Localizable Effective Theories, Bootstrap and the Parameters of Hadron Resonances

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Abstract. We discuss the basic principles of constructing a meaningful perturbative scheme for effective theory. The main goal of this talk is to explain the approach and to present recent results [1] – [5] obtained with the help of the method of Cauchy forms in several complex variables.

Our work is aimed to develop a field-theoretic scheme providing the basis for dual models and, simultaneously, a link between two methods – the quantum field theory and the analytic theory of S-matrix. Besides, it concerns with the problem of renormalization of conventionally nonrenormalizable theories.

We rely on Weinberg's general scheme (see Chs. 2-5 of [6]) of constructing a quantum theory in terms of field operators that create and annihilate the *true* asymptotic states (corresponding to stable particles). Following [7], we call a theory effective if the interaction Hamiltonian (in the interaction picture) contains *all* terms consistent with the requirements of a given algebraic (linear) symmetry. The scheme is quantum *ab initio* and the problem of dynamical (nonlinear) symmetries requires special consideration. Here we do not discuss it.

We only consider a special class of *localizable* effective theories. An initial effective theory (constructed according to Weinberg's scheme solely from the fields of true asymptotic states) is called localizable if its tree level amplitudes can be reproduced in the framework of the well-defined tree level approximation based on the Hamiltonian of extended effective theory which – along with the fields of stable particles – also contains complementary auxiliary fields corresponding to fictitious unstable particles (resonances) of arbitrary high spin and mass. This implies that infinite sums of graphs providing formal expressions for tree level amplitudes of the initial theory converge, at least in certain small domains of the corresponding complex spaces. The existence of such domains is, in any case, necessary to assign meaning to the initial theory. It is precisely those domains where the tree level amplitudes of both theories must coincide with each other. Thus the extended theory just provides an analytic continuation of the tree level amplitudes constructed in the framework of the initial effective theory. In fact, our approach is just an attempt to extend Weinberg's quasiparticle method (see [8] and refs. therein) to the case of relativistic quantum theory.

It should be stressed that we only consider the case when the extended effective theory (as well as the initial one) does not contain massless particles of spin $J > 1/2$. Besides, we are only interested in constructing the effective *scattering* theory – the calculation of

Green functions is not implied. This means that the renormalization procedure possesses certain specific features allowing one to avoid attracting unnecessary renormalization prescriptions. The divergences which might occur in Green functions off the mass shell never bother us.

The last point deserves comment. In the case of customary renormalizable theories the number of renormalization prescriptions (RP) fixing the finite parts of counterterms is equal to that of coupling constants in the Hamiltonian (including mass and kinetic terms), *this latter one being finite*. "Hidden" couplings create no problem (see, e.g., [9]). In case of effective theories the situation looks quite different. They are renormalizable by construction because all possible local monomials are presented in the Hamiltonian. The problem is that the number of coupling constants is essentially infinite. This means that, to obtain finite results for all Green functions, one needs to point out a complete self-consistent infinite set of independent RP's. The structure of this set must provide a guarantee of convergence of infinite series appearing at every given order of loop expansion due to the presence of field derivatives of arbitrary high degree and order in the effective Hamiltonian. Otherwise, the resulting amplitudes would make no sense. The problem of constructing of such a set looks unsolvable until we have no regularity fixing its structure (possibly, up to a finite number of independent constants). The requirement of localizability is extremely useful in this very respect. As shown in [4], [5], in case of effective scattering theory it is possible to perform a detailed classification of parameters (combinations of coupling constants) appearing in expressions for the renormalized S-matrix elements of extended effective theory. This allows one to separate a group of *resultant* parameters. This group only contains the parameters which do contribute to renormalized S-matrix elements and thus require formulating the RP's. Finally, it is possible to show that crossing symmetry together with the requirement of convergence impose strong limitations on the allowed values of resultant parameters. In the case of tree level parameters describing the amplitudes of binary processes these limitations take a form of an infinite system of (algebraic) *bootstrap* equations connecting the values of the resultant coupling constants and the mass parameters appearing in the Hamiltonian. It is a direct consequence of the postulated properties of meromorphy and polynomial boundedness of the tree level amplitudes in every 3-dimensional band $B_x\{x \in \mathbf{R},\ x \sim 0;\ \nu_x \in \mathbf{C}\}$, where x stands for (real) momentum transfer and ν_x – for the corresponding (complex) energy-like variable.

We demand such properties for the following reasons. First, the polynomial boundedness property of the full (non-perturbative) amplitudes follows from the general axiomatic requirements. Hence it makes sense to construct the perturbation series in such a way that at every step the perturbative amplitude possesses the property of polynomial boundedness with the same degree of bounding polynomial as that of the full (non-perturbative) amplitude – this gives a chance to avoid strong corrections from the higher order terms. Besides, this requirement provides a guarantee that we deal with tempered distributions. Second, the property of meromorphy of tree level amplitudes follows from the *summability* requirement. The meaning of this term can be explained as follows. At every given order of loop expansion we have to take account of an infinite number of graphs. In absence of guiding principle fixing the summation order there is a danger either to get diverging series or to fall in contradiction with basic properties like crossing symmetry. That is why it looks reasonable to impose the summability requirement in the

following form. *In every sufficiently small domain of the complex space of kinematical variables there must exist an appropriate order of summation of the formal sum of contributions coming from the graphs of a given loop order, such that the reorganized series happens convergent. Altogether, these series must define a unique analytic function with only those singularities which are presented in contributions of individual graphs or can be reproduced in the framework of the same loop order of extended theory containing auxiliary fields corresponding to unstable particles.* This definition can be considered as a generalization (or, better, detailing) of the localizability requirement. From this formulation it follows that to study the most general localizable effective theory it is quite sufficient to consider the case of extended theory with no *ad hoc* limitations on the structure of set of resonance parameters. In turn, this means that the tree level amplitudes of the extended theory must be meromorphic functions of kinematical variables. They must be polynomially bounded in certain energy-like variables at fixed values of the other ones,the degree of bounding polynomials being dictated by unitarity.

Here it is pertinent to stress one point. To present the results of resonance saturation of dispersion relations (DR) for the amplitudes with nondecreasing asymptotic behavior the following form is often used:

$$A(z,x) = P(z,x) + \sum_{k=1}^{\infty} \left[\frac{r_k(x)}{z - p_k(x)} \right]. \tag{1}$$

Here z stands for energy-like variable, x – for any real parameter (say, the momentum transfer) and $P(z,x)$ – for so-called subtracting polynomial in z with the coefficients depending on x. However, it should be noted that in the case of infinite number of resonances this form is to be taken just as a formal one. Usually, it is tacitly assumed that the series of pole contributions in the RHS converges. Unfortunately, this is not always true (see, e.g., [11]). This suggestion imposes too strong limitations on the values of resonance parameters $r_k(x)$ and $p_k(x)$. In fact, it states that resonances do not affect the asymptotic behavior of the amplitude in question. In case of strong interactions this looks too restrictive. For this reason in our work we use the method of Cauchy forms specially adjusted for the case of several variables (see [2], [4]); this allows us to avoid implicit postulating of the resonance spectrum properties. Using this method it is easy to write down the most general form of the result. In case when the amplitude has only simple poles at $z_k = p_k(x) \neq 0$ with residues $r_k(x)$ and behaves, say, like a constant (it only makes sense to speak about the contour asymptotics) for $x \in (a,b)$ and $|z| \to \infty$, it looks as follows

$$A(z,x) = A(0,x) + \sum_{k=1}^{\infty} \left[\frac{r_k(x)}{z - p_k(x)} + \frac{r_k(x)}{p_k(x)} \right], \qquad x \in (a,b). \tag{2}$$

It is possible to show (see [2]) that in this case the functional series $S_n = \sum_{k=1}^{\infty} \frac{r_k(x)}{p_k^n(x)}$ is certainly convergent for $n \geq 2$ and may happen divergent for $n < 2$. In case when $S_1 \leq M < \infty$ the only possibility to fulfil the postulated asymptotic behavior is to demand that $A(0,x) \neq 0$. This means that the asymptotics is completely formed by $A(0,x)$, i.e., the contribution of resonances is irrelevant. These problems disappear if one uses the Cauchy form (2).

The summability requirement imposes certain restrictions on the parameters of theory. As shown in cited above papers, those restrictions for the tree level parameters follow from the condition of identical coincidence of two Cauchy forms representing the amplitudes of cross-conjugated processes. Each one of these forms is only applicable in the corresponding 3-dimensional band (layer). In the intersection of those layers both forms are equally applicable and thus must coincide. This requirement leads to an infinite system of algebraic relations between the resultant parameters.

The corresponding mechanism may be illustrated by the following example. Consider a rational function of two complex variables $F(x,y)$. Let us suppose it has a single simple pole (in x) in the layer $B_y\{\ y\in\mathbf{R},\ y\in(-\eta,+\eta);\ x\in\mathbf{C}\}$ and also a single simple pole (in y) in the 'orthogonal' layer $B_x\{\ x\in\mathbf{R},\ x\in(-\xi,+\xi);\ y\in\mathbf{C}\}$. Let us also assume that $F(x,y)$ is decreasing at infinity in each layer and that it is regular at the origin $M(0,0)$. Now let us try to answer the following question: what is the structure of the set of numerical parameters providing a complete description of functions that possess these properties?

Every function regular at the origin is completely fixed by the coefficients f_{ij} of its series expansion $F(x,y)=\sum f_{ij}x^iy^j$. The above question can be rephrased in a more concrete way: how many independent combinations of these coefficients can be arbitrary and what are these combinations? Or, in terms of field theory: how many independent renormalization prescriptions is it necessary to impose in order to completely fix the amplitude $F(x,y)$ and what is the explicit form of those prescriptions?

In the layer B_y we have $F(x,y)=\frac{\rho(y)}{x-\pi(y)}$. By condition, the functions $\rho(y)$ and $\pi(y)$ are regular in the vicinity of the origin. Hence, $\pi(y)=\sum\pi_iy^i$, $\rho(y)=\sum\limits_{i=0}\rho_iy^i$. By analogy, in B_x: $F(x,y)=\frac{r(x)}{y-p(x)}$, where $p(x)=\sum p_ix^i$,$r(x)=\sum r_ix^i$. Hence in the intersection domain $B_x\cap B_y\equiv D_{xy}$:

$$\frac{r(x)}{y-p(x)}=\frac{\rho(y)}{x-\pi(y)}\ ,\qquad (x,y)\in D_{xy}\{x\in(-\xi,+\xi),\ y\in(-\eta,+\eta)\}\ . \tag{3}$$

Substituting $\pi(y)$, $\rho(y)$, $p(x)$ and $r(x)$ into (3), we obtain an infinite system of conditions on the coefficients p_k,r_k,π_k,ρ_k:

$$r_{i+1}\pi_0-p_{i+1}\rho_0=r_i,\quad \rho_{i+1}p_0-\pi_{i+1}r_0=\rho_i,\quad r_{i+1}p_{j+1}=\rho_{i+1}\pi_{j+1}\quad i,j=0,1,... \tag{4}$$

This system provides an example of what we call the bootstrap equations. Once solved, it permits to express the parameters p_i,r_i in terms of π_i,ρ_i. So it gives an answer to the question wether it is possible to carry out the analytic continuation from one layer to another. This is an infinite system of equations with respect to $2\times\infty$ (formal notation!) unknown parameters, needed to reexpress the function $F(x,y)$ in the layer B_x in terms of the parameters defining it in the layer B_y. In general, it is very difficult to find solutions of such systems and even to show their solvability. Fortunately, in this simple example it turns out possible to write down the solution in explicit form. Separating the variables in (3), taking derivatives and solving the corresponding ordinary differential equations, one finds:

$$F(x,y)=\frac{ad+bc}{-d+axy+bx+cy}. \tag{5}$$

The important property of this result is that it contains only 4 arbitrary parameters! This means that the infinite system (4) only happens consistent if the function $F(x,y)$ defined in the layer B_y belongs to the four-parametric family (5). This is the only case when there exists the analytic continuation of this function from B_y into B_x with the desired properties. It is clear that in this case the continuation is unique.

This exercise gives an idea of the "power" of bootstrap restrictions. The direct analysis of the system (4) would lead to the same conclusion. In this simple example it happens possible. Unfortunately, the regular method of solving infinite algebraic systems is not known, except few trivial cases.

With the help of (5), one can express the parameters $f_{ij} = f_{ij}(a_1, a_2, a_3, a_4)$ in terms of "fundamental constants" a_i $(i = 1, \ldots, 4)$. Then one can choose four arbitrary coefficients f_k $(k = 1,2,3,4)$ (or four arbitrary combinations) that allow the inversion $a_i = a_i(f_1, ..., f_4)$, and impose arbitrary "renormalization prescriptions" for these four quantities. The values of all other parameters should respect the conditions (4).

So, *to fix the amplitude $F(x,y)$ uniquely it is sufficient to impose four independent renormalization prescriptions defining the "fundamental" constants a, b, c, d.*

Precisely the same mechanism provides the system of bootstrap constraints for the parameters of pion-nucleon resonances (see [10]). The most important feature of this system is the renormalization invariance: *bootstrap equations are nothing but the restrictions for renormalization prescriptions*. This very property allows us to compare the bootstrap equations directly with known experimental data.

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