

A NEW LOOK AT HERRMANN'S FORMULATION OF INCOMPRESSIBILITY

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SUMMARY

An exposition of Herrmann's formulation is given first, and its extension to the anisotrope case is presented following a new way. Using this method, the matrix of material coefficients can be calculated in a fully automated manner.

The existence of the solution is briefly discussed next. A necessary and sufficient condition is obtained which turns to be of unpractical use. Fortunately, this condition is in most practical cases equivalent to another one, which is of fairly simple use.

These concepts are illustrated by an example.

1. INTRODUCTION

In 1965, a mixed pressure-displacement formulation for compressible and incompressible isotropic solids was proposed by HERRMANN [1]. The orthotropic case has been considered by TAYLOR, PISTER and HERRMANN [2], and the anisotropic case a short time later by KEY [3].

A new way of extending Herrmann's formulation to anisotropic solids [4,5] is proposed. It differs from Key's approach by the fact that the variables used are the pressure and the strains, and not the deviatoric strains. Given the compliance matrix, the kernel of the functional is constructed by a method which can be fully automatized. This possibility turns out to be of great practical interest.

Unfortunately, when passing from the compressible case to the incompressible case, a discontinuity can occur in Herrmann's formulation since the existence and unicity of the solution are not always guaranteed. A theoretical study of this difficulty has been achieved [5] which shows that account must be taken of certain restrictions to ensure correct results. This theory is rather technical, and only its main results will be given here. The proof will be given next that the theoretical criterium is in most cases equivalent to a much simpler one due to NAGTEGAAL et al. [6].

2. GENERALIZED HERRMANN'S FORMULATION

2.1. In the following text, the stresses and strains will be noted in matrix form

$$t^T = (\tau_{11}, \tau_{22}, \tau_{33}, \tau_{12}, \tau_{13}, \tau_{23}), \quad e^T = (\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{12}, \epsilon_{13}, \epsilon_{23}).$$

Reissner's principle can then be written

$$\int_V (t^T e - \frac{1}{2} t^T B t) dV - \int_{S_2} \bar{t}_i u_i dS \text{ stat,}$$

where B is the compliance matrix. Varying the stress vector t yields the relation $e = Bt$ (1)

The strains of an incompressible structure verify following $r^T e = 0$,

where $r^T = (1, 1, 1, 0, 0, 0)$. For any stress-vector t , one has thus $r^T B t = 0$, and this means that $B r = 0$. This singularity involves the lack of existence of a displacement formulation.

2.2. Let us now define a "deviatoric projector" D by the relation

$$D = I - \frac{1}{3} r r^T .$$

where I is the identity matrix. It is easy to see that this matrix has the properties $D = D^T$ (symmetry), $DD = D$ (idempotency), $Dr = 0$ (projectivity). Any vector p admits the following decomposition in a deviatoric and an isotropic vector:

$$p = Dp + \frac{1}{3} (r^T p) r$$

2.3. If the structure is compressible, we may invert the stress-strain relation (1) to give $t = B^{-1} e$. This is of course not possible in the incompressible case. But one can write

$$e = Bt = B(Dt + \frac{1}{3} r^T t r) = BDt .$$

We are therefore looking for a matrix A having the following property:

$$ABD = D \quad (2)$$

If possible, A should also be invertible and well-conditioned. Such a matrix can be constructed by considering the one parameter matrix family

$$\bar{B}_\alpha = B + \frac{\alpha}{3} r r^T , \quad \alpha > 0$$

Obviously, if $B r = 0$, one has

$$\bar{B}_\alpha r = \frac{\alpha}{3} r r^T r = \alpha r \quad (3)$$

i.e. the vector r is no longer singular, and, for any deviatoric vector Dp ,

$$\bar{B}_\alpha Dp = BDp + \frac{\alpha}{3} r r^T Dp = BDp$$

If $B r = 0$ has no other non trivial solution than r , \bar{B}_α is thus invertible, and one can set

$$A_\alpha = \bar{B}_\alpha^{-1} . \quad (4)$$

Making use of (3) yields then expected property (2)

$$D = A_\alpha \bar{B}_\alpha D = A_\alpha B D + \frac{\alpha}{3} A_\alpha r r^T D = A_\alpha B D ,$$

The free parameter α allows for an optimization of the condition number of A_α . A suitable choice is given by the relation

$$\frac{r^T \bar{B}_\alpha r}{r^T r} = \text{mean of the eigenvalues of } \bar{B}_\alpha = \frac{1}{6} \text{tr} (\bar{B}_\alpha) , \text{ where the notation}$$

tr states for the trace of a matrix. Noting that

$$r^T (\bar{B}_\alpha) r = \text{tr}(B) + \alpha , \quad r^T \bar{B}_\alpha r = r^T B r + \frac{\alpha}{3} (r^T r)^2 = r^T B r + 3\alpha ,$$

one obtains immediately the optimum value α_0

$$5\alpha_0 = \text{tr}(B) - 2 r^T B r \quad (5)$$

In what follows, we will write A for A_{α_0} and α for α_0 . Consider for instance the isotropic case. The matrix B is given by

$$B = \frac{1+\nu}{\epsilon} I - \frac{\nu}{\epsilon} r r^T ,$$

Where ϵ and ν are Young's modules and Poisson's ratio respectively. After some elementary calculations, one obtains

$$\alpha = \frac{3\nu}{\epsilon} \quad , \quad A = \frac{\epsilon}{1+\nu} I \quad (6)$$

The particular choice (6) can be seen to lead to Herrmann's isotropic formulation.

The "optimum" matrix A has some remarkable properties. For sake of brevity, they are given without their (elementary) proof:

$$(a) \quad BAr = \mu r, \quad \text{with } \mu = 1 - \frac{1}{3} \alpha \quad r^T Ar \quad (7)$$

$$(b) \quad \text{in the incompressible case, } Ar = \frac{1}{\alpha} r \quad \text{and } \mu = 0 \quad (8)$$

$$(c) \quad (AB-I) = -\frac{\alpha}{3} Arr^T \quad (9)$$

$$(d) \quad (AB-I) r = -\alpha Ar \quad (10)$$

$$(AB-I) D = 0 \quad (11)$$

$$(BA-I) BD = 0 \quad (12)$$

2.4. We are now ready to transform Reissner's principle. Starting from (1), which can be rewritten in the form

$$e = BDt + \frac{1}{3} Brr^T t, \quad (13)$$

let us multiply both sides of equation (13) by A. By (2), this leads to

$$Ae = Dt + \frac{1}{3} ABrr^T t \quad (14)$$

As it appears immediatly, the deviatoric stress-vector has a real dimension equal to 5 since the subsidiary condition $r^T Dt = 0$ holds. If this is automatically verified, one may define an independant mean pressure $\pi = -\frac{1}{3} r^T t$. Equation (14) becomes then

$$Dt = Ae + \pi ABr,$$

yielding thus the result

$$t = Dt - \pi r = Ae + \pi (AB-I)r. \quad (15)$$

The first term of Reissner's functional is then easy to transform

$$t^T e = e^T Ae + \pi r^T (BA-I)e$$

Transforming the complementary energy density requires somewhat more care: the key is to transform it from one side, and then from the other one:

$$\begin{aligned} \frac{1}{2} t^T Bt &= \frac{1}{2} e^T ABt + \frac{1}{2} \pi r^T (BA-I) Bt \\ &= \frac{1}{2} e^T ABDt - \frac{1}{2} e^T ABr\pi + \frac{1}{2} \pi r^T (BA-I) BDt - \frac{1}{2} \pi r^T (BA-I) Br\pi \end{aligned}$$

(a) (b) (c) (d)

and we get

$$(a) \rightarrow \frac{1}{2} e^T Dt = \frac{1}{2} e^T Ae + \frac{1}{2} \pi ABr, \quad (c) \rightarrow 0, \quad \text{from (12)}.$$

The resulting functional is

$$\int_V [\phi(Du, \pi) - F_i u_i] dV - \int_{S_2} \bar{t}_i u_i dS, \quad (16)$$

Where

$$\phi(e, \pi) = \frac{1}{2} e^T A e + \pi r^T (BA-I) e + \frac{1}{2} r^T (BA-I) B r \pi^2, \quad (17)$$

whose stationarity restitutes the elastic solution.

In fact, $\frac{\partial \phi}{\partial e} = A e + (AB-I) r \pi = t$,

from what follows the fact that equilibrium will be satisfied. Varying π , one obtains the condition

$$r^T (BA-I) (e + B r \pi) = 0,$$

which, by (10), is equivalent to

$$- \alpha r^T (A e + A B r \pi) = 0, \quad (18)$$

that is, $(A e + A B r \pi)$ is effectively a deviatoric stress vector. It is the condition of independancy of the pressure. This relation admits another interpretation. Following (7), we have $\alpha r^T A B r \pi = 3 \alpha \mu \pi$, and combining this relation with (18), one obtains

$$\pi = - \frac{r^T A e}{3 \mu}, \quad (19)$$

that is, the relation between the pressure and the deformations. In the isotropic case, it reduces to the classical compressibility equation

$$\pi = - \frac{\epsilon}{3(1-2\nu)} r^T e. \quad (20)$$

The function ϕ can be set in the matrix form

$$\phi(e, \pi) = \frac{1}{2} (e, \pi)^T \bar{H} \begin{pmatrix} e \\ \pi \end{pmatrix},$$

with

$$\bar{H} = \begin{bmatrix} A & (AB-I)r \\ r^T (BA-I) & r^T (BA-I) B r \end{bmatrix} = \begin{bmatrix} A & - \alpha A r \\ - \alpha r^T A & - 3 \alpha \mu \end{bmatrix} \quad (21)$$

In the incompressible case, this matrix becomes

$$\bar{H}(\text{in c.}) = \begin{bmatrix} A & -r \\ -r^T & 0 \end{bmatrix} \quad (22)$$

For isotropic structures, it follows from (6) that

$$\mu = \frac{1-2\nu}{1+\nu}, \quad \bar{H} = \begin{bmatrix} \frac{\epsilon}{1+\nu} I & - \frac{3\nu}{1+\nu} r \\ - \frac{3\nu}{1+\nu} r^T & - \frac{3\nu}{1+\nu} \frac{3(1-2\nu)}{\epsilon} \end{bmatrix} \quad (23)$$

We recognize Herrmann's isotropic formulation which can thus be considered as optimum in the sense defined above.

3. SOME REMARKS

3.1. The conditions of applicability of Herrmann's formulation are first that the derivatives of the displacements are square-integrable. This is verified if the displacement are continuous at the interelement boundaries. But the pressure should simply exist in the L^2 sense. In particular, one can

choose a piecewise continuous polynomial, with discontinuities at the interelement boundaries. Many authors use a continuous pressure. We find it is not very logical and, moreover, it causes great difficulties when the material is discontinuous. With element polynomials including at least a constant term, the volume of each element is preserved, and this property seems the most natural way of ensuring the incompressibility.

3.2. In practical applications, it is necessary to rescale the element pressure to ensure a good conditioning of the final matrix. The problem lies in the fact that the pressure and the displacements are d.o.f. of different nature. A different scaling factor must be used in each element taking in account the orders of magnitude of the moduli and of the size of the element in concern. This possibility of preserving the condition number is obviously lost if the pressure is taken continuous.

3.3. Herrmann's formulation leads to a nonstandard system of equations, because the corresponding matrix is not positive definite. Therefore, some authors have tried to eliminate these cumbersome pressure d.o.f. This is possible only if the structure is not exactly incompressible, that is $\mu \neq 0$. Consider for simplicity the case of a constant pressure in each element. One can write

$$\pi = - \frac{1}{\int_K 3\alpha \mu dV} \int_K \alpha r^T Ae dV = - \lambda \int_K \alpha r^T Ae dV \quad (24)$$

and this leads to the displacement -like functional

$$\Sigma_K \int_K \left\{ \frac{1}{2} e^T Ae + \frac{1}{2} \lambda \left(\int_K \alpha r^T Ae dV \right)^2 \right\} dK \quad (25)$$

For the quasi-incompressible case, $|r^T e| \ll ||De||$, and one can replace e by De in the first term of the functional. This leads to a generalization of a principle due to NAGTEGAAL et al. [6]. In fact, in the isotropic homogeneous case, we obtain

$$\Sigma_K \int_K \left\{ \frac{1}{2} e^T Ae + \frac{1}{2} \frac{3\nu}{1+\nu} \frac{\epsilon}{3(1-2\nu)} \left(\int_K r^T e dV \right)^2 \right\} dV, \quad (26)$$

and if $\nu = 0,5$, this can be approximated by

$$\Sigma_K \int_K \left\{ \frac{1}{2} e^T D A De + \frac{1}{2} \frac{\epsilon}{3(1-2\nu)} \left(\int_K r^T e dV \right)^2 \right\} dV, \quad (27)$$

which is precisely Nagtegaal's formulation. It is thus no more than a variant of Herrmann's principle, where the element incompressibility conditions are now obtained by a penalty method when $\nu \rightarrow 0,5$.

Nagtegaal's two field principle is also intimately connected to Herrmann's one, by the substitution of π by $\psi = 3 \alpha \mu \pi$ and the above approximations when $\nu \rightarrow 0,5$.

However, it should be emphasized that the pressure d.o.f. do not cause "insurmountable implementation problems" [7]. In a frontal elimination process, it is only necessary to select a suitable order of pivoting the different terms.

4. PROBLEMS CONNECTED TO THE EXISTENCE AND UNICITY OF THE SOLUTION

Herrmann's variational principle is unfortunately not minimal, and the corresponding formulation does not necessary admit a solution. A complete discussion of this problem has been done [5]. But it is rather technical, and we shall here restrict ourselves to the main results, limiting the justifications at the intuitive level. Three cases can occur:

(a) $\alpha = 0$: in this case, which corresponds, in the isotropic case to $\nu = 0$, the pressure does not appear in the principle, and the corresponding degrees of freedom must be fixed. The corresponding variational principle is then minimal, and the solution always exists and is unique.

(b) standard compressible case, $\alpha \neq 0$, $\mu \neq 0$. Then, as it has been seen, the pressure d.o.f. can be eliminated at the element level, and this will reduce the principle to a displacement-like form. The principle is then minimal. The solution always exists and is unique.

(c) The only problematic case is the incompressible one, $\alpha \neq 0$, $\mu = 0$. The pressure plays the role of a Lagrange multiplier, and we are in presence of a saddle point problem. Following BREZZI's theory [8], the solution exists and is unique if and only if the following condition is verified.
"There exists a constant $\beta > 0$ such that for any $p \in L^2(V)$

$$\sup_{\substack{u \in U \\ u \neq 0}} \frac{\int_V p \operatorname{div} \vec{u} \, dV}{\left(\int_V e^T A e \, dV \right)^{\frac{1}{2}}} \geq \beta \left(\int_V p^2 \, dV \right)^{\frac{1}{2}}, \quad (28)$$

where U is the space of admissible displacements". It can be shown that in the undiscretized case, this condition is verified, except when the normal displacements are fixed on the whole boundary. In this special case, the solution is correctly defined if one add the extra-condition that if p is constant, it must be equal to zero.

Since the finite element discretization consists of selecting finite dimensional subspaces $U_h \subset U$ and $P_h \subset L^2$ and finding a solution (u_h, p_h) with $u_h \in U_h$ and $p_h \in P_h$, it is clear that the same condition (28) must hold for the discretized problem, where U is replaced by U_h and β by another number β_h . Unfortunately, this condition does not always hold in the present case, and, in principle, must be verified in any practical case. But the quite complicate form of this condition is somewhat discouraging.

However, there exists another, very simple condition, which is

$$r = \text{dimension}(P_h) \leq \text{dimension}(U_h) = n, \quad (29)$$

i.e. the number of unfixed displacements is greater than the number of unfixed pressures. The necessity of this condition will be proven first. Let us develop u and p in their basis functions :

$$u = \sum_{k=1}^n \vec{v}_k u^k(x), \quad p = \sum_{k=1}^r q_k \phi_k(x),$$

and suppose that $n < r$. One writes

$$C_{k\ell} = \int_V p_k \operatorname{div}(\vec{v}_\ell) \, dV, \quad \vec{v}^T = (v_1, \dots, v_n), \quad q^T = (q_1, \dots, q_r).$$

If (w_1, \dots, w_n) is a basis of the n -dimensional euclidean space R^n , the vectors $s_k = \sum w_k$ form the basis of a n -dimensional subspace of R^r . Taking p in the k th orthogonal complement of this subspace, one has necessarily $q^T C v = 0$,

and this contradicts (28). But is the condition (29) also sufficient? The answer is almost always affirmative. In fact, if the matrix C is of rank r , that is, if all incompressibility relations are independent - and this is the usual case - the condition (28) is also sufficient. To prove this, we decompose v in

$$v = C^T y + \hat{v}, \text{ with } C \hat{v} = 0,$$

There,

$$\sup_v \frac{q^T C v}{\|v\|} = \sup_v \frac{q^T C C^T y}{(y^T C C^T y + \hat{v}^T \hat{v})^{\frac{1}{2}}} = (q^T C C^T q)^{\frac{1}{2}},$$

where $C C^T$ is a positive definite matrix. The relation (28) follows then from the fact that all norms of a finite dimensional space are equivalent.

The condition (29) is very simple, and therefore, it can be considered as a very useful and effective criterium. Its application can be systematized as follows :

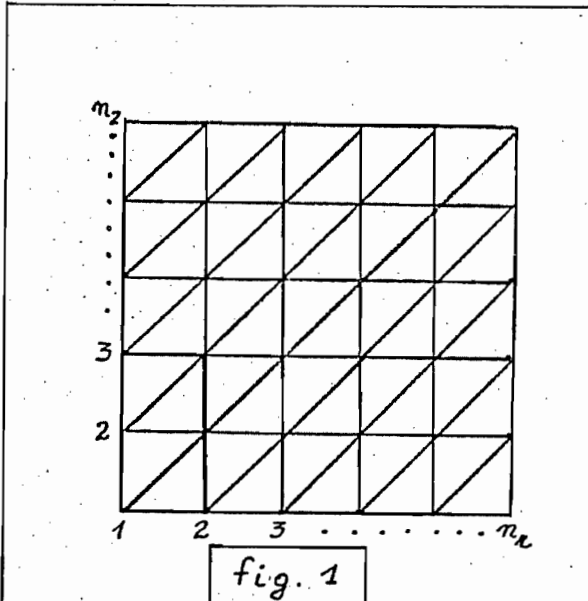
- (i) numbering of the n unfixed displacements
- (ii) numbering of the r pressure
- (iii) definition of the discretization ratio $\eta = \frac{n-r}{n}$

This ratio represents the proportion of "active" displacements. If negative or equal to zero, the solution does not exist. The greater η is, the better is the discretization, at least from the safety point of view. To characterize the performance of an element in more general terms, one may use the following method, which was introduced by NAGTEGAAL, PARKS and RICE [6] for their plasticity problems.

- (i) Define a regular mesh with $n_x \times n_y \times n_z$ nodes
- (ii) Compute n, r and η in terms of n_x, n_y, n_z
- (iii) Compute the asymptotic discretization ratio

$$\bar{\eta} = \lim_{n_x, n_y, n_z \rightarrow \infty} \eta$$

As an illustration, the case of axisymmetrical triangular tori elements, of various degrees is considered (fig.1).



a) degree 1, constant pressure :

$$n = 2n_r n_z, \quad r = 2(n_r - 1)(n_z - 1)$$

$$\eta_1 = \frac{1}{n_r} + \frac{1}{n_z} - \frac{1}{n_r n_z} \quad \bar{\eta}_1 = 0$$

The discretization is thus generally somewhat poor. Note that this evaluation does not take the fixed d.o.f. in account. It is relatively easy to obtain examples where $\eta_1 < 0$.

b) degree 2, constant pressure

$$n = 8 n_r n_z - 4 n_r - 4 n_z + 2, \quad r = 2(n_r - 1)(n_z - 1)$$

$$\eta_2 = \frac{6n_r n_z - 2n_r - 2n_z}{8n_r n_z - 4n_r - 4n_z + 2}, \quad \bar{\eta}_2 = \frac{3}{4}$$

c) degree 3, linear pressure : the linear pressure terms can be eliminated with the "bubble modes" in each element, and we omit these two kinds of d.o.f. Then,

$$n = 14 n_r n_z - 8n_r - 8n_z + 4, \quad r = 2(n_r - 1)(n_z - 1)$$

$$\eta_3 = \frac{12n_r n_z - 6n_r - 6n_z + 2}{14n_r n_z - 8n_r - 8n_z + 4}, \quad \bar{\eta}_3 = \frac{6}{7}$$

d) degree 1, linear continuous pressure : we have already said that, from our point of view, the procedure is not justified. But it gives a better discretization ratio than (a) :

$$\eta_4 = \bar{\eta}_4 = \frac{1}{2}$$

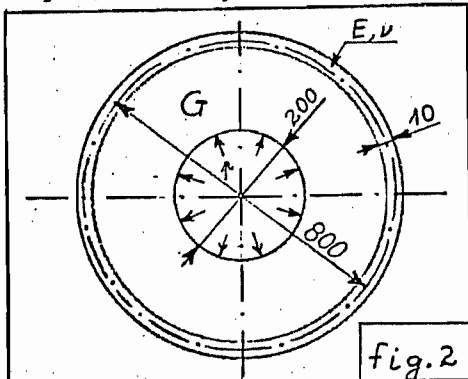
e) when all vertical displacements are fixed, η is reduced. The results are

$$\bar{\eta}_1 = -1 \text{ (bad)}, \quad \bar{\eta}_2 = \frac{1}{2} \text{ (correct)}, \quad \bar{\eta}_3 = \frac{5}{7} \text{ (correct)}, \quad \bar{\eta}_4 = 0 \text{ (bad)}$$

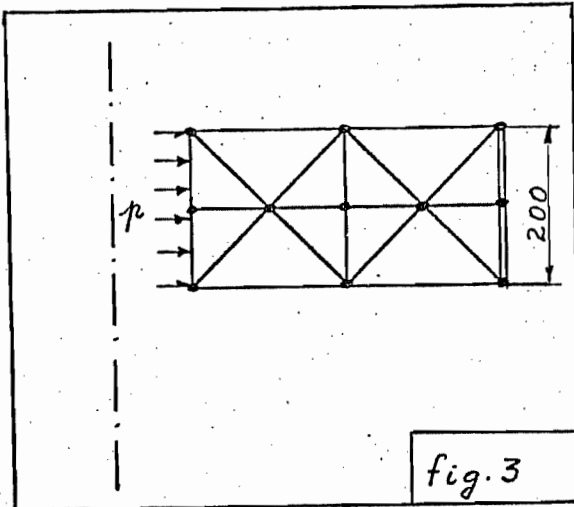
Finally, it should be noted that, as is the case for the undiscretized problem, when the fixations ensure the constance of the global volume, one has to add the extra-condition that if p is constant, it must be equal to zero. This can be done by fixing arbitrarily a pressure to zero. This condition admits a very simple interpretation when the pressure remains constant on each element. In fact, each pressure is "responsible" for the volume conservation of its element. When the last element has to be connected to the others, its volume cannot vary since it is equal to the difference between the total volume and the invariant volume of the rest of the structure. Therefore, the last pressure parameter plays no role and must be fixed.

5. NUMERICAL EXAMPLE

As an example, the plane strain problem of a pressurized thick-walled incompressible cylinder contained in a thin case (fig.2) is considered.



It is discretized by axisymmetrical triangular core elements, as represented on fig.3. The plane strain condition is obtained by fixing all longitudinal displacements. The idealization contains 12 elements and 11 nodes. At the first degree, it leads to 11 displacements and 12 pressure d.o.f., and the solution does not exist. This prediction is confirmed numerically. (In Gauss method, after some operations, the non-pivoted sub-matrix has all zero diagonal terms).



By contrast, a second-degree idealization, with constant pressures leads to a discretization ratio equal to $21/33 = 0.636$, and with third degree elements with a linear pressure, the value 0.78 is obtained. Therefore, both idealization will be satisfactory. With the following data,

$$E=21000 \text{ hb}, t=10\text{mm}, (1-\nu^2)=0.91,$$

$$h=200 \text{ mm},$$

$$b=400\text{mm}, a=100\text{mm}, G=7,692300\text{hb},$$

$$p=1 \text{ hb}$$

the exact analytical solution yields the following results:

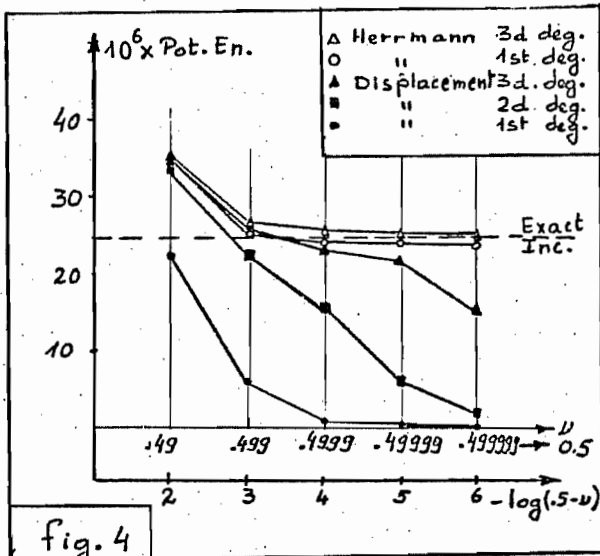
$$\text{potential energy } U=2,4893389 \cdot 10^5$$

$$\text{hoop stress resultant in the case: } N_{\theta\theta} = 285,71437$$

With Herrmann's third degree elements, having a linear (discontinuous) pressure, we have obtained

$$U = 2,489198 \cdot 10^5, N_{\theta\theta} = 285,7$$

The correspondance is quite good. Figure 3 shows the results obtained with Herrmann's first, second, and third degree elements, when $\nu \rightarrow 0.5$,



and compares them to the results of the classical displacement formulation. As demonstrated in [10], the extremely poor results of this formulation - the energy converges to zero when $\nu \rightarrow 0.5$ - , are due to the fact that the finite element model contains no incompressible mode. It is equivalent to say that Fried's K_1 matrix [9] is not singular. Fried's method consists in underintegrating the compressibility energy terms. It can be considered as a disguised form of Nagtegaal-Herrmann's method, where the independant strain parameter is connected to the displacements by an interpolation condition at Gauss points.

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