EFFECTIVE MODAL PARAMETERS TO EVALUATE STRUCTURAL STRESSES

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ABSTRACT

Computing the structural stresses induced during vibration qualification tests in a space equipment is a difficult task, as the finite element codes have to use time-varying loads. Alternative methods are proposed for the stress evaluation during vibration tests. Effective modal parameters are used to identify 3 types of stresses: static stresses, *quasistatic* stresses from base acceleration and dynamic stresses resulting from modal development. These stresses can be simply evaluated by static and modal analyses with FEM. Their combination and summation with the proposed formulation yields the real structural stresses produced during the vibration test.
Effects of truncature of the modal series are considered and several methods are proposed to evaluate the different types of structural stresses. These methods are tested with the study of the optical system of the Optical Monitoring Camera (OMC) of the INTEGRAL spacecraft.

1. INTRODUCTION

The structural design of subsystems for space instruments is based on specifications on the environmental conditions that will be seen by the equipment. The design is then verified by analyses and qualification on structural models. Generally, the mechanical environment is specified in terms of static loads called design loads, and uniaxial vibrations applied on the mounting interface through a rigid fixture. The qualification test can then easily be performed with a shaking machine whose interface represents the rigid fixture. Sine and random tests are often specified. For space equipments, the static load specification generates generally lower stress levels than the ones produced with the dynamic environment. In some frequency ranges, or in flexible structures, the stresses in the vibration tests can exceed the static stresses. Therefore, the dynamic stresses can become the driving parameter in the design phase. It means that the vibration loads become the dimensioning loads for the equipment. For this reason, we emphasize the need to evaluate efficiently the dynamic stresses resulting from vibration tests.
This driven-base dynamic environment allows the decoupling of junction degrees of freedom (dof) from the other structural dof. Using a dedicated formulation [1], one can derive the effective parameters of the dynamic model. These parameters play a significant role in the system response. A very first approach using the effective masses of a dynamic system is the well known concept of dynamic mass. It can be used [ref 2 and 4] to derive a first evaluation of an equivalent dynamic force on the interface dof. This approach is also very useful for the dimensioning of the fixtures at the base junction.
To evaluate formally the dynamic stresses, the concept of effective parameters will be used to derive the stress equation. With this formulation, 3 types of stresses will be well identified. They all can be computed with standard FE softwares, with a simple static analysis and a single modal analysis. After performing these analyses, our original formulation of the stress equation can be used to any type of uniaxial driven-base vibration environment. The junction dof can be submitted to sine acceleration, shock acceleration as well as random excitation defined by a PSD.

2. THE GENERALIZED DIRAC NOTATION

Structural analyses lead to use non square matrices of various sizes: m x n, (m + n) x n, ... In order to clearly identify the matrix size, we will use an extension of the Dirac notation. In this formalism, the symbols (table 2.1) are written before and after the matrix name to identify the number of lines (left symbol) and columns (right symbols).

Left	Right	Size	Left	Right	Size
|	|	m + n	<	>	1 (one)
[	]	n	«	»	n (non symmetric)
{	}	m	◄	►	m + n (non symm.)
			‡	‡	N (non symmetric)

Table 2.1: matrix symbols

It means that the matrix |A| is a (m+n).(m+n) symmetric matrix, [p> is a n components column vector, <q} is a m

components line vector, |B► is a (m+n).(m+n) non symmetric matrix.
When two matrices are associated, adjacent symbols must be identical. The meaning of this association is the product of the 2 matrices. When several matrices are multiplied, the final result has the size determined by the external symbols (for example <A| |B} {C] [D> is the product of 4 matrices and the result is a scalar).
The transposition operation consists in inverting the symbols $[A\}^T = \{A]$. When the matrix is composed of sub-matrices, this operation is defined by inverting all symbols and transposing the main matrix.

$$\begin{vmatrix} [A] & [B\} \\ \{C] & \{D\} \end{vmatrix}^T = \begin{vmatrix} [A] & [C\} \\ \{B] & \{D\} \end{vmatrix}$$

Using this notation, the dynamic equilibrium equation of the dof of a given structure can be written as

$$|M|\,|\ddot{q}> + |C|\,|\dot{q}> + |K|\,|q> = |F> \qquad (2.1)$$

with |q> the displacements, |M|, |C|, and |K| respectively the associated mass, damping and stiffness matrices.
Let [q> be the n internal dof, {q> the m junction dof (6 for an isostatic mount, 1 when using a shaker with a rigid interface). The equation can then be written:

$$\begin{vmatrix} [M] & [M\} \\ \{M] & \{M\} \end{vmatrix} \begin{vmatrix} [\ddot{q}\rangle \\ \{\ddot{q}\rangle \end{vmatrix} + \begin{vmatrix} [C] & [C\} \\ \{C] & \{C\} \end{vmatrix} \begin{vmatrix} [\dot{q}\rangle \\ \{\dot{q}\rangle \end{vmatrix} + \begin{vmatrix} [K] & [K\} \\ \{K] & \{K\} \end{vmatrix} \begin{vmatrix} [q\rangle \\ \{q\rangle \end{vmatrix} = \begin{vmatrix} [F\rangle \\ \{F\rangle \end{vmatrix} \qquad (2.2)$$

3. RESOLUTION OF THE EQUILIBRIUM EQUATION

3.1. Rigid modes.

The rigid mode matrix is defined to be the result of the equation |K| |ϕ} = |F'}. In the solution we impose an unitary displacement of each junction dof, one by one without applying any force on the internal dof. By this way we obtain a solution of the type $|\phi\} = \begin{vmatrix} [\phi\} \\ \{I\} \end{vmatrix}$.

If, as in our application, the support is infinitely rigid or isostatic, the force will be null (|F} = |0}) and the rigid modes (m = 6) will be defined as $[\phi\} = -[K]^{-1}[K\}$ if $[K]^{-1}$ exists. In this case, we can also observe than $\{K\} - \{K][K]^{-1}[K\}$ is null.

3.2. Elastic modes

The elastic modes |ϕ» (» because non-symmetrical) corresponding to the structure clamped at the junction dof can be written as $|\phi» = \begin{vmatrix} [\phi» \\ \{0» \end{vmatrix}$. The matrix [ϕ» is square but non symmetric and is composed of the n eigenvectors of the [M] [K] system and defined by

$$[\,[-\mathrm{diag}(\omega_k^2)][M] + [K]\,]\,[\phi» = 0 \qquad k = 1,2,\ldots n$$

By this way, $[\phi_k>$ (each column of [ϕ») diagonalizes simultaneously [M] and [K] :

$k_k = <\phi_k]\,[K]\,[\phi_k>$ are the modal stiffness and
$m_k = <\phi_k]\,[M]\,[\phi_k>$ are the modal masses.

Note that the eigenvectors are defined with a free normalization and so the modal parameters have no physical meaning. Nevertheless, their ratio, equal to ω_k^2 represents the pulsation of the eigenmode ($2\pi\, f_k$ where f_k is the frequency of mode k).
All theses modes (rigid and elastic) will be grouped in the (m + n)x(m + n) non symmetric matrix |ϕ► which constitutes a complete basis on which the solution of the dynamic equation can be developed.

3.3. Computation of miscellaneous matrices

Using these definitions, we can compute the following expressions:

$$\{\,\phi \mid K \mid \phi\,\} = \{K\} - \{K]\,[K]^{-1}[K\} \qquad \text{if } [K]^{-1} \text{ exists}$$
$$\{\,\phi \mid M \mid \phi\,\} = \{K][K]^{-1}[M][K]^{-1}[K\} - \{M][K]^{-1}[K\} - \{K][K]^{-1}[M\} + \{M\} \quad \text{if } [K]^{-1} \text{ exists}$$
$$= \{M_0\}$$

This matrix $\{M_0\}$ represents the masses and inertia of the structure and is generally named Guyan mass matrix.

$$« \phi \mid K \mid \phi » = « \mathrm{diag}(\ldots\, k_k \ldots) »$$
$$« \phi \mid M \mid \phi » = « \mathrm{diag}(\ldots\, m_k \ldots) »$$
$$\{\,\phi \mid M \mid \phi » \equiv \{L» = \{M][\phi» + \{\phi][M][\phi»$$
$$« \phi \mid M \mid \phi\,\} \equiv «L\} = «\phi][M\} + «\phi][M][\phi\}$$

These two last matrices are the coupling matrices between junction modes and eigenmodes (also called participation factors).

$$\{\,\phi \mid K \mid \phi » = \{\,0 »$$
$$« \phi \mid K \mid \phi\,\} = «\,0\,\}$$

3.4. Variable transformation.

For further analysis, we will develop the solution on the rigid/elastic modes basis:

$$|q(t)> \;=\; |\phi\blacktriangleright \blacktriangleleft\tau(t)> \;=\; |\phi\}\,\{\tau(t)> + |\phi»\,«\tau(t)> \quad (3.1)$$

Resulting from this, {q(t)> = {τ(t)>, the displacement at the junction. The internal displacements are defined by [q(t)> :

$$[q(t)> = [\phi\}\,\{\tau(t)> + [\phi»«\tau(t)> = [\phi\}\;\{q(t)> + [\phi»«\tau(t)> \qquad (3.2)$$

The equilibrium equation can be rewritten

$$|M|\,|\phi\blacktriangleright \blacktriangleleft\tau(t)>'' + |K|\,|\phi\blacktriangleright \blacktriangleleft\tau(t)> = |F(t)> \qquad (3.3)$$

Pre-multiplying this equation by $|\phi\blacktriangleright^T = \blacktriangleleft\phi|$, we obtain

$$\blacktriangleleft\phi|M|\phi\blacktriangleright \blacktriangleleft\tau(t)>'' + \blacktriangleleft\phi|K|\phi\blacktriangleright \blacktriangleleft\tau(t)> = \blacktriangleleft\phi \mid F(t) > \quad (3.4)$$

Where the unknowns are «τ(t)> and {F(t)>.
Let's analyze the new matrices:

$$\blacktriangleleft\phi \mid M \mid \phi\blacktriangleright = \blacktriangleleft\begin{pmatrix} «\mathrm{diag}(...m_k...)» & «L\} \\ \{L» & \{M_0\} \end{pmatrix}\blacktriangleright$$

$$\blacktriangleleft\phi \mid K \mid \phi\blacktriangleright = \blacktriangleleft\begin{pmatrix} «\mathrm{diag}(...k_k...)» & «0\} \\ \{0» & \{\phi K\phi\} \end{pmatrix}\blacktriangleright$$

And in the case of a rigid/isostatic mounting, we showed earlier that $\{\phi|K|\phi\} = \{0\}$.

3.5. Damping

The damping of the structure is a quite difficult parameter to evaluate. It will not be discussed here and we will only use the assumption of small diagonal structural damping, meaning that we can introduce the matrix $\blacktriangleleft\phi|C|\phi\blacktriangleright$ in the equilibrium equation with the following definition:

$$\blacktriangleleft\phi \mid C \mid \phi\blacktriangleright = \blacktriangleleft\begin{pmatrix} «\mathrm{diag}(...c_k...)» & «0\} \\ \{0» & \{0\} \end{pmatrix}\blacktriangleright$$

Where $c_k = 2\,\varepsilon_k\, m_k\, \omega_k$, ε_k being the damping factor of the mode k.

3.6. Results computation

Let's the excitation be described by $[F(t)> = [F_0> e^{i\omega t}$ and $\{q(t)> = \{q_0> e^{i\omega t}$. In this case the answer has the same layout $«\tau(t)> = «\tau_0> e^{i\omega t}$ and $\{F(t)> = \{r(t)> = \{R_0> e^{i\omega t}$.

The solution becomes

$$«\mathrm{diag}(-m_k\omega^2 + i\omega\, 2\varepsilon_k\omega_k m_k + k_k)»\, «\tau_0> = «\phi][F_0> + \omega^2\, «L\}\{q_0>$$

so that

$$«\tau_0> = «\mathrm{diag}(\ldots H_k(\omega)/k_k\ldots)»\; «\phi][F_0> + \omega^2\, «L\}\{q_0> \qquad (3.5)$$

with

$$H_k(\omega) = \frac{1}{1 - \dfrac{\omega^2}{\omega_k^2} + 2\, i\, \varepsilon_k \dfrac{\omega}{\omega_k}}$$

is called the dynamic amplification factor.

The $[q_0>$ and $\{R_0>$ definitions are thus

$$[q_0> = [\ [\phi\} + [\phi»«\mathrm{diag}(\ldots\omega^2 H_k/k_k\ldots)»«L\}\ \}\{q_0> + [\phi»«\mathrm{diag}(\ldots H_k/k_k\ldots)»«\phi]\, [F_0> \qquad (3.6)$$

and

$$\begin{aligned}\{R_0> &= -\omega^2\, \{L»«\tau_0> - \omega^2\{\phi M\phi\}\{q_0> + \{\phi K\phi\} - \{\phi][F_0> \\ &= \{-\{L»«\mathrm{diag}(\ldots\omega^2 H_k/k_k\ldots)»«\phi] - \{\phi]][F_0> \\ &\quad + \{\ -\omega^2\{L»«\mathrm{diag}(\ldots\omega^2 H_k/k_k\ldots)»«L\} \\ &\quad - \omega^2\{\phi M\phi\} + \{\phi K\phi\}\ \}\ \{q_0> \qquad (3.7)\end{aligned}$$

3.7. Introduction of the dynamic matrices.

We introduce the following dynamic matrices:

$$[G(\omega)] = [\phi»«\mathrm{diag}(\ldots H_k/k_k\ldots)»«\phi]$$

the dynamic flexibility matrix,

$$\{M(\omega)\} = \{L»«\mathrm{diag}(\ldots T_k/m_k\ldots)»«L\}$$

the dynamic mass matrix,

$$[T(\omega)\} = [\phi»«\mathrm{diag}(\ldots T_k/m_k\ldots)»«L\}$$

the dynamic transmissibility matrix, with

$$T_k(\omega) = 1 + \left(\frac{\omega}{\omega_k}\right)^2 H_k(\omega)$$

Using these new matrices, the answer can be written as :

$$\left|\begin{matrix}[q_0> \\ \{R_0>\end{matrix}\right\rangle = \left|\begin{matrix}[G(\omega)] & [T(\omega)\} - [M]^{-1}[M\} \\ -\{T(\omega)] + \{M][M]^{-1} & -\omega^2\{M(\omega)\} + \{\phi K\phi\}\end{matrix}\right|\left|\begin{matrix}[F_0> \\ \{q_0>\end{matrix}\right\rangle$$

3.8. Effective modal parameters

These matrices can be deduced from simple modal parameters that are often really useful.

$$[G(\omega)] = \Sigma_k\, H_k(\omega)\, [G_k]$$
$$\{M(\omega)\} = \{MB\} + \Sigma_k\, T_k(\omega)\, \{M_k\}$$
$$[T(\omega)\} = \Sigma_k\, T_k(\omega)\, [T_k\}$$

where

$\{MB\} = \{M\} - \{M][M]^{-1}[M\}$ is the mass affected to the junction (often small, null in continuous system).

$[G_k] = [\phi_k> 1/k_k <\phi_k]$ is the effective flexibility matrix of mode k.

$\{M_k\} = \{L_k> 1/m_k <L_k\}$ is the effective mass matrix of mode k.

$[T_k\} = [\phi_k> 1/m_k <L_k\}$ is the effective transmissibility matrix of mode k.

Note that these expressions have a physical significance and represent for each dof the part of the total mass/rigidity/transmissibility affected to the mode. It can be verified that

$$\Sigma_k\, [G_k] = [G] = [K]^{-1}$$
$$\Sigma_k\, \{M_k\} = \{M_0\} - \{MB\}$$
$$\Sigma_k\, [T_k\} = [\phi\} + [M]^{-1}[M\}$$

4. INTRODUCTION OF EFFECTIVE MODES

First of all, we will assume that the movement only occurs along a single direction. It means that the $\{q_0>$ vector has only one non-null component. In this case

$$«L\}\{\omega^2 q_0> = «1_k \sqrt{m_{eff,k}\, m_k} > \omega^2 q_0 \qquad (4.1)$$

where the scalar q_0 defines the amplitude of the shaker movement and 1_k is ± 1 depending on the mode, then

$$«\mathrm{diag}(\ldots H_k/k_k\ldots)»«1_k \sqrt{m_{eff,k}\, m_k} > \omega^2 q_0 =$$

$$«\frac{H_k}{\omega_k^2} \pm \sqrt{\frac{m_{eff,k}}{m_k}} > \omega^2\, q_0 = «\,(T_k - 1)\,.\,1_k \sqrt{\frac{m_{eff,k}}{m_k}} > q_0 \qquad (4.2)$$

The factor $1_k \sqrt{\dfrac{m_{eff,k}}{m_k}}$ is non-dimensional factor that can be inserted in the eigenmodes matrix. The sign $(1_{k)}$ of this parameter must be defined in order that the center of gravity of each mode or the reaction implied by each mode on the support has the same sign. So we define the new vectors $[\phi_{eff,k}> = [\phi_{e,k}> \,.\, 1_k \sqrt{\frac{m_{eff,k}}{m_k}}$ that will be called *effective elastic modes* for the specific direction of excitation. It can be verified that the modal masses and the participation factors in this new basis are equal to the effective mass. The displacement can be written

$$[q_0> = [\ [\phi\}\{\omega^2 q> + [\phi_{eff}»«H_k/\omega_k^2> \omega^2 q_0 > \quad (4.3)$$

5. EXACT STRESS COMPUTATION

We define $\ddagger\sigma(t)>$as a list of selected linear combination of the stress tensor components calculated or interpolated at any point of the structure (mainly at these parts where these stresses are supposed to reach a maximum). The number of elements of this vector is arbitrary ($N \neq n$). The equation

$$\ddagger\sigma(t)> = \ddagger S|\ |q(t)> \quad (5.1)$$

simply states that these linear combinations of the stress tensor elements is a linear relation of displacements in the approximation of the linear finite element model, assuming small deformations.
$\ddagger S|$ is a Nx$(m + n)$ matrix. Using the definition of $|q(t)>$, we can write

$$\ddagger\sigma(t)> = \ddagger S|\ |\phi\blacktriangleright\blacktriangleleft\tau(t)>$$

$$\begin{aligned}\ddagger\sigma> = &\ \ddagger S|\ |\phi\}\{q_0> \\ &+ \ddagger S]\ [\phi»«\mathrm{diag}(\ldots\omega^2 H_k/k_k\ldots)»«L\}\ \{q_0> \\ &+ \ddagger S]\ [\phi»«\mathrm{diag}(\ldots H_k/k_k\ldots)»«\phi][F_0> \quad (5.2)\end{aligned}$$

The first term $\ddagger S|\ |\phi\}\{q_0>$ represents the hyperstatic stress, which is null in our application due to the isostatic/rigid mount. The second term is the dynamic stresses produced by the junction movement $\{q_0>$ and the third term represents the dynamic stresses resulting from the internal forces $[F_0>$.
In the frame of driven-base environments we impose only a junction displacement. So the stresses can be simply expressed as

$$\ddagger\sigma> = \ddagger\sigma_e»«\mathrm{diag}(\ldots H_k/k_k\ldots)»«L\}\ \{\omega^2 q_0> \quad (5.3)$$

The $\ddagger\sigma_{e,k}>$ vectors will be called stress modes. As the eigenmodes, the stress modes are defined with a free normalization.
Similarly to the effective elastic modes, we define the *effective modal stresses* with:

$$\ddagger\sigma_{eff,k}> = \ddagger\sigma_{e,k}> .\pm \sqrt{\frac{m_{eff,k}}{m_k}} \quad (5.4)$$

By this way, the stresses in the structure for a base-driven excitation in the x-direction can be obtained by equation (5.5) where a_0 is the shaker acceleration.

$$\ddagger\sigma> = \Sigma_k\ \ddagger\sigma_{eff,k}> \frac{H_k}{\omega_k^2}\,\omega^2 q_0 = \Sigma_k\ \ddagger\sigma_{eff,k}> \frac{H_k}{\omega_k^2}\ a_0 \quad (5.5)$$

Note that in the case of static loading ($\omega = 0$), the stress in the structure when applying a volumic acceleration a_0 is equal to

$$\ddagger\sigma_{static}> = \Sigma_k\ \ddagger\sigma_{eff,k}> a_0/\omega_k^2 \quad (5.6)$$

6. APPROXIMATIONS FOR STRESS EVALUATION

6.1. Limitation to the useful modes

The equation (5.5) is exact if we use all the modes to evaluate the stresses. In practical applications, only the first modes are correctly known after a finite element model analysis. We have to deal with a finite set of modes.
A first approach is to use only the first modes and to neglect the effect of the high frequency modes. If the stress is evaluated at eigenfrequencies far lower than ω_K, we can approximate the $H_{k>K}$ by 1. It means that the $\sigma_{residual}$ is equal to the static contribution of the high frequency modes.

To improve the result, we propose to extract the static contribution from each term of the sum (5.5). This solution is formally exact and is best suited for truncature with small values of ω ($\omega<\omega_{K'}$) . Therefore, if we want to limit the number of modes involved in the development, we must use the next equation resulting from the "*summation rules of the stresses*":

$$\begin{aligned}\ddagger\sigma> = &\ \ddagger\sigma_{static}\ (\text{volumic acceleration} = \omega^2 q_0) > \\ &+ \Sigma_{k\leq K'}\ [\sigma_{eff,k}> \frac{H_k - 1}{\omega_k^2}\ \omega^2 q_0 \quad (6.1)\end{aligned}$$

where $K' \leq K$. In this case, $\lim_{\omega\to\infty} (H_k(\omega)-1) = 0$ for large k, this minimizes the truncature error. The static stresses can then be computed by a simple static FEM analysis or by formula's table.

6.3. Practical use

The expression (6.1) must be compared with the following approximation, often made in a preliminary stage of the study. The approximation method, which is faster to apply, is very useful at a preliminary stage of the study with a rough model of the structure:
- to verify the integrity of the primary structure;
- to evaluate roughly the acceleration at the center of gravity corresponding to qualification tests, which should be compared to the flight;
- to identify which qualification test is the most stringent and on which the design should be optimized.

In a preliminary stage, we proceed with the computation of the acceleration at the instantaneous center of gravity provided with the help of the dynamic mass (ref [1] and [2]). This acceleration at the center of gravity is used as a general acceleration statically applied to the whole structure. The stresses obtained under these assumptions are given by

$$\ddagger\sigma_{approximated}(\omega)> = \ddagger\sigma_{static;\omega^2q=I}> q_0(\omega)\ \omega^2\ M_{dyn}(\omega)$$

$$=\ddagger\sigma_{static;\omega^2q=I}> q_0(\omega)\ \omega^2\ (1 + \Sigma_k\ m_{eff,k,x}/M_{stat}\ H_k\ \omega^2/\omega_k^2\ ) \tag{6.2}$$

to be compared with (6.1). As it can readily be seen, theses two formulations are equivalent for low values of ω but they are quite different at the first resonances of the structure.

6.4. Approximation by the deformation energy

We propose here a simple way to make a preliminary evaluation of the mean stress in a vibrating structure. During vibrations, there is an exchange between deformation energy and kinetic energy. When the structure reaches a non-deformed state, the velocities are the highest and when the structure stops its movement (the center of gravity speed is null), the deformations are the highest (note that at this moment, the work of the shaker is non-null). The kinetic energy can be written with (6.3) where M_{stat} is the static mass of the equipment,

$$K(\omega) = ½\ M_{stat}\ .\ v_{cog}(\omega)^2\ = ½\ M_{stat}\ .\ a_{cog}(\omega)^2/\omega^2 \tag{6.3}$$

a_{cog} is given by

$$a_{cog} = \frac{M(\omega)}{M_{stat}} a_{base} \tag{6.4}$$

The deformation energy can be written

$$D(\omega) = \frac{1}{2}\int_V \varepsilon\sigma dV = \frac{1}{2}\int_V \frac{\sigma^2}{E} dV = \frac{1}{2}\frac{\sigma^2_{average\ on\ volume}(\omega)}{E} V \tag{6.5}$$

The work performed by the shaker can be written

$$\mathrm{T}\ (\omega) = -\ ½\ M_{stat}.a_{cog}.a_{base}\ /\omega^2 \tag{6.6}$$

We can write that the maximum deformation energy is equal to the maximum kinetic energy plus the shaker work. So

$$K(\omega) + \mathrm{T}\ (\omega) = D(\omega)$$

$$M_{stat}\ .\ a_{cog}^2/\omega^2\ -\ M_{stat}.a_{cog}.a_{base}.1/\omega^2 = \sigma_{aver^2}\ .V/E \tag{6.7}$$

and,

$$\sigma_{aver}(\omega)^2 = \rho_{aver}\ E\ (a_{cog} - a_{base})\ a_{cog}\ /\omega^2 \tag{6.8}$$

This average value on the volume can be used as a rough estimation of the maximum stress. For example in a cantilever beam, $\sigma_{max} = 6\ \sigma_{aver}$.

7. EXTENSION TO RANDOM VIBRATIONS

7.1. Effective modes combination in random vibration.

We have seen (eq 5.5) that stresses can be expressed as

$$\ddagger\sigma> = \Sigma_k\ \ddagger\sigma_{eff,k}>\ a_0\ H_k/\omega_k^2 \tag{7.1}$$

It can also be written $\ddagger\sigma> = \ddagger\sigma(\omega)> a_0$ where $\ddagger\sigma(\omega)>$ is a transfer function.

The random vibrations tests are defined by they power spectral density (PSD). It is well known that in a linear system the PSD of the response is linked to the excitation PSD by the product of the conjugate of the transfer function with the transfer function itself. So

$$\ddagger PSD_\sigma\ddagger = \ddagger\sigma(\omega)^*>PSD_{a0} <\sigma(\omega)\ddagger \tag{7.2}$$

$$PSD_{\sigma i^2} = \|\sigma(\omega)_i\|^2\ PSD_{a0} \tag{7.3}$$

where $\|\sigma(\omega)_i\|$ is the modulus of the i^{th} component of the vector.

Statistical computation shows us that the integral of the PSD on all frequencies is the square of the RMS value of random variable. So,

$$\sigma_{i,RMS}^{\ 2} = \int_{-\infty}^{+\infty} \left\|\sigma(\omega)_i\right\|^2 PSD_{a0} d\omega. \tag{7.4}$$

The evaluation of this expression leads to equation (7.5) when eigenfrequencies are in the frequency range of the PSD.

$$\sigma_{i,RMS}^2 \approx \Sigma_k\ |\sigma_{eff,k,i}|^2/\omega_k^4\ \frac{\pi}{2} . \frac{1}{2\varepsilon_k} .f_k.PSD_{\ a0}\ (f_k) \tag{7.5}$$

When there are no eigenfrequencies in the range, the solution of (7.4) is defined by equation (7.6)

$$\sigma_{i,RMS}^2 \approx\ \sigma_{stat,i}^2\ a_{base,RMS}^2 \tag{7.6}$$

7.2. Deformation energy formula in random vibration.

The same operation can be performed on the deformation energy formula. The transfer function from (6.7) is

$$\sqrt{\rho_{aver}\ E} \sum_k \frac{m_{eff,k}}{M_{stat}} \frac{T(\omega)}{\omega} \tag{7.7}$$

so

$$PSD_{\sigma aver} = \left\| \sqrt{\rho_{aver}\ E} \sum_k \frac{m_{eff,k}}{M_{stat}} \frac{T(\omega)}{\omega} \right\|^2 PSD_{a0} \tag{7.8}$$

And the integration on all frequency gives

$$\sigma_{RMS,moy}^{\ 2} = \rho_{aver}.E.\Sigma_k\ (m_{eff,k}/M_{stat})^2.\frac{1}{16\pi\varepsilon_k f_k}\ PSD_{a0}(f_k) \tag{7.9}$$

8. TEST ON SIMPLE STRUCTURE

The different approximations proposed to evaluate the stress are tested here on a simple system. This system consists in a cantilever beam, 150 mm long, 5 mm thick and 10 mm large. This beam is made of Aluminium with a volumic mass of 2700 kg/m³ and a Young modulus of 72500 MPa. The modal analysis with FEM code, gave the results of table 8.1:

Frequency	Effective mass along x	Modal mass
187 Hz	$1.238\ 10^{-2}$ (61.1 %)	$5.054\ 10^{-3}$
1165 Hz	$3.817\ 10^{-3}$ (18.8 %)	$5.085\ 10^{-3}$
3241 Hz	$1.321\ 10^{-3}$ (6.5 %)	$5.120\ 10^{-3}$
6304 Hz	$6.845\ 10^{-4}$ (3.4 %)	$5.149\ 10^{-3}$

Table 8.1: Beam analysis result

Assuming a damping factor of 2% on each elastic mode and acceleration at the clamping junction of 119 m/s², we perform different computations that we detail hereafter.

8.1. Sinusoidal excitation

8.1.1. Reference computation

The reference computation for the sinus loading will be performed with a dynamical analysis FEM software using time varying loads. The maximum value is located at the clamping area and is 105.1 MPa.

8.1.2. Elastic energy

Using the formula (6.13), the mean $\sigma$is 21.1 MPa, to obtain the maximum stress, we can multiply this value by 6 and we obtain 126.6 MPa.

8.1.3. Comparison between static stress and modal sum of stress

It was demonstrated earlier that the static stress for an unitary volumic acceleration must be identical to the sum of the effective stress modes divided by the square of their pulsation.

8.1.4. Stresses modes combination

Using the stress vectors given by FE modal analysis, we can derive the effective stress vectors. The addition of all vectors with their adequate multiplicative coefficient would lead to the exact solution if the modes were correctly estimated (which is not the case in FEM). It was seen that, at the first eigenfrequency, the first mode represents almost all the total maximum stress.
At the second frequency, the second mode is dominant and at low frequency, several modes are needed to reach the correct value.

8.1.5. Static constraint and addition of stress modes.

It was shown in paragraph 6.1 that computing the static stress with a unitary volumic acceleration could improve the results of the computation. This was applied to our test model. On this simple model, the difference is not obvious but it was observed that the error made on higher modes are suppressed.

8.2. Random excitation

8.2.1. Reference computation.

The use of a FEM code dedicated to random vibrations can give us the PSD of the stress at different location of the system. From this we can compute the RMS value of the stress. This will be the reference for the following comparison. This computation will be performed along two axes (lateral and axial) in order to have two cases: with and without eigenfrequencies in the excitation range.

8.2.2. Elastic energy

The use of the simple elastic energy formula leads to a RMS maximum stress of 29 MPa for the lateral direction and ??? MPa for axial direction.

8.2.3. Stress modes combination

Formulas ??? were used to evaluate the rms stress in the beam. The following values were observed: for lateral, using formula 7.5 and for axial using formula 7.6:

9. TESTS ON OMC LENS BARREL

The Optical Monitoring Camera (OMC) is an instrument that will be mounted on the INTEGRAL spacecraft. Its goal is to observe the visible counterpart of what will be observed by the gamma and X rays telescopes. Optical subsystem of the camera, which consists in a 6-lens system, is designed and manufactured under CSL responsibility. The lenses are mounted in a titanium lens barrel. The goal of the study is to verify that the maximum stress is lower than the elastic limit of titanium during qualification vibration test.
Using the methods developed in this article, the maximum estimated stress is ??? what is largely lower than the elastic limit.

10. CONCLUSIONS

This paper presents a new way to evaluate the effects of mechanical tests on equipments mounted on vibration machines. This was tested on a simple model (cantilever beam) and than applied on the OMC optical subsystem.

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