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A GENERAL THEORY OF DUAL ERROR BOUNDS BY FINITE ELEMENTS

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Rapport LMF/D5, 1983

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Summary - This paper presents a general error bound theory for linear elliptic problems, which constitutes an extension of particular results obtained earlier by B. Fraeijs de Veubeke.

1. INTRODUCTION

The dual analysis technique, which was mainly developed by Fraeijs de Veubeke and his collegas /1 to 9/ , consists in a double computation of the same linear elliptic problem by two different ways. The first or primal one is based on displacement models while the second or dual one makes use of equilibrium models. Under certain conditions, such a dual analysis leads to an useful energetic measure of the discretization errors of both approaches.

Up to now, the applicability of this dual error bound technique seemed to be restricted to two particular cases corresponding, if we take stress analysis as an example, to zero applied loads or zero imposed displacements.

In this paper, an analysis is proposed, which proves the possibility to generalize dual analysis concepts to all cases of boundary conditions, by a slightly different evaluation of the error bounds. The theory is developed in a general abstract frame and includes as particular cases the two classical results due to Fraeijs de Veubeke /3,6/. The last section illustrates these concepts in the case of linear elasticity.

2. VARIATIONAL PROBLEMS

Let E be a Hilbert space and E_o a closed subspace of E. A bilinear form a(p,q) is given on E, which verifies the following conditions:

$$\begin{cases} |a(p,q)| \leq C \; \|p\|_E \; \|q\|_E & \text{(boundedness)} \quad (1) \\ a(p,q) = a(q,p) & \text{(symmetry)} \quad (2) \\ a(p,p) \geq \beta \, \|p\|_E^2 \; , \quad \beta > 0 \; , \quad \forall \, p \in E_o & (E_o\text{- ellipticity}) \quad (3) \end{cases}$$

As usual, symbol E' will denote the dual space of E.

In what follows, we will be concerned by variational problems which have the following general form:

V.P. - Let be given $q_o \in E$ and $f \in E'$. Find $\hat{q} \in E_o$ such that

$$a(q_o + \hat{q} \, , \, q) = f(q) \qquad (4)$$

whatever $q \in E_o$.

The existence and unicity of the solution of such a problem is guaranteed by Lax-Milgram's theorem /10,13/. Moreover, taking $q = \hat{q}$ in (4) leads to

$$a(\hat{q}, \hat{q}) = f(\hat{q}) - a(q_o, \hat{q}) \; ,$$

which implies by the properties of the bilinear form

$$\| q \|_E \leq \frac{1}{\beta} \left(\| f \|_{E'} + C \; \| q_o \|_E \right) \; , \qquad (5)$$

an inequality which proves that this problem is a well-posed one.

We emphasize that most interest is in general devoted to the complete solution

$$\hat{r} = q_o + \hat{q} \qquad (6)$$

In this view, note that there are a lot of different q_o's which lead to the same complete solution. Consider for instance $q_{o1} \neq q_{o2}$. With q_{o1}, the variational equation will be

$$a(q_{o1} + \hat{q}_1, q) = f(q) \qquad \forall \; q \in E_o.$$

With q_{o2}, it will be

$$a(q_{o2} + \hat{q}_2, q) = f(q) \qquad \forall \; q \in E_o.$$

Taking the difference between these two equations, one obtains

$$a\left((q_{o1} + \hat{q}_1) - (q_{o2} + \hat{q}_2), q \right) = 0 \qquad \forall \, q \in E_o.$$

Under the condition

$$(q_{o1} + \hat{q}_1) - (q_{o2} + \hat{q}_2) \in E_o \; , \qquad (7)$$

this element itself may be taken as a particular q. This implies by (3)

$$0 = a\left((q_{o1} + \hat{q}_1) - (q_{o2} + \hat{q}_2), (q_{o1} + \hat{q}_1) - (q_{o2} + \hat{q}_2) \right)$$

$$\geqslant \beta \, \|(q_{o1} + \hat{q}_1) - (q_{o2} + \hat{q}_2)\|_E^2 ,$$

that is to say

$$q_{o1} + \hat{q}_1 = q_{o2} + \hat{q}_2 .$$

As condition (7) is equivalent to

$$q_{o1} - q_{o2} \in E_o ,$$

we shall say that two elements q_{o1} and q_{o2} of E are E_o- equivalent if they differ only by an element of E_o. Thus, two variational problems involving the same bilinear form, the same f and two E_o-equivalent q_o's lead to the same complete solution.

Now, a very simple lemma which will be useful in the following. Suppose that $f(E_o) \neq \{0\}$ unless f = 0. Then, among all q_o^* which are E_o-equivalent to a given q_o, it is possible to select (at least) one particular $\tilde{q}_o$ verifying $f(\tilde{q}_o) = 0$.

Remark first that if $f(E_o) = \{0\}$, f is equivalent to zero in E_o, and may be replaced by zero without change of the solution. Our restriction is thus natural.

If f = 0 , the lemma is of trivial nature. In other cases, consider $p \in E_o$ such that $f(p) \neq 0$. Taking

$$\tilde{q}_o = q_o - \frac{f(q_o)}{f(p)} \, p \tag{9}$$

leads the announced result.

As is well known, every symmetric variational problem of the form V.P. is equivalent to the minimization of the functional

$$\mathcal{A}(q_o + q) = \tfrac{1}{2} \, a\,(q_o + q,\, q_o + q) - f(q_o + q)$$

among all $q \in E_o$ or, in other words, to the following minimization problem:

M.P. - Find $\hat{r} \in q_o + E_o$ which minimizes in this linear manifold the functional

$$\mathcal{A}(r) = \tfrac{1}{2} \, a(r,r) - f(r). \tag{10}$$

3. RAYLEIGH-RITZ APPROXIMATIONS

The basis of Rayleigh-Ritz approximations (including conforming finite element techniques) is to replace the space E by a finite dimensional subspace E_h. Subspace E_o is then replaced by $E_{oh} = E_h \cap E_o$. It is necessary to make the three following restrictions:

R_1 : $E_{oh} \neq \{0\}$

R_2 : $f(E_{oh}) \neq \{0\}$

R_3 : The given q_o is E_o- equivalent to some $\bar{q}_o \in E_h$.

Conditions R_1 and R_2, which only exclude "too little" subspaces leading to meaningless solutions, are fulfilled in all reasonable idealizations. In contrast, condition R_3 is a real restriction. Its purpose is to preserve the inclusion

$$\bar{q}_o + E_{oh} \subset \bar{q}_o + E_o .$$

Rayleigh-Ritz' technique consists to minimize $\mathcal{A}(r)$ on $\bar{q}_o + E_{oh}$. If the three preceeding conditions are verified, this approximate problem will be called in the following a regular internal approximation. For simplicity, overbars on q_o will be omitted.

The approximate solution $\hat{r}_h = q_o + \hat{q}_h$ verifies

$$a(q_o + \hat{q}_h, q_h) = f(q_h) \tag{11}$$

whatever $q_h \in E_{oh}$. Since $q_o + E_{oh} \subset q_o + E_o$, one has the inequality

$$\mathcal{A}(\hat{r}_h) = \min_{r_h \in q_o + E_{oh}} \mathcal{A}(r_h) \geqslant \min_{r \in q_o + E_o} \mathcal{A}(r) = \mathcal{A}(\hat{r}) \tag{12}$$

Now, the quantity

$$d(\hat{r}_h, \hat{r}) = \left(a(\hat{r}_h - \hat{r}, \hat{r}_h - \hat{r}) \right)^{\frac{1}{2}}$$

will be called the energetic distance between $\hat{r}$ and $\hat{r}_h$. The word distance is justified by the fact that, since $\hat{r} - \hat{r}_h \in E_o$,

$$d^2(\hat{r}, \hat{r}_h) \geqslant \beta \, \|\hat{r} - \hat{r}_h\|_E^2 .$$

Calculating this distance, one has

$$\begin{aligned} d^2(\hat{r}, \hat{r}_h) &= a\left((q_o + \hat{q}_h) - (q_o + \hat{q}), (q_o + \hat{q}_h) - (q_o + \hat{q})\right) \\ &= a(q_o + \hat{q}_h, q_o + \hat{q}_h) + a(q_o + \hat{q}, q_o + \hat{q}) \\ &\qquad - 2\, a(q_o + \hat{q}, q_o + \hat{q}_h). \end{aligned}$$

Noting that

$$\begin{aligned} 2a(q_o + \hat{q}, q_o + \hat{q}_h) &= 2a(q_o + \hat{q}, q_o + \hat{q}) + 2a(q_o + \hat{q}, \hat{q}_h - \hat{q}) \\ &= 2a(q_o + \hat{q}, q_o + \hat{q}) + 2f(\hat{q}_h - \hat{q}), \end{aligned}$$

where account has been taken of relation (4) and that $\hat{q}_h - \hat{q} \in E_o$, one obtains

$$\begin{aligned} d^2(\hat{r}, \hat{r}_h) = a(q_o + \hat{q}_h, q_o + \hat{q}_h) - 2\, f(q_o + \hat{q}_h) + 2\, f(q_o + \hat{q}) \\ - a(q_o + \hat{q}, q_o + \hat{q}), \end{aligned}$$

i.e. the following fundamental result:

$$d^2(\hat{r}, \hat{r}_h) = 2 \left(\mathcal{A}(\hat{r}_h) - \mathcal{A}(\hat{r}) \right). \quad (13)$$

Note finally that the three restrictions R_1 to R_3 permit to suppose, as in the undiscretized problem, that $f(q_o) = 0$, q_o being in E_{oh}.

4. GENERAL DUAL ANALYSIS SITUATIONS

The general dual analysis frame may be described as follows. Let V and H be two Hilbert spaces, equipped with scalar products $(u,v)_V$ and $(\varepsilon, \eta)_H$. Let ∂ be a linear operator from V to H, which is bounded,

$$\|\partial u\|_H \leqslant M_1 \|u\|_V \quad (14)$$

Let now V_o be a closed subspace of V, in which the following ellipticity condition is verified

$$(\partial u, \partial u)_H \geqslant \alpha \|u\|_V^2 \qquad \forall u \in V_o \quad , \alpha > 0 \quad (15)$$

This inequality implies that the image ∂V_o of V_o in H is a closed subspace of H.

Finally, let C be a linear operator from H to H which exhibits the following properties:

$$\begin{cases} \cdot \ \|C\varepsilon\|_H \leqslant M_2 \|\varepsilon\|_H \quad \text{(boundedness)} & (16) \\ \cdot \ (C\varepsilon, \eta)_H = (\varepsilon, C\eta)_H \quad \text{(hermiticity)} & (17) \\ \cdot \ (C\varepsilon, \varepsilon)_H \geqslant \gamma \ \|\varepsilon\|_H^2, \gamma > 0 \quad \text{(ellipticity)} & (18) \end{cases}$$

These conditions implie the existence of an inverse operator C^{-1} which is also bounded, hermitic and elliptic on H. In fact, for a given $\sigma \in H$, the equation

$$C\varepsilon = \sigma \quad (19)$$

corresponds to seeking an element $\varepsilon \in H$ such that

$$(C\varepsilon - \sigma, \eta)_H = 0 \qquad \forall \eta \in H$$

or equivalently,

$$(C\varepsilon, \eta)_H = (\sigma, \eta)_H \qquad \forall \eta \in H$$

By virtue of Lax-Milgram's theorem, this problem admits an unique solution verifying

$$\|\varepsilon\|_H \leqslant \frac{1}{\gamma} \|\sigma\|_H ,$$

that is to say

$$\|C^{-1}\sigma\|_H \leqslant \frac{1}{\gamma} \|\sigma\|_H . \quad (20)$$

Now, taking $\varepsilon = C^{-1}\sigma$ in (16) leads

$$\|C^{-1}\sigma\|_H \geqslant \frac{1}{M_2} \|\sigma\|_H$$

and this implies by (18) the ellipticity relation

$$(\sigma , C^{-1}\sigma)_H = (C\, C^{-1}\sigma , C^{-1}\sigma)_H \geq \gamma \; \| C^{-1}\sigma\|^2_H \;\geq\; \frac{\gamma}{M_2^2} \|\sigma\|^2_H \tag{21}$$

Now, by setting $\varepsilon = C^{-1}\sigma$ and $\eta = C^{-1}\tau$ in (17), one obtains

$$(\sigma , C^{-1}\tau)_H = (C^{-1}\sigma , \tau)_H \tag{22}$$

that is,the hermiticity relation for C^{-1}.

5. THE PRIMAL PROBLEM

The primal problem consists as follows

P.P. – $u_o \in V$ and $f \in V'$ being given, find $\hat{u} \in V_o$ such that

$$(C(\partial u_o + \partial \hat{u}), \partial u)_H = f(u) \tag{23}$$

whatever $u \in V_o$

It is easy to see that this problem is a particular form of problem V.P. described in section 2. This results from the inequalities

$$\begin{cases} |(C \partial u, \partial v)_H| \leq M_2 \|\partial u\|_H \|\partial v\|_H \leq M_2\, M_1^2 \;\|u\|_V \;\|v\|_V \\ (C\, \partial u, \partial u)_H \geq \gamma \|\partial u\|^2_H \geq \alpha\gamma \;\|u\|^2_V \quad \forall u \in V_o \end{cases}$$

The primal problem is also equivalent to the minimization of the primal functional

$$\mathcal{A}_1(w) = \tfrac{1}{2} (C\, \partial w, \partial w) - f(w)$$

on the linear manyfold $u_o + V_o$. Consequently, any regular internal approximation of this problem will lead to

$$d_1^2(\hat{w}_h, \hat{w}) = (C(\partial \hat{w}_h - \partial \hat{w}), \partial \hat{w}_h - \partial\hat{w}) = 2 \; (\mathcal{A}_1(\hat{w}_h) - \mathcal{A}_1(\hat{w})) \tag{25}$$

6. THE DUAL PROBLEM

Let us introduce the subspace S_o, defined as the orthogonal subspace of ∂V_o in H,

$$S_o = \{\sigma \in H \mid (\sigma , \partial v)_H = 0 \quad \forall v \in V_o \} \;. \tag{26}$$

This is naturally a closed subspace. Owing the fact that ∂V_o is also closed, one concludes to the interrelations

$$S_{o\perp} = \partial V_o \;, \qquad \partial V_{o\perp} = S_o . \tag{27}$$

For any arbitrary functional $f \in V'$, it is possible to find an element $\sigma_o \in H$ such that

$$(\sigma_o , \partial v)_H = f(v) \qquad \forall \; v \in V_o . \tag{28}$$

In fact, any

$$\sigma_o = C(\partial u_o + \partial \hat{u}) + \tau \quad ,$$

where $(u_o + \hat{u})$ is the solution of the primal problem and $\tau \in S_o$ visibly satisfies to condition (28).

Now, the dual problem is as follows

D.P. - A functional $f \in V'$ and an element $u_o \in V$ being given, let σ_o be such that

$$(\sigma_o, \partial v)_H = f(v) \qquad \forall \; v \in V_o .$$

Find $\hat{\sigma} \in S_o$ verifying

$$(C^{-1}(\sigma_o + \hat{\sigma}), \; \sigma)_H = (\partial u_o, \sigma)_H \tag{29}$$

whatever $\sigma \in S_o$

This problem is also a particular form of problem V.P. described in section 2, since

$$\begin{cases} |(C^{-1}\sigma \; , \tau)_H| \leqslant \frac{1}{\gamma} \|\sigma\|_H \quad \|\tau\|_H \\ (C^{-1}\sigma \; , \sigma)_H \geqslant \frac{\gamma}{M_2^2} \|\sigma\|_H^2 \qquad \forall \sigma \in H \supset S_o . \end{cases}$$

The fact that the solution $\hat{\sigma}$ verifies

$$(C^{-1}(\sigma_o + \hat{\sigma}) - \partial u_o, \sigma)_H = 0 \qquad \forall \sigma \in S_o$$

implies

$$C^{-1}(\sigma_o + \hat{\sigma}) - \partial u_o \in S_{o\perp} = \partial V_o ,$$

that is to say, there exists a $v \in V_o$ such that

$$C^{-1}(\sigma_o + \hat{\sigma}) = \partial u_o + \partial v ,$$

or

$$\sigma_o + \hat{\sigma} = C(\partial u_o + \partial v).$$

It follows from the definition of σ_o and $\hat{\sigma}$ that

$$f(u) = (\sigma_o + \hat{\sigma}, \partial u)_H = (C(\partial u_o + \partial v), \partial u)_H \qquad \forall u \in V_o$$

and this proves that v is precisely the solution $\hat{u}$ of the primal problem. In other words, the primal and the dual problem lead to two different forms of the same solution.

The dual problem is equivalent to the minimization of the dual functional

$$\mathcal{A}_2(\rho) = \tfrac{1}{2} (C^{-1}\rho \; , \rho)_H - (\partial \; u_o, \rho)_H \tag{30}$$

on the linear manyfold $\sigma_o + S_o$. Consequently, any regular internal approximation of this problem will lead to the following error bound:

$$d_2^2(\hat{\rho}_h, \hat{\rho}) = (C^{-1}(\hat{\rho}_h - \hat{\rho}), \hat{\rho}_h - \hat{\rho})_H = 2(\mathcal{A}_2(\hat{\rho}_h) - \mathcal{A}_2(\hat{\rho})) \tag{31}$$

7. DUAL ERROR BOUNDS

It is possible to uniformize the two distances d_1 and d_2 by noting that

$$d_1^2(\hat{w}_h, \hat{w}) = (C(\partial\hat{w}_h - \partial\hat{w}), \partial\hat{w}_h - \partial\hat{w})_H = (C^{-1}C(\partial\hat{w}_h - \partial\hat{w}), C(\partial\hat{w}_h - \partial\hat{w}))_H$$
$$= d_2^2(C\partial\hat{w}_h, C\partial\hat{w}). \tag{32}$$

Now, the fact that $\hat{\rho} = C\partial\hat{w}$ implies

$$0 = d_2^2(\hat{\rho}, C\partial\hat{w}) = (C^{-1}(\hat{\rho} - C\partial\hat{w}), \hat{\rho} - C\partial\hat{w})_H$$
$$= (C^{-1}\hat{\rho}, \hat{\rho})_H + (C\partial\hat{w}, \partial\hat{w})_H - 2(\hat{\rho}, C\partial\hat{w})_H$$

The last term may be transformed as follows:

$$-2(\hat{\rho}, C\partial\hat{w})_H = -2(\sigma_o + \hat{\sigma}, \partial u_o + \partial\hat{u})_H =$$
$$= -2(\sigma_o + \hat{\sigma}, \partial u_o)_H - 2(\sigma_o + \hat{\sigma}, \partial\hat{u})_H$$
$$= -2(\sigma_o + \hat{\sigma}, \partial u_o)_H - f(\hat{u})$$
$$= -2(\sigma_o + \hat{\sigma}, \partial u_o)_H - f(u_o + \hat{u}),$$

where account has been taken to (28) and the fact that one may always assume that $f(u_o) = 0$. As a result, one obtains

$$\mathcal{A}_1(\hat{w}) + \mathcal{A}_2(\hat{\rho}) = 0 \tag{33}$$

Let now $\hat{w}_h$ and $\hat{\rho}_h$ be regular internal approximations of $\hat{w}$ and $\hat{\rho}$. The fact that $\hat{\rho} = C\partial\hat{w}$ implies

$$d_2^2(\hat{\rho}_h, C\partial\hat{w}_h) = (C^{-1}(\hat{\rho}_h - C\partial\hat{w}_h), \hat{\rho}_h - C\partial\hat{w}_h)_H$$
$$= (C^{-1}(\hat{\rho}_h - \hat{\rho} + C\partial\hat{w} - C\partial\hat{w}_h), \hat{\rho}_h - \hat{\rho} + C\partial\hat{w} - C\partial\hat{w}_h)_H$$
$$= d_2^2(\hat{\rho}_h, \hat{\rho}) + d_2^2(C\partial\hat{w}, C\partial\hat{w}_h) + 2(\hat{\rho}_h - \hat{\rho}, \partial\hat{w} - \partial\hat{w}_h)_H$$

Given that

$$\hat{\rho}_h - \hat{\rho} = \hat{\sigma}_h - \hat{\sigma} \in S_o$$
$$\partial\hat{w}_h - \partial\hat{w} = \partial\hat{u}_h - \partial\hat{u} \in \partial V_o = S_{o\perp},$$

the last term vanisches, so that

$$d_2^2(\hat{\rho}_h, C\partial\hat{w}_h) = d_2^2(\hat{\rho}_h, \hat{\rho}) + d_1^2(\hat{w}, \hat{w}_h)$$
$$= 2(\mathcal{A}_2(\hat{\rho}_h) - \mathcal{A}_2(\hat{\rho})) + 2(\mathcal{A}_1(\hat{w}_h) - \mathcal{A}_1(\hat{w}))$$

and from (33), we find the following expression for the distance

$$d_2^2(\hat{\rho}_h, C\partial\hat{w}_h) = 2\,(\mathcal{A}_1(\hat{w}_h) + \mathcal{A}_2(\hat{\rho}_h)) \qquad (34)$$

Of course,

$$d_1^2(\hat{w}, \hat{w}_h) \leq d_1^2(\hat{w}, \hat{w}_h) + d_2^2(\hat{\rho}, \hat{\rho}_h) = 2(\mathcal{A}_1(\hat{w}_h) + \mathcal{A}_2(\hat{\rho}_h)) \qquad (35)$$

and similarly,

$$d_2^2(\hat{\rho}, \hat{\rho}_h) \leq 2(\mathcal{A}_1(\hat{w}_h) + \mathcal{A}_2(\hat{\rho}_h)) \qquad (36)$$

Results (34) to (36) form the basis of a general dual error bound analysis. For the same physical situation, perform two computations, the first one being a regular internal approximation of the primal problem, and the second one, a regular internal approximation of the dual problem. Then, inequalities (35) and (36) provide error bounds for both discretization errors. This result, obtained here in the general inhomogeneous case, is of great interest at a practical point of view, from it permits a secure evaluation of the actual error, independently of any convergence result. In order to qualify a given analysis, the above error bounds may be compared to the computed energetic norms, i.e. $(C^{-1}\hat{\rho}_h, \hat{\rho}_h)_H$ and $(C\partial\hat{w}_h, \partial\hat{w}_h)_H$.

Two particular results of this type were obtained by Fraeijs de Veubeke /1,2,3,6/ and widely used by his collegas/ 4,5,7,8/. They correspond to the following situations

a) $u_o = 0$. In this case, $\hat{w} = \hat{u}$, $\hat{w}_h = \hat{w}$ and, by(4) ,

$$\mathcal{A}_1(\hat{w}_h) = \tfrac{1}{2}\,(C\partial\hat{w}_h, \partial\hat{w}_h)_H - f(\hat{w}_h) = -\tfrac{1}{2}(C\partial\hat{w}_h, \partial\hat{w}_h)_H$$

and similarly,

$$\mathcal{A}_1(\hat{w}) = -\tfrac{1}{2}\,(C\partial\hat{w}, \partial\hat{w})_H ,$$

while

$$\mathcal{A}_2(\hat{\rho}_h) = \tfrac{1}{2}\,(C^{-1}\hat{\rho}_h, \hat{\rho}_h)_H \;,\quad \mathcal{A}_2(\hat{\rho}) = \tfrac{1}{2}\,(C^{-1}\hat{\rho}, \hat{\rho})_H,$$

so that

$$d_2^2(\hat{\rho}_h, C\partial\hat{w}_h) = (C^{-1}\hat{\rho}_h, \hat{\rho}_H)_H - (C\partial\hat{w}_h, \partial\hat{w}_h)_H \qquad (37)$$

From equations (25), (31) and (33), one can deduce

$$(C\partial\hat{w}_h, \partial\hat{w}_h)_H \leq (C\partial\hat{w}, \partial\hat{w})_H = (C^{-1}\hat{\rho}, \hat{\rho})_H \qquad (C^{-1}\hat{\rho}_h, \hat{\rho}_h)_H \qquad (38)$$

b) $\sigma_o = 0$. Here, the situation is reversed. One has

$$\mathcal{A}_2(\hat{\rho}_h) = -\tfrac{1}{2}\,(C^{-1}\hat{\rho}_H, \hat{\rho}_h)_H \;,\quad \mathcal{A}_2(\hat{\rho}) = -\tfrac{1}{2}\,(C^{-1}\hat{\rho}, \hat{\rho})_H$$

and

$$\mathcal{A}_1(\hat{w}_h) = \tfrac{1}{2}\,(C\partial\hat{w}_h, \partial\hat{w}_h)_H \;,\quad \mathcal{A}_1(\hat{w}) = \tfrac{1}{2}\,(C\partial\hat{w}, \partial\hat{w})_H$$

so that

$$d_2^2(\hat{\rho}_h, C\partial\hat{w}_h) = (C\partial\hat{w}_h, \partial\hat{w}_h)_H - (C^{-1}\hat{\rho}_h, \hat{\rho}_h)_H \tag{39}$$

and

$$(C^{-1}\hat{\rho}_h, \hat{\rho}_h)_H \leq (C^{-1}\hat{\rho}, \hat{\rho})_H = (C\partial\hat{w}, \partial\hat{w})_H \leq (C\partial\hat{w}_h, \partial\hat{w}_h)_H \tag{40}$$

Relations (38) and (40) are known by structural analysts as <u>bounds on the direct influence coefficients</u>.

8. APPLICATION TO STRESS ANALYSIS

Let Ω be an open set of R^3, and define

$$V = (H^1(\Omega))^3 = \{ u = (u_1, u_2, u_3) \mid u_i \in L^2(\Omega), D_i u_j \in L^2(\Omega) \}, \tag{41}$$

equipped with the sealar product

$$(u,v)_V = \int_\Omega (u_i v_i + D_j u_i D_j v_i)\, dx \tag{42}$$

where repeated indices implie summation over the three values 1,2,3. The boundary Γ of Ω consists of two parts Γ_1 and Γ_2, in such a manner that

$$\begin{cases} \Gamma \subset \overline{\Gamma_1 \cup \Gamma_2} \\ \int_{\Gamma_1} dS \neq 0 , \quad \int_{\Gamma_1 \cap \Gamma_2} dS = 0 \end{cases} \tag{43}$$

Let V_o be the subspace of displacement whose traces on Γ_1 vanische

$$V_o = \{ u \in V \mid u|_{\Gamma_1} = 0 \} \tag{44}$$

The operator ∂ is defined as

$$\partial_{ij} u = \tfrac{1}{2}(D_i u_j + D_j u_i) \tag{45}$$

and transforms V into the space given by

$$H = \{ \varepsilon_{ij} \mid \varepsilon_{ij} = \varepsilon_{ji}, \quad \varepsilon_{ij} \in L^2(\Omega) \} \tag{46}$$

From Korn's inequality /11,13/ , any $u \in V_o$ verifies

$$\int_\Omega \partial_{ij} u \ \partial_{ij} u \ dx \geq \alpha \|u\|_V^2 , \quad \alpha > 0 \tag{47}$$

Finally, let the operator C be defined by its 21 independent components (which may depend upon x)

$$C_{ijkl}(x) = C_{jikl}(x) = C_{ijlk}(x) = C_{klij}(x) \in L^\infty(\Omega), \tag{48}$$

with an uniform positive definiteness condition on Ω :

$$C_{ijkl}(x)\ \varepsilon_{ij}\ \varepsilon_{kl} \geq \gamma\ \varepsilon_{ij}\ \varepsilon_{ij} , \quad \gamma > 0 \tag{49}$$

The classical formulation of elasticity problems consists to solve (in the distributional sense) the following equations:

$$\left\{\begin{array}{lll} D_j(C_{ijkl}\ \partial_{kl}u) + f_i = 0 & \text{in } \Omega, & f_i \in L^2(\Omega) \\ n_j(C_{ijkl}\ \partial_{kl}u) = t_i & \text{on } \Gamma_2, & t_i \in H^{-\frac{1}{2}}(\Gamma_2) \\ u_i = \bar{u}_i & \text{on } \Gamma_1, & \bar{u}_i \in H^{\frac{1}{2}}(\Gamma_1) \end{array}\right. \qquad (49)$$

(For the definition of $H^{\frac{1}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$, see e.g. /10,11/)

The primal formulation consists to solve this system in the virtual work sense. Let firt $u^o \in V$ be such that

$$u_i^o = \bar{u}_i \quad \text{on} \quad \Gamma_1 . \qquad (51)$$

As pointed in section 2, one may assume that

$$\int_\Omega f_i\ u_i^o\ dx + \int_{\Gamma_2} t_i\ u_i^o\ dS = 0 \qquad (52)$$

In the primal formulation, the functional to minimize is

$$\mathcal{A}_1(u) = \tfrac{1}{2}\int_\Omega C_{ijkl}\,\partial_{ij}u\ \partial_{kl}u\ dx - \int_\Omega f_i\ u_i\ dx - \int_{\Gamma_2} t_i u_i\ dS \qquad (53)$$

This minimization has to be performed in the manifold $u^o + V_o$.
It is the well-known <u>total energy principle</u>.

In the dual formulation, our first task is to seek a particular field $\sigma^o \in H$ such that

$$\int_\Omega \sigma_{ij}^o\ \partial_{ij}u\ dx = \int_\Omega f_i u_i\ dx + \int_{\Gamma_2} t_i u_i\ dS \quad \forall u \in V_o \qquad (54)$$

i.e. verifying the equilibrium conditions in the virtual work sense. Noting that the space $(\mathcal{D}(\Omega))^3$ of smooth displacements with compact support within Ω is contained in V_o, one can see that (54) implies

$$D_j\sigma_{ji}^o + f_i = 0 \qquad \text{in } \Omega , \qquad (55)$$

in the distributional sense. Therefore, the vectors

$$\sigma_i = (\sigma_{1i}, \sigma_{2i}, \sigma_{3i})$$

are elements of the space

$$H_{div}(\Omega) = \left\{ p = (p_1, p_2, p_3) \mid p_i \in L^2(\Omega),\ \text{div}\ p \in L^2(\Omega) \right\}$$

On this space, it is possible /11/ to define boundary traces

$$n_j\sigma_{ji} \in H^{-\frac{1}{2}}(\Gamma)$$

which, by virtue of (54) and (55), will here verify

$$n_j\sigma_{ji} = t_i \qquad \text{on } \Gamma_2 , \qquad (56)$$

as can be verified by a formal integration by parts.

Subspace S_o is defined as

$$S_o = \left\{ \sigma \in H \mid \int_\Omega \sigma_{ij}\ \partial_{ij}u\ dx = 0 \quad \forall u \in V_o \right\} \qquad (57)$$

By a similar argument as for σ°, each $\sigma \in S_o$ verifies

$$\begin{cases} \sigma_i = (\sigma_{1i}, \sigma_{2i}, \sigma_{3i}) & H_{div}(\Omega) \\ D_j\sigma_{ji} = 0 & \text{in } \Omega \\ n_j\sigma_{ji} = 0 & \text{on } \Gamma_2 \end{cases} \tag{58}$$

The dual problem consists thus in the minimization in $\sigma^\circ + S_o$ of the dual functional

$$\mathcal{A}_2(\sigma) = \frac{1}{2}\int_\Omega C^{-1}_{ijkl}\,\sigma_{ij}\sigma_{kl}\,dx - \int_\Omega \sigma_{ij}\,\partial_{ij}u^\circ\,dx$$

This expression may be reduced to a more familiar one by transforming the last term in the following way. First, an integration by parts gives

$$\int_\Omega \sigma_{ij}\,\partial_{ij}u^\circ\,dx = \frac{1}{2}\int_\Omega \sigma_{ij}(D_iu^\circ_j + D_ju^\circ_i)\,dx =$$
$$\int_\Gamma \sigma_{ij}n_ju^\circ_i\,dS - \int_\Omega (D_j\,\sigma_{ji})\,u^\circ_i\,dx$$

Recalling that $\sigma_{ij} \in \sigma^\circ + S_o$ and taking account to relations (55) to (58), one obtains

$$\int_{\Gamma_2} t_i\,u^\circ_i\,dS + \int_{\Gamma_1} n_j\,\sigma_{ji}\,u^\circ_i\,dS + \int_\Omega f_i\,u^\circ_i\,dx = \int_{\Gamma_1} n_j\,\sigma_{ji}\,u^\circ_i\,dS,$$

the last expression resulting from assumption (54). Finally,

$$\mathcal{A}_2(\sigma) = \frac{1}{2}\int_\Omega C^{-1}_{ijkl}\sigma_{ij}\sigma_{kl}\,dx - \int_{\Gamma_1} n_j\,\sigma_{ji}\,u^\circ_i\,dS\,. \tag{59}$$

The reader will recognize the <u>complementary energy principle</u>.

We now turn our attention on finite element approximations of these two problems. The case of the primal problem (displacement model) is rather simple. Condition $V_h \subset V$ is the well-known conformity condition. Restriction R_3 of section 3 means that the Dirichlet conditions on Γ_1 have to be taken in account <u>exactly</u> in the finite element model.

The dual problem requires somewhat more care. Condition $H_h \subset H$ requires only

$$\sigma_{ij} \in L^2(\Omega),$$

a condition which is fulfilled by all local polynomial approximations on a bounded set. The necessity of ensuring interelement transmissions appears from the definitions of σ° and S_o. If T_h is the set of finite elements K, one may write

$$\int_\Omega \sigma^\circ_{ij}\, \partial_{ij} u\, dx = \sum_{K\in T_h} \int_K \sigma^\circ_{ij}\, \partial_{ij} u\, dx =$$

$$\sum_{K\in T_h} \int_{\partial K} n_j \sigma^\circ_{ji} u_i\, dS - \sum_{K\in T_h} \int_K (D_j\, \sigma^\circ_{ji})\, u_i\, dx$$

the integration by parts being justified by the fact that σ° is smooth in each element. The conditions are now

$$\begin{cases} D_j \sigma^\circ_{ji} + f_i = 0 & \text{in each } K\in T_h \\ n_j\, \sigma^\circ_{ji} & \text{equilibrated at interelement boundaries} \\ n_j\, \sigma^\circ_{ji} = t_i & \text{on } \Gamma_2 \end{cases} \qquad (60)$$

By a similar way, one finds that a stress field σ is contained in S_o if the preceeding relations are verified with $f_i = 0$ and $t_i = 0$. These are the classical relations for equilibrium elements. Restriction R_3 has here the meaning that equations (60) can be solved exactly by the finite element model.

Under these conditions, if $\hat{u}^h$ and $\hat{\sigma}^h$ are the finite element solutions of a same elastic problem whose exact solution is given by $\hat{u}$ and $\hat{\sigma}$, the dual error bounds are

a) $$\int_\Omega C^{-1}_{ijkl}(\hat{\sigma}^h_{ij} - C_{ijpq}\, \partial_{pq}\hat{u}^h)(\hat{\sigma}^h_{kl} - C_{klrs}\, \partial_{rs}\hat{u}^h)\, dx = 2(\mathcal{A}_1(\hat{u}^h) + \mathcal{A}_2(\hat{\sigma}^h)) \qquad (61)$$

b) $$\int_\Omega C^{-1}_{ijkl}(\hat{\sigma}^h_{ij} - \hat{\sigma}_{ij})(\hat{\sigma}^h_{kl} - \hat{\sigma}_{kl})\, dx \leq 2(\mathcal{A}_1(\hat{u}^h) + \mathcal{A}_2(\hat{\sigma}^h)) \qquad (62)$$

c) $$\int_\Omega C_{ijkl}(\partial_{ij}\hat{u}^h - \partial_{ij}\hat{u})(\partial_{kl}\hat{u}^h - \partial_{kl}\hat{u})\, dx \quad 2(\mathcal{A}_1(\hat{u}^h) + \mathcal{A}_2(\hat{\sigma}^h)) \qquad (63)$$

9. CONCLUSION

A general dual error bound theory has been presented, which works in all inhomogeneous problems and includes earlier results due to Fraeijs de Veubeke.

Presented in a general abstract frame, this theory is applicable to many physical situations including, among others, elasticity, thermal conduction problems, diffusion, electrostatics, incompressible lubrication, etc. The application to elasticity has been treated as an illustration.

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