to its original direction, but with any other than normal incidence it is shifted through a distance proportional nearly to the tangent of the angle of incidence, up to quite considerable values of that angle.

I have already published an account of a use of this property in obtaining a fine adjustment for a cathetometer telescope. (*Phil. Trans.* vol. 182 (1891), A., p. 588.) A parallel plate movable round a horizontal axis is placed just in front of the object-glass. The telescope is brought nearly to the level of the point to be sighted, and the final adjustment is completed by tilting the plate until the point appears on the cross wire.

We have for some time used a similar device with a microscope on the Mason College Physical Laboratory for the measurement of small objects, placing the parallel plate between the stage and objective. It is very easy and very rapid in use. The plate might be placed, for the measurement of smaller objects, between the objective and eyepiece, but we have never required the sensitiveness which would thus be obtained.

I have lately applied the principle in the construction of a double-image micrometer for a telescope, and as the instrument, as far as I can test it, appears to be successful, it may be worthy of description.

A circular parallel plate of glass, rather more than 1 inch in diameter and about $\frac{1}{8}$ inch thick, being cut down the middle, one half is fixed in one semicircle of a ring, while the other is placed in the other semicircle, attached, however, to an axis passing through a bearing in the ring, so that the plate can be rotated, the axis being in the plane of the fixed plate and perpendicular to its edge at the middle point. The ring is