Linear formulation of identifying codes in graphs

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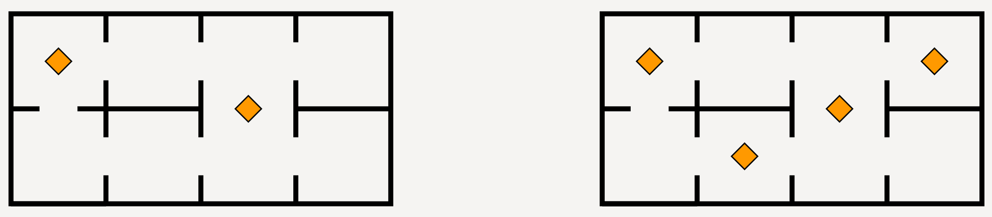
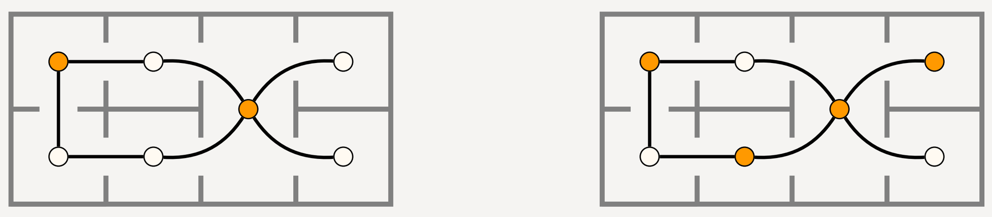
Fire in a building

Consider a sensor-placement in a building such that each sensor can detect a fire in its room and in its neighbouring rooms.
If there is a fire in one room, can we determine exactly where the fire is ?



Modelization with a graph : vertices : rooms,
edges : between two neighbouring rooms.



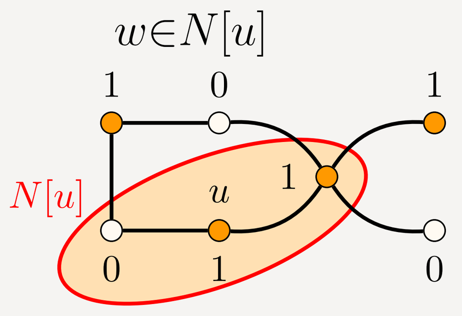
Identifying code of a graph : set C of vertices such that

- (Domination) each vertex must contain at least a vertex of C in its neighbourhood,
- (Separation) any pair of vertices can not have the same set of vertices of C in their neighbourhoods.

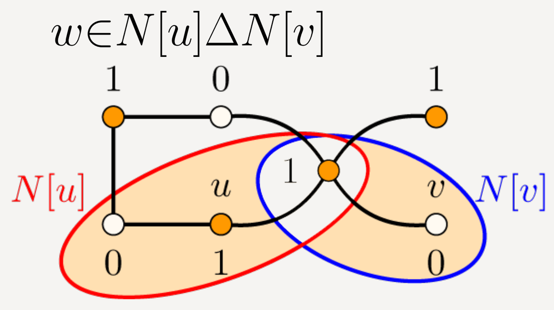
Linear formulation

We put weights $x_u \in \{0,1\}$ on the vertices with the convention that for a set of vertices $C \subseteq V$, $x_u = 1 \Leftrightarrow u \in C$.
The set $C \subseteq V$ is an identifying code if for any $u, v \in V$ with $u \neq v$,

$$\sum_{w \in N[u]} x_w \geq 1 \quad \text{and} \quad \sum_{w \in N[u] \Delta N[v]} x_w \geq 1.$$



Domination



Separation

Then $\gamma^{\text{ID}}(G) := \min\{\#C = \sum_{u \in V} x_u | \, C \text{ is an identifying code of } G\}$.

Finding an identifying code with minimum cardinality is **NP-hard** in general !

How to obtain a natural bound on $\gamma^{\text{ID}}(G)$?
Allow the weights x_u to be fractions in $[0,1]$ in the above problem and denote by $\gamma_f^{\text{ID}}(G)$ the smallest possible $\sum_{u \in V} x_u$. Then

$$\gamma_f^{\text{ID}}(G) \leq \gamma^{\text{ID}}(G).$$

Case of transitive graphs

For a vertex-transitive graph G on n vertices, there exists an optimal solution to the fractional problem where $x_u = \frac{\gamma_f^{\text{ID}}(G)}{n}$ for any $u \in V$.

Let k be the degree of vertices and d the smallest cardinal of symmetric differences between two neighbourhoods. The domination and separation conditions imply that

$$\frac{n}{\gamma_f^{\text{ID}}(G)} \leq \min(k+1, d).$$

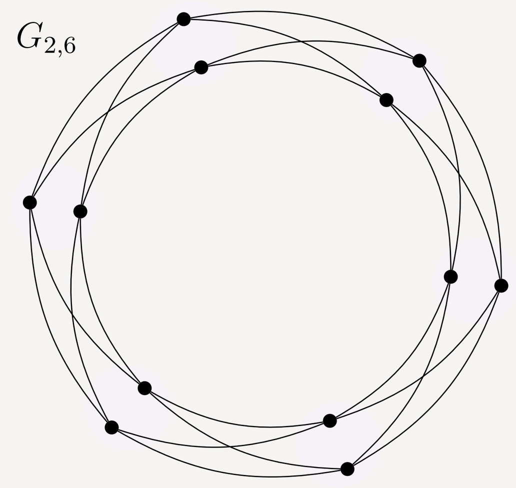
It is even an equality. Hence $\gamma_f^{\text{ID}}(G) = \max\left(\frac{n}{k+1}, \frac{n}{d}\right) \leq \gamma^{\text{ID}}(G)$.

NB : The bound $\frac{n}{k+1}$ is never reached. Indeed, Karpovsky, Chakrabarty and Levitin showed in 1998 that $\left\lceil \frac{2n}{k+2} \right\rceil \leq \gamma^{\text{ID}}(G)$.

When the separation condition prevails

How close is the bound $\gamma_f^{\text{ID}}(G) = \frac{n}{d}$ to $\gamma^{\text{ID}}(G)$?

A family of circulant graphs : $G_{m,p}$ with $m \geq 2, p > 4$



Number of vertices : $n = mp$
Degree of vertices : $k = 2m$
Smallest symmetric difference : $d = 2$
$\gamma^{\text{ID}}(G) = p(m-1)$

$$\frac{\gamma^{\text{ID}}(G)}{\gamma_f^{\text{ID}}(G)} = 2 - \frac{2}{m}$$

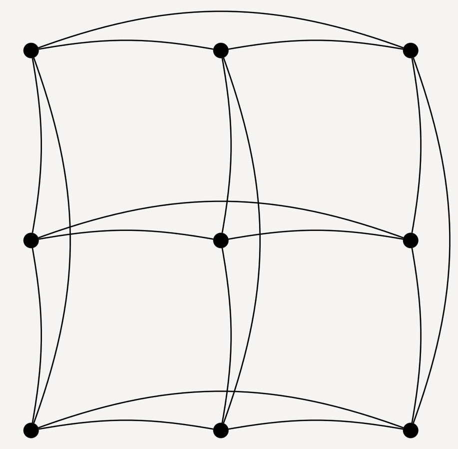
If $m = 2$, the bound is tight ! In contrast, the integer solution is nearly twice the fractional solution for big enough m. This corresponds to the worst case scenario with $d = 2$ since

$$\gamma^{\text{ID}}(G) \leq n = d\gamma_f^{\text{ID}}(G).$$

Cartesian product of complete graphs : $K_p \square K_p$

Number of vertices : $n = p^2$
Degree of vertices : $k = 2p - 1$
Smallest symmetric difference : $d = 2p - 2$
$\gamma^{\text{ID}}(G) = \frac{3p}{2}$ [Gravier, Moncel, Semri (2008)]

$$\frac{\gamma^{\text{ID}}(G)}{\gamma_f^{\text{ID}}(G)} = 3 - \frac{3}{p}$$



$K_3 \square K_3$

Q : Can we find examples with $d \neq 2$ corresponding to the "worst case"?

When the domination condition prevails

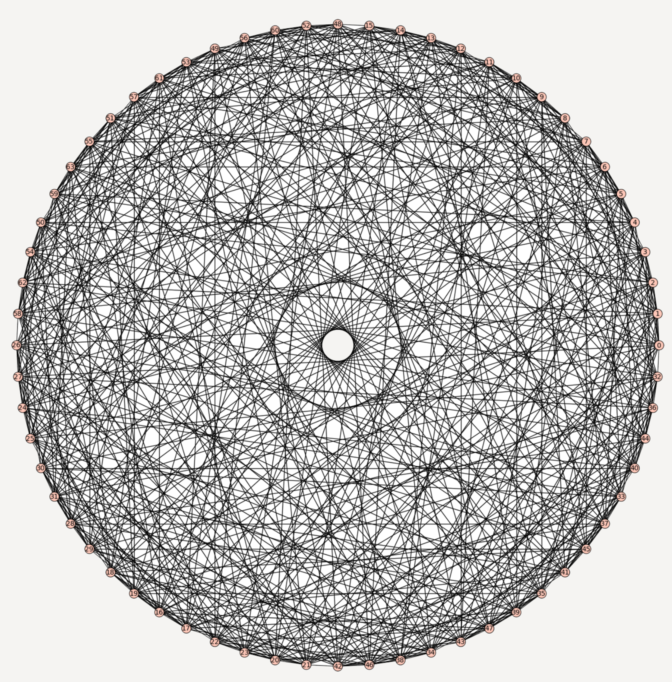
Generalized quadrangles $GQ(s,t)$

$GQ(s,t)$ is an incidence structure, i.e. a set of points and lines such that :

- there are $s+1$ points on each line,
- there are $t+1$ lines passing through a point,
- for a point P that does not belong to a line L, there is exactly one line passing through P and intersecting L.

Consider its incidence graph where points are vertices and there is an edge between two points if they belong to the same line.
For example, $K_p \square K_p$ corresponds to $GQ(p-1, 1)$.

The domination condition prevails in $GQ(s,t)$ iff $s > 1$ and $t > 1$.



$GQ(3,5)$
Vertices : points of $\mathbb{F}_4^3 = \{0, 1, z, z^2\}^3$
Edges : between two points A and B if the direction (AB) belongs to

$$\{(1,0,0), (0,1,0), (0,0,1), (1,1,1), (1,z,z^2), (1,z^2,z)\}$$

Deg. of vertices : $k = 3 \cdot 6 = 18$
Smallest sym. diff. : $d = 26$

$$\gamma_f^{\text{ID}}(GQ(3,5)) = \frac{64}{19} \approx 3,36 \text{ and } \gamma^{\text{ID}}(GQ(3,5)) = 9$$

Q : Identifying code of $GQ(7,9)$? Generalization to $GQ(2^\ell - 1, 2^\ell + 1)$?

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