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Generic results in classes of ultradifferentiable functions

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ABSTRACT

Let E be a Denjoy–Carleman class of ultradifferentiable functions of Beurling type on the real line that strictly contains another class F of Roumieu type. We show that the set S of functions in E that are nowhere in the class F is large in the topological sense (it is residual), in the measure theoretic sense (it is prevalent), and that $S \cup \{0\}$ contains an infinite dimensional linear subspace (it is lineable). Consequences for the Gevrey classes are given. Similar results are also obtained for classes of ultradifferentiable functions defined imposing conditions on the Fourier–Laplace transform of the function.

1. Introduction

Let E be a Denjoy–Carleman class of ultradifferentiable functions of Beurling type on the real line $\mathbb{R}$ that strictly contains another class F of Roumieu type. The aim of this paper is to investigate how large is the set of functions in the class E that are nowhere in the class F, i.e. such that the restriction of the function to any open subset of $\mathbb{R}$ does not belong to this class. In this way we complement work by Schmets and Valdivia [25], Bernal-González [5], Bastin, Nicolay and the author [3] and by Bastin, Conejero, Seoane-Sepúlveda and the author [4]. In order to be more precise, we need some definitions and notations.

Given an open subset Ω of $\mathbb{R}^n$, let $\mathcal{E}(\Omega)$ be the set of all complex-valued smooth functions on Ω. If K is a compact subset of $\mathbb{R}^n$, let $\mathcal{E}(K)$ denote the set of all complex-valued smooth functions on the interior of K such that $D^\alpha f$ can be continuously extended to K for all $\alpha \in \mathbb{N}_0^n$. Moreover, if $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, we use the notation $|\alpha| = \alpha_1 + \dots + \alpha_n$.

An arbitrary sequence of positive real numbers $M = (M_k)_{k\in\mathbb{N}_0}$ is called a *weight sequence*. For every weight sequence M, every compact subset K of $\mathbb{R}^n$ and every $h > 0$, we define the space $\mathcal{E}_{M,h}(K)$ as the space of functions $f \in \mathcal{E}(K)$ such that

$$\|f\|_{K,h} := \sup_{\alpha\in\mathbb{N}_0^n} \sup_{x\in K} \frac{|D^\alpha f(x)|}{h^{|\alpha|} M_{|\alpha|}} < +\infty.$$

Endowed with the norm $\|\cdot\|_{K,h}$, the space $\mathcal{E}_{M,h}(K)$ is a Banach space.

Definition 1.1. If Ω is an open subset of $\mathbb{R}^n$, the space $\mathcal{E}_{\{M\}}(\Omega)$ is defined by

$$\mathcal{E}_{\{M\}}(\Omega) := \left\{ f \in \mathcal{E}(\Omega)\colon \forall K \subset \Omega \text{ compact } \exists h > 0 \text{ such that } \|f\|_{K,h} < +\infty \right\}.$$

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If $f \in \mathcal{E}_{\{M\}}(\Omega)$, we say that f is *M-ultradifferentiable of Roumieu type* on Ω. We obtain a locally convex topology on these spaces via the representation

$$\mathcal{E}_{\{M\}}(\Omega) = \underset{\overleftarrow{K \subset \Omega}}{\text{proj}}\ \underset{\overrightarrow{h>0}}{\text{ind}}\ \mathcal{E}_{M,h}(K).$$

Fundamental examples of Roumieu spaces are given by the weight sequences $(k!)_{k \in \mathbb{N}_0}$ and $((k!)^\alpha)_{k \in \mathbb{N}_0}$ with $\alpha > 1$. They correspond respectively to the space of real analytic functions on Ω and the space of Gevrey differentiable functions of order α on Ω.

On weight sequences, the following conditions are usually considered:

- A weight sequence M is *logarithmically convex* (or shortly *log-convex*) if $M_k^2 \leqslant M_{k-1}M_{k+1}$ for every $k \in \mathbb{N}$. The Gorny theorem [14] states that for every weight sequence M, there is a log-convex weight sequence L such that $\mathcal{E}_{\{M\}}(\Omega) = \mathcal{E}_{\{L\}}(\Omega)$. If the sequence M is log-convex, then the sequence $(\frac{M_k}{M_{k-1}})_{k \in \mathbb{N}}$ is increasing and one has $M_k M_l \leqslant M_{k+l}$ for every $k, l \in \mathbb{N}_0$. This implies that the space $\mathcal{E}_{\{M\}}(\Omega)$ is an algebra.
- Since we have $\mathcal{E}_{\{M\}}(\Omega) = \mathcal{E}_{\{\frac{M}{M_0}\}}(\Omega)$, we can assume that any weight sequence M is such that $M_0 = 1$.
- We say that the sequence M is *quasianalytic* if it satisfies one of the two following equivalent conditions

$$1.\quad \sum_{n=1}^{+\infty} \frac{M_{n-1}}{M_n} = +\infty, \qquad 2.\quad \sum_{n=1}^{+\infty} (M_n)^{-1/n} = +\infty.$$

  If this is not the case, we say that the sequence is *non-quasianalytic*. The Denjoy–Carleman theorem states that if M is log-convex, then the class $\mathcal{E}_{\{M\}}(\Omega)$ is quasianalytic (i.e. 0 is the unique function f in the space for which there is a point $x \in \Omega$ such that $D^\alpha f(x) = 0$ for every $\alpha \in \mathbb{N}_0^n$) if and only if the sequence M is quasianalytic (see for example [23, Theorem 19.11]). Note that the class $\mathcal{E}_{\{M\}}(\Omega)$ is quasianalytic if and only if there is no non-trivial function in $\mathcal{E}_{\{M\}}(\Omega)$ with compact support (a proof of this result can be found in [23, Theorem 19.10]). Then, if the class is non-quasianalytic, given an open subset Ω of $\mathbb{R}^n$ and a compact $K \subset \Omega$, there exists a function of $\mathcal{E}_{\{M\}}(\mathbb{R}^n)$ having a compact support included in Ω and being identically equal to 1 in K.

Agreement. In this paper, we will always assume that any weight sequence M is log-convex and $M_0 = 1$.

Let us now introduce the second type of Denjoy–Carleman classes.

Definition 1.2. If Ω is an open subset of $\mathbb{R}^n$, we define the space $\mathcal{E}_{(M)}(\Omega)$ by

$$\mathcal{E}_{(M)}(\Omega) := \{ f \in \mathcal{E}(\Omega) \colon \forall K \subset \Omega \text{ compact},\ \forall h > 0,\ \|f\|_{K,h} < +\infty \}.$$

If $f \in \mathcal{E}_{(M)}(\Omega)$, we say that f is *M-ultradifferentiable of Beurling type* on Ω and we use the representation

$$\mathcal{E}_{(M)}(\Omega) = \underset{\overleftarrow{K \subset \Omega}}{\text{proj}}\ \underset{\overleftarrow{h>0}}{\text{proj}}\ \mathcal{E}_{M,h}(K)$$

to endow $\mathcal{E}_{(M)}(\Omega)$ with a structure of Fréchet space.

Of course, we always have $\mathcal{E}_{(M)}(\Omega) \subset \mathcal{E}_{\{M\}}(\Omega)$. Moreover, conditions on two weight sequences M and N to have the inclusion $\mathcal{E}_{\{M\}}(\Omega) \subset \mathcal{E}_{(N)}(\Omega)$ are known and presented in the second section of this paper. Let us consider the following definition.

Definition 1.3. We say that a function is *nowhere in* $\mathcal{E}_{\{M\}}$ if its restriction to any open and non-empty subset Ω of $\mathbb{R}$ never belongs to $\mathcal{E}_{\{M\}}(\Omega)$.

We want to handle the question of how large the subset of $\mathcal{E}_{(N)}(\mathbb{R})$ formed by the functions which are nowhere in $\mathcal{E}_{\{M\}}$ is. We will use three different notions of genericity. Let us recall their definitions here.

First, let us recall this classical definition of residuality from a topological point of view.

Definition 1.4. If X is a Baire space, then a subset $A \subset X$ is called *residual* (or *comeager*) if A contains a countable union of dense open sets of X, or equivalently if $X \setminus A$ is included in a countable union of closed sets of X with empty interior.

From a measure-theoretical point of view, we will use the notion of prevalence. It was introduced by Christensen and rediscovered later by Hunt, Sauer and Yorke in order to generalize the notion of "Lebesgue almost everywhere" to infinite dimensional spaces. More precisely, we use the following definition.

Definition 1.5. (See [13,17].) A Borel set B in a complete metric linear space E is said to be *shy* if there exists a Borel probability measure μ on E with compact support such that $\mu(B+x)=0$ for any $x\in E$. A set is said to be *prevalent* if it is the complement of a shy set.

Finally, for the last decade there has been an increasing interest toward the search for large algebraic structures of special objects (see [7] for a review). In this paper, we use the following definition introduced by Aron, Gurariy and Seoane-Sepúlveda.

Definition 1.6. (See [1].) Let X be a topological vector space, M a subset of X, and μ a cardinal number. We say that M is μ-lineable if $M\cup\{0\}$ contains a vector space of dimension μ. At times, we shall simply be referring to the set M as lineable if the existing subspace is infinite dimensional. When the linear space can be chosen to be dense in X, we say that M is μ-dense-lineable.

In the first part of this paper, given two weight sequences N and M such that $\mathcal{E}_{\{M\}}(\mathbb{R})$ is strictly included in $\mathcal{E}_{(N)}(\mathbb{R})$ and such that M is non-quasianalytic, we construct a function of $\mathcal{E}_{(N)}(\mathbb{R})$ which is nowhere in $\mathcal{E}_{\{M\}}$. We obtain then generic results about the set of functions of $\mathcal{E}_{(N)}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{M\}}$. We extend this result using any countable union of Roumieu classes included in $\mathcal{E}_{(N)}(\mathbb{R})$. An application to the classes of Gevrey functions is given. In the second part, the same question is handled but working with ultradifferentiable functions defined imposing conditions on the Fourier–Laplace transform of the function. Our main result is Theorem 2.10.

2. Generic results in Denjoy–Carleman classes

Let us start by defining some relations on weight sequences. If M and N are two weight sequences, we use the following notations from [21]:

$$\begin{cases} M\preccurlyeq N & \iff \quad \exists C,\rho>0 \text{ such that } M_k\leqslant C\rho^k N_k\ \forall k \quad \iff \quad \sup_{k\in\mathbb{N}_0}\left(\frac{M_k}{N_k}\right)^{\frac{1}{k}}<+\infty,\\ M\approx N & \iff \quad M\preccurlyeq N \text{ and } N\preccurlyeq M,\\ M\rhd N & \iff \quad \forall\rho>0\ \exists C>0 \text{ such that } M_k\leqslant C\rho^k N_k\ \forall k \quad \iff \quad \lim_{k\to+\infty}\left(\frac{M_k}{N_k}\right)^{\frac{1}{k}}=0. \end{cases}$$

Of course, for any open subset Ω of $\mathbb{R}$, if $M\preccurlyeq N$, then $\mathcal{E}_{\{M\}}(\Omega)\subset\mathcal{E}_{\{N\}}(\Omega)$ and $\mathcal{E}_{(M)}(\Omega)\subset\mathcal{E}_{(N)}(\Omega)$. Moreover, if $M\rhd N$, then $\mathcal{E}_{\{M\}}(\Omega)\subset\mathcal{E}_{(N)}(\Omega)$. All the converse implications are true as proved in [21], using the assumption that the weight sequence M is log-convex. Let us recall the two following lemmas of Rainer and Schindl which directly imply that in the case $M\rhd N$, the inclusion is even strict.

Lemma 2.1. *(See [21].) Let M and N be two weight sequences satisfying $M\rhd N$ and $(k!M_k)^{\frac{1}{k}}\to+\infty$ as $k\to+\infty$. There exists a sequence L satisfying $(k!L_k)^{\frac{1}{k}}\to+\infty$ as $k\to+\infty$ such that*

$$M\rhd L\rhd N.$$

Remark 2.2.

1. The assumption $(k!M_k)^{\frac{1}{k}}\to+\infty$ as $k\to+\infty$ is automatically satisfied since we have assumed that the weight sequence M is log-convex. Indeed, if M is log-convex, the sequence $(M_k^{\frac{1}{k}})_k$ is increasing as proved in [24].
2. We can assume that the sequence L is log-convex. Indeed, given a weight sequence M, we set first

$$M_j^i:=\inf_{k\geqslant j}M_k^{\frac{1}{k}}$$

and we introduce the sequence M^c by

$$\begin{cases} M_0^c:=M_0=1,\\ M_j^c:=\inf\left\{M_k^{\frac{l-j}{l-k}}M_l^{\frac{j-k}{l-k}}:\ k\leqslant j\leqslant l,\ k\neq l\right\}. \end{cases}$$

Then, from [24], M^c is the largest log-convex minorant (for $\leqslant$) of the sequence M. Moreover, a simple computation shows that if M and N are two positive sequences such that $M\rhd N$, then $M^c\rhd N^c$.

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Lemma 2.3. *(See [26].) Let M be a weight sequence and θ be the function defined on $\mathbb{R}$ by*

$$\theta(x) = \sum_{k=0}^{+\infty} \frac{M_k}{2^k} \left(\frac{M_{k-1}}{M_k}\right)^k \exp\left(2i \frac{M_k}{M_{k-1}} x\right).$$

Then $\theta \in \mathcal{E}_{\{M\}}(\mathbb{R})$ and $|D^j \theta(0)| \geqslant M_j$ for all $j \in \mathbb{N}_0$. In particular, this function belongs to $\mathcal{E}_{\{M\}}(\mathbb{R}) \setminus \mathcal{E}_{(M)}(\mathbb{R})$.

The next result follows directly.

Proposition 2.4. *Let M, N be two weight sequences and let Ω be an open subset of $\mathbb{R}$. Then*

$$M \rhd N \quad \Longleftrightarrow \quad \mathcal{E}_{\{M\}}(\Omega) \subset \mathcal{E}_{(N)}(\Omega)$$

and in this case, the inclusion is strict.

Let us consider two weight sequences M and N such that $M \rhd N$. In order to study the set of functions of $\mathcal{E}_{(N)}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{M\}}$, let us first start by an explicit construction of such a function.

Proposition 2.5. *Assume that M and N are two weight sequences such that $M \rhd N$. If M is non-quasianalytic, there exists a function of $\mathcal{E}_{(N)}(\mathbb{R})$ which is nowhere in $\mathcal{E}_{\{M\}}$.*

Proof. From Lemma 2.1, there is a log-convex weight sequence $N^\star$ such that $M \rhd N^\star \rhd N$. Applying recursively this lemma, we get a sequence $(L^{(p)})_{p\in\mathbb{N}}$ of log-convex weight sequences such that

$$M \rhd L^{(1)} \rhd L^{(2)} \rhd \cdots \rhd L^{(p)} \rhd \cdots \rhd N^\star \rhd N.$$

For every $p \in \mathbb{N}$, Lemma 2.3 allows us to consider a function f_p that belongs to the class $\mathcal{E}_{\{L^{(p)}\}}(\mathbb{R})$ and such that $|D^j f_p(0)| \geqslant L_j^{(p)}$ for every $j \in \mathbb{N}_0$. Since M is non-quasianalytic, there exists $\phi \in \mathcal{E}_{\{M\}}(\mathbb{R})$ with compact support and identically equal to 1 in a neighbourhood of the origin. If we consider a countable dense subset $\{x_p\colon\ p \in \mathbb{N}\}$ of $\mathbb{R}$, then for every $p \in \mathbb{N}$, we can find $k_p > 0$ such that the function

$$\phi_p(x) := \phi\big(k_p(x - x_p)\big)$$

has its support disjoint from $\{x_0, \ldots, x_{p-1}\}$. We introduce the function g_p defined on $\mathbb{R}$ by

$$g_p(x) := f_p(x - x_p)\phi_p(x).$$

Since $f_p \in \mathcal{E}_{\{L^{(p)}\}}(\mathbb{R}) \subset \mathcal{E}_{(N^\star)}(\mathbb{R})$ and $\phi_p \in \mathcal{E}_{\{M\}}(\mathbb{R}) \subset \mathcal{E}_{(N^\star)}(\mathbb{R})$, we obtain that g_p is a function with compact support that belongs to $\mathcal{E}_{(N^\star)}(\mathbb{R})$. Then, there exists $\gamma_p > 0$ such that

$$\sup_{x\in\mathbb{R}} \left|D^j g_p(x)\right| \leqslant \gamma_p N_j^\star \quad \forall j \in \mathbb{N}_0.$$

We define the function g by

$$g := \sum_{p=1}^{+\infty} \frac{1}{\gamma_p 2^p} g_p.$$

First, let us show that $g \in \mathcal{E}_{(N)}(\mathbb{R})$. For every $j \in \mathbb{N}_0$ and every $x \in \mathbb{R}$, we have

$$\sum_{p=1}^{+\infty} \frac{1}{\gamma_p 2^p} \left|D^j g_p(x)\right| \leqslant \sum_{p=1}^{+\infty} \frac{1}{2^p} N_j^\star \leqslant N_j^\star$$

so that g belongs to $\mathcal{E}_{\{N^\star\}}(\mathbb{R})$. Since $N^\star \rhd N$, we get that $g \in \mathcal{E}_{(N)}(\mathbb{R})$.

Let us now prove that the function g is nowhere in $\mathcal{E}_{\{M\}}$. We proceed by contradiction and we assume that there exists an open subset Ω of $\mathbb{R}$ such that $g \in \mathcal{E}_{\{M\}}(\Omega)$. Since the subset $\{x_p\colon\ p \in \mathbb{N}\}$ is dense in $\mathbb{R}$, there is $p_0 \in \mathbb{N}$ such that $x_{p_0} \in \Omega$. Remark that the function $\sum_{p=1}^{p_0-1} \frac{1}{\gamma_p 2^p} g_p$ belongs to $\mathcal{E}_{(L^{(p_0)})}(\mathbb{R})$ and that $\mathcal{E}_{\{M\}}(\Omega) \subset \mathcal{E}_{(L^{(p_0)})}(\Omega)$. Consequently, the function

$$\sum_{p=p_0}^{+\infty} \frac{1}{\gamma_p 2^p} g_p = g - \sum_{p=1}^{p_0-1} \frac{1}{\gamma_p 2^p} g_p$$

belongs to $\mathcal{E}_{(L^{(p_0)})}(\Omega)$. But, since the support of g_p is disjoint of x_{p_0} for every $p > p_0$, we also have

$$\begin{aligned}\left|\sum_{p=p_0}^{+\infty}\frac{1}{\gamma_p 2^p}D^j g_p(x_{p_0})\right| &= \frac{1}{\gamma_{p_0}2^{p_0}}\left|D^j g_{p_0}(x_{p_0})\right| \\ &= \frac{1}{\gamma_{p_0}2^{p_0}}\left|D^j f_{p_0}(0)\right| \\ &\geqslant \frac{1}{\gamma_{p_0}2^{p_0}}L_j^{p_0}\end{aligned}$$

for every $j \in \mathbb{N}$, hence a contradiction. □

In order to get generic results from the measure-theoretical sense, let us recall the following lemma that gives a sufficient condition for a Borel subset to be prevalent.

Lemma 2.6. *(See [3].) If A is a non-empty Borel subset of E such that the complement of A is a linear subspace of E, then A is prevalent.*

Proposition 2.7. *Assume that M and N are two weight sequences such that $M \rhd N$. If M is non-quasianalytic, the set of functions of $\mathcal{E}_{(N)}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{M\}}$ is prevalent in $\mathcal{E}_{(N)}(\mathbb{R})$.*

Proof. The set of functions of $\mathcal{E}_{(N)}(\mathbb{R})$ which are somewhere in $\mathcal{E}_{\{M\}}$ is given by

$$\bigcup_{I\subset\mathbb{R}}\bigcup_{m\in\mathbb{N}}E(I,m),$$

where I denotes rational subintervals of $\mathbb{R}$ and

$$E(I,m) := \left\{f \in \mathcal{E}_{(N)}(\mathbb{R})\colon \exists C>0 \text{ such that } \sup_{x\in I}\left|D^j f(x)\right| \leqslant Cm^j M_j\ \forall j \in \mathbb{N}_0\right\}.$$

Since any countable union of shy sets is shy [17], we just have to prove that $E(I,m)$ is shy for every I and every m. It is clear that $E(I,m)$ is a linear subspace of $\mathcal{E}_{(N)}(\mathbb{R})$ which is proper using Proposition 2.5. Moreover, it is a Borel subset of $\mathcal{E}_{(N)}(\mathbb{R})$. Indeed, we have

$$E(I,m) = \bigcup_{s\in\mathbb{N}}\left\{f \in \mathcal{E}_{(N)}(\mathbb{R})\colon \sup_{x\in I}\left|D^j f(x)\right| \leqslant sm^j M_j\ \forall j \in \mathbb{N}_0\right\}$$

which is a countable union of closed sets in $\mathcal{E}_{(N)}(\mathbb{R})$. Lemma 2.6 gives the conclusion. □

Proposition 2.8. *Assume that M and N are two weight sequences such that $M \rhd N$. If M is non-quasianalytic, the set of functions of $\mathcal{E}_{(N)}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{M\}}$ is residual in $\mathcal{E}_{(N)}(\mathbb{R})$.*

Proof. As in the previous proof, the set of functions of $\mathcal{E}_{(N)}(\mathbb{R})$ which are somewhere in $\mathcal{E}_{\{M\}}$ is

$$\bigcup_{I\subset\mathbb{R}}\bigcup_{m\in\mathbb{N}}\bigcup_{s\in\mathbb{N}}\left\{f \in \mathcal{E}_{(N)}(\mathbb{R})\colon \sup_{x\in I}\left|D^j f(x)\right| \leqslant sm^j M_j\ \forall j \in \mathbb{N}_0\right\}.$$

Each closed set $\{f \in \mathcal{E}_{(N)}(\mathbb{R})\colon \sup_{x\in I}|D^j f(x)| \leqslant sm^j M_j\ \forall j \in \mathbb{N}_0\}$ has empty interior since it is included in $E(I,m)$ which is a proper linear subspace of the locally convex space $\mathcal{E}_{(N)}(\mathbb{R})$. The conclusion follows. □

The next construction used to prove the lineability follows an idea of Schmets and Valdivia [25]. Fix two weight sequences M and N such that M is non-quasianalytic and $M \rhd N$. For every $t \in]0,1[$, we define a weight sequence $L^{(t)}$ by

$$L_k^{(t)} := (M_k)^{1-t}(N_k)^t \quad \forall k \in \mathbb{N}_0.$$

Since N, M are log-convex, it is straightforward to see that $L^{(t)}$ is also log-convex. Moreover, the assumption $M \rhd N$ leads directly to the relations $M \rhd L^{(t)} \rhd N$ for all $t \in]0,1[$ and $L^{(t)} \rhd L^{(s)}$ if $t < s$. For every $p \in \mathbb{N} \setminus \{1\}$ and for every $t \in]0,1[$, using Lemma 2.3, we consider a function $f_{p,t} \in \mathcal{E}_{\{L^{((1-\frac{1}{p})t)}\}}(\mathbb{R})$ such that $|D^j f_{p,t}(0)| \geqslant L_j^{((1-\frac{1}{p})t)}$ for every $j \in \mathbb{N}_0$.

Since M is non-quasianalytic, we can choose a function $\phi \in \mathcal{E}_{\{M\}}(\mathbb{R})$ with compact support and identically equal to 1 in a neighbourhood of 0. Let us consider a countable dense subset $\{x_p\colon p \in \mathbb{N}\setminus\{1\}\}$ of $\mathbb{R}$. For every $p \geqslant 2$, we fix $k_p > 0$ such that the function

$$\phi_p(x) := \phi\big(k_p(x - x_p)\big)$$

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has its support disjoint from $\{x_2, \ldots, x_{p-1}\}$ and we introduce for every $t \in]0,1[$ the function $g_{p,t}$ defined by

$$g_{p,t} := f_{p,t}(\cdot - x_p)\phi_p.$$

We know that $f_{p,t} \in \mathcal{E}_{\{L^{((1-\frac{1}{p})t)}\}}(\mathbb{R}) \subset \mathcal{E}_{(L^{(t)})}(\mathbb{R})$ and $\phi \in \mathcal{E}_{\{M\}}(\mathbb{R}) \subset \mathcal{E}_{(L^{(t)})}(\mathbb{R})$. Consequently there exists $\gamma_{p,t} > 0$ such that

$$\sup_{x\in\mathbb{R}} \left| D^j g_{p,t}(x) \right| \leqslant \gamma_{p,t} L_j^{(t)} \quad \forall j \in \mathbb{N}_0$$

and we define for every $t \in]0,1[$ the function g_t by

$$g_t := \sum_{p=2}^{+\infty} \frac{1}{\gamma_{p,t} 2^p} g_{p,t}.$$

Remark that we are in the same situation as in the proof of Proposition 2.5 since

$$M \rhd L^{(\frac{t}{2})} \rhd L^{(\frac{2t}{3})} \rhd L^{(\frac{3t}{4})} \rhd \cdots \rhd L^{(t)} \rhd N, \quad \forall t \in]0,1[.$$

Therefore, as done previously, the function g_t belongs then to $\mathcal{E}_{\{L^{(t)}\}}(\mathbb{R})$ and is not in $\mathcal{E}_{(L^{((1-\frac{1}{p_0})t)})}(\Omega)$, for any open neighbourhood Ω of x_{p_0} and for any $p_0 \geqslant 2$. This leads to the following lemma.

Lemma 2.9. *If $\mathcal{D}$ denotes the subspace of $\mathcal{E}_{(N)}(\mathbb{R})$ spanned by the functions g_t, $t \in]0,1[$, then $\dim \mathcal{D} = \mathfrak{c}$ and every non-zero function of $\mathcal{D}$ is nowhere in $\mathcal{E}_{\{M\}}$.*

Proof. First, assume there exist $\alpha_1, \ldots, \alpha_N \in \mathbb{C}$ with $\alpha_N \neq 0$ and $t_1 < \cdots < t_N$ in $]0,1[$ such that $\sum_{n=1}^{N} \alpha_n g_{t_n} = 0$. Then

$$g_{t_N} = \frac{-1}{\alpha_N} \sum_{n=1}^{N-1} \alpha_n g_{t_n}$$

and since $g_{t_n} \in \mathcal{E}_{\{L^{(t_n)}\}}(\mathbb{R}) \subset \mathcal{E}_{\{L^{(t_{N-1})}\}}(\mathbb{R})$ for every $n \leqslant N-1$, we get that

$$g_{t_N} \in \mathcal{E}_{\{L^{(t_{N-1})}\}}(\mathbb{R}) \subset \mathcal{E}_{(L^{((1-\frac{1}{p_0})t_N)})}(\mathbb{R})$$

if p_0 is such that $(1 - \frac{1}{p_0})t_N > t_{N-1}$. This is a contradiction and it follows that the functions f_t, $t \in]0,1[$, are linearly independent.

It remains to show that every non-zero linear combination of the functions g_t, $t \in]0,1[$, is nowhere in $\mathcal{E}_{\{M\}}$. Let us fix $\alpha_1, \ldots, \alpha_N \in \mathbb{C}$ with $\alpha_N \neq 0$ and $t_1 < \cdots < t_N$ in $]0,1[$, and let us consider the function

$$G = \sum_{n=1}^{N} \alpha_n g_{t_n}.$$

Assume that there exists an open subset Ω of $\mathbb{R}$ such that $G \in \mathcal{E}_{\{M\}}(\Omega)$. We fix $p_0 \in \mathbb{N}$ such that $x_{p_0} \in \Omega$ and $t_{N-1} < (1 - \frac{1}{p_0})t_N$. Again, the function g_{t_n} belongs to $\mathcal{E}_{\{L^{(t_{N-1})}\}}(\mathbb{R})$ for every $n \leqslant N-1$ and it follows that the function

$$g_{t_N} = \frac{1}{\alpha_N} \left(G - \sum_{n=1}^{N-1} \alpha_n g_{t_n} \right)$$

belongs to $\mathcal{E}_{\{L^{(t_{N-1})}\}}(\Omega)$. From the choice of p_0, we have $\mathcal{E}_{\{L^{(t_{N-1})}\}}(\Omega) \subset \mathcal{E}_{(L^{((1-\frac{1}{p_0})t_N)})}(\Omega)$ and this leads to a contradiction with the construction of g_{t_N}. $\square$

In order to obtain the dense-lineability in $\mathcal{E}_{(N)}(\mathbb{R})$ of the set of functions which are nowhere in $\mathcal{E}_{\{M\}}$, we will slightly modify the uncountable subspace $\mathcal{D}$. Let $(t_m)_{m\in\mathbb{N}}$ be a sequence of different elements of $]0,1[$. Since $\mathcal{E}_{(N)}(\mathbb{R})$ is a Fréchet space, there exists a countable basis $\{U_m\colon m \in \mathbb{N}\}$ of convex balanced absorbing neighbourhoods of 0 in $\mathcal{E}_{(N)}(\mathbb{R})$. Using the continuity of the multiplication by scalars, we choose for every $m \in \mathbb{N}$ a positive constant k_m such that $k_m g_{t_m} \in U_m$. Moreover, from [19] (Theorem 7.3), we know that the set of polynomials is dense in $\mathcal{E}_{(N)}(\mathbb{R})$. Let $(P_{t_m})_{m\in\mathbb{N}}$ be a dense sequence of polynomials in $\mathcal{E}_{(N)}(\mathbb{R})$. We consider the linear space $\mathcal{D}_d$ spanned by

$$\left\{ P_t + k_t g_t\colon t \in]0,1[\right\}$$

where $k_t = 1$ and $P_t = 0$ if $t \neq t_m$ for every $m \in \mathbb{N}$.

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Theorem 2.10. *Assume that N and M are two weight sequences such that M is non-quasianalytic and $M \rhd N$. Then $\mathcal{D}_d$ is dense in $\mathcal{E}_{(N)}(\mathbb{R})$, $\dim \mathcal{D}_d = \mathfrak{c}$ and any non-zero function of $\mathcal{D}_d$ is nowhere in $\mathcal{E}_{\{M\}}$. In particular, the set of functions of $\mathcal{E}_{(N)}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{M\}}$ is $\mathfrak{c}$-dense-lineable in $\mathcal{E}_{(N)}(\mathbb{R})$.*

Proof. By construction, the family $\{P_m + k_{t_m} g_{t_m}\colon m \in \mathbb{N}\}$ is dense in $\mathcal{E}_{(N)}(\mathbb{R})$ and therefore, the same holds for $\mathcal{D}_d$. Moreover, as shown in Lemma 2.9, the generating functions g_t, $t \in]0,1[\setminus \{t_m\colon m \in \mathbb{N}\}$ are linearly independent and we get that $\dim \mathcal{D}_d = \mathfrak{c}$. Finally, any non-zero function $f \in \mathcal{D}_d$ can be written as the sum of a polynomial P and a linear combination g of the functions g_t, $t \in]0,1[$. Since any polynomial belongs to the linear space $\mathcal{E}_{\{M\}}(\mathbb{R})$, the function f is nowhere in $\mathcal{E}_{\{M\}}$. Indeed, otherwise the function $g = f - P$ would also belong to $\mathcal{E}_{\{M\}}(\Omega)$ for some open subset Ω of $\mathbb{R}$, which is impossible by Lemma 2.9. This concludes the proof. □

Remark 2.11. This last result follows the proof of Theorem 2.2 and Remark 2.5 of [2]. Nevertheless we have rewritten it to show that the dense subspace can still be chosen with a maximal dimension. Alternatively, Lemma 2.1 of [6] can also be used.

Lemma 2.12. *Let N be a weight sequence and let $(M^{(n)})_{n\in\mathbb{N}}$ be a sequence of weight sequences such that $M^{(n)} \rhd N$ for every $n \in \mathbb{N}$. Then, there exists a weight sequence P such that*

$$M^{(n)} \preccurlyeq P \quad \forall n \in \mathbb{N} \text{ and } P \rhd N.$$

Proof. By assumption, we know that $M^{(n)} \rhd N$ for every $n \in \mathbb{N}$ and then there exists a sequence $(C_n)_{n\in\mathbb{N}}$ of positive numbers such that

$$M_k^{(n)} \leqslant C_n n^{-k} N_k \quad \forall k \in \mathbb{N}_0,\ n \in \mathbb{N}.$$

Then, for every $k \in \mathbb{N}_0$, $\sup\{\frac{M_k^{(n)}}{C_n}\colon n \in \mathbb{N}\} < +\infty$ and we define a weight sequence P by setting

$$P_k := \sup\left\{\frac{M_k^{(n)}}{C_n}\colon n \in \mathbb{N}\right\}, \quad k \in \mathbb{N}_0.$$

It is clear that $M^{(n)} \preccurlyeq P$ for every $n \in \mathbb{N}$. Moreover, let us fix $\rho > 0$. Then, there exists $N \in \mathbb{N}$ such that $\rho \geqslant \frac{1}{n}$ for every $n \geqslant N$. We get that

$$M_k^{(n)} \leqslant C_n n^{-k} N_k \leqslant C_n \rho^k N_k \quad \forall k \in \mathbb{N}_0$$

if $n \geqslant N$. Moreover, if $n < N$, the assumption $M^n \rhd N$ gives a constant $D > 0$ such that

$$M_k^{(n)} \leqslant D \rho^k N_k \quad \forall k \in \mathbb{N}_0,\ \forall n < N.$$

It follows that the constant $C := \max\{1, \max\{\frac{D}{C_n}\colon n < N\}\} > 0$ is such that

$$P_k \leqslant C \rho^k N_k \quad \forall k \in \mathbb{N}_0.$$

Moreover, it is straightforward to see that the sequence P is log-convex. This leads to the conclusion. □

Proposition 2.13. *Let N be a log-convex weight sequence and let $(M^{(n)})_{n\in\mathbb{N}}$ be a sequence of log-convex weight sequences such that $M^{(n)} \rhd N$ for every $n \in \mathbb{N}$. If there is $n_0 \in \mathbb{N}$ such that the weight sequence $M^{(n_0)}$ is non-quasianalytic, then the set of functions of $\mathcal{E}_{(N)}(\mathbb{R})$ which are nowhere in $\bigcup_{n\in\mathbb{N}} \mathcal{E}_{\{M^{(n)}\}}$ is prevalent, residual and $\mathfrak{c}$-dense-lineable in $\mathcal{E}_{(N)}(\mathbb{R})$.*

Proof. From Lemma 2.12, there is a log-convex weight sequence P such that

$$\bigcup_n \mathcal{E}_{\{M^{(n)}\}}(\Omega) \subset \mathcal{E}_{\{P\}}(\Omega) \subsetneq \mathcal{E}_{(N)}(\Omega)$$

for every open subset Ω of $\mathbb{R}$. Moreover, since the weight sequence $M^{(n_0)}$ is non-quasianalytic and $M^{(n_0)} \preccurlyeq P$, the weight sequence P is also non-quasianalytic. The result follows then directly from Propositions 2.7, 2.8 and Theorem 2.10. □

As mentioned before, an important example of ultradifferentiable functions of Roumieu type is given by the classes of Gevrey differentiable functions of order $\alpha > 1$. They correspond to the weight sequences

$$M_k := (k!)^\alpha, \quad k \in \mathbb{N}_0.$$

Remark that for every $\alpha > 1$, the class $\mathcal{E}_{\{(k!)^\alpha\}}(\mathbb{R})$ is non-quasianalytic. Moreover, for every α, β such that $1 < \beta < \alpha$, we have

$$\mathcal{E}_{\{(k!)^\beta\}}(\mathbb{R}) \subset \mathcal{E}_{((k!)^\alpha)}(\mathbb{R}).$$

In [25], the following result is proved.

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Proposition 2.14. *(See [25].) Let $\alpha > 1$. The set of functions of $\mathcal{E}_{((k!)^\alpha)}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{(k!)^\beta\}}$ for every $\beta \in]1,\alpha[$, is residual in $\mathcal{E}_{((k!)^\alpha)}(\mathbb{R})$.*

This result can be seen as a consequence of Proposition 2.13 applied to the weight sequences $M^{(n)}$, $n \in \mathbb{N}$, given by

$$M_k^{(n)} := (k!)^{\beta_n}, \quad k \in \mathbb{N}_0,\ n \in \mathbb{N},$$

where $(\beta_n)_{n\in\mathbb{N}}$ is an increasing sequence of $]1,\alpha[$ that converges to α.

Here is another direct consequence of our results which improves Proposition 2.14.

Proposition 2.15. *Let $\alpha > 1$. The set of functions of $\mathcal{E}_{((k!)^\alpha)}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{(k!)^\beta\}}$ for every $\beta \in]1,\alpha[$ is prevalent and $\mathfrak{c}$-dense-lineable in $\mathcal{E}_{((k!)^\alpha)}(\mathbb{R})$.*

3. Generic results in Braun, Meise and Taylor classes

In the present section, we handle the same kind of question as previously but in the context of non-quasianalytic classes of ultradifferentiable functions which have been introduced by Beurling [8], see Björck [9] for more details. They pointed out that decay properties of the Fourier–Laplace transform of a C^∞ compactly supported function and weight functions ω can also be used to measure the smoothness of the function. This method was modified by Braun, Meise and Taylor [12] who showed that these classes can also be defined by the decay properties of their derivatives through the Young conjugate of the function $t \mapsto \omega(e^t)$. It is in this context that we will work in this section. Let us first start by introducing the weight functions we will use, following Braun, Meise and Taylor.

Definition 3.1. (See [12].) A function $\omega : [0,+\infty[\to [0,+\infty[$ is called a *weight function* if it is continuous, increasing and satisfies $\omega(0) = 0$ as well as the following conditions

(α) there exists $L \geqslant 1$ such that $\omega(2t) \leqslant L\omega(t) + L$, $t \geqslant 0$,
(β) $\int_1^{+\infty} \frac{\omega(t)}{t^2}\,dt < +\infty$,
(γ) $\log(t) = o(\omega(t))$ as t tends to infinity,
(δ) $\varphi_\omega : t \mapsto \omega(e^t)$ is convex on $[0,+\infty[$.

The Young conjugate of φ_ω is defined by

$$\varphi_\omega^*(x) := \sup\{xy - \varphi_\omega(y):\ y > 0\}, \quad x \geqslant 0.$$

Remark 3.2. Condition (β) implies the following condition

$$(\beta'):\quad \omega(t) = O(t) \quad \text{as } t \text{ tends to infinity}.$$

If a weight function ω with (β') also satisfies

$$\int_1^\infty \frac{\omega(t)}{t^2}\,dt = \infty, \tag{Q}$$

it is called a *quasianalytic weight function*. Otherwise (i.e. if condition (β) holds), it is called non-quasianalytic. In this paper, we will only work with non-quasianalytic weights as in Definition 3.1.

With these notations, we can introduce function spaces of Beurling and Roumieu type associated with a weight function ω. For a compact subset K of $\mathbb{R}^n$ and every $m \in \mathbb{N}$, we define the space $\mathcal{E}_\omega^m(K)$ as the space of functions $f \in \mathcal{E}(K)$ such that

$$\|f\|_{K,m} := \sup_{\alpha\in\mathbb{N}_0^n} \sup_{x\in K} \left|D^\alpha f(x)\right| \exp\left(-\frac{1}{m}\varphi_\omega^*\left(m|\alpha|\right)\right) < +\infty.$$

Clearly, it is a Banach space.

Definition 3.3. If ω is a weight function and if Ω is an open subset of $\mathbb{R}^n$, we define the space $\mathcal{E}_{\{\omega\}}(\Omega)$ of *ω-ultradifferentiable functions of Roumieu type* on Ω by

$$\mathcal{E}_{\{\omega\}}(\Omega) := \left\{f \in \mathcal{E}(\Omega):\ \forall K \subset \Omega \text{ compact } \exists m \in \mathbb{N} \text{ such that } \|f\|_{K,m} < +\infty\right\}.$$

It is endowed with the topology given by the representation

$$\mathcal{E}_{\{\omega\}}(\Omega) = \operatorname*{proj}_{\overleftarrow{K\subset\Omega}} \operatorname*{ind}_{\overrightarrow{m\in\mathbb{N}}} \mathcal{E}_\omega^m(K),$$

where K runs over all compact subsets of Ω.

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Definition 3.4. If ω is a weight function and if Ω is an open subset of $\mathbb{R}^n$, the space $\mathcal{E}_{(\omega)}(\Omega)$ of *ω-ultradifferentiable functions of Beurling type* on Ω is defined by

$$\mathcal{E}_{(\omega)}(\Omega) := \left\{ f \in \mathcal{E}(\Omega)\colon \forall K \subset \Omega \text{ compact}, \ \forall m \in \mathbb{N}, \ p_{K,m}(f) < +\infty \right\},$$

where for every compact subset K of $\mathbb{R}^n$ and every $m \in \mathbb{N}$

$$p_{K,m}(f) := \sup_{\alpha \in \mathbb{N}_0^n} \sup_{x \in K} \left| D^\alpha f(x) \right| \exp\left(-m\varphi_\omega^*\left(\frac{|\alpha|}{m} \right) \right).$$

We endow the space $\mathcal{E}_{(\omega)}(\Omega)$ with its natural Fréchet space topology.

From the properties of a weight function, both spaces $\mathcal{E}_{\{\omega\}}(\Omega)$ and $\mathcal{E}_{(\omega)}(\Omega)$ are algebras [12]. Moreover, those spaces contain some non-trivial functions with compact support. Therefore given an open subset Ω of $\mathbb{R}^n$ and a compact $K \subset \Omega$, it is possible to find a function in $\mathcal{E}_{\{\omega\}}(\mathbb{R}^n)$ (resp. in $\mathcal{E}_{(\omega)}(\mathbb{R}^n)$) with compact support included in Ω and identically equal to 1 on K [12].

Remark 3.5. For the weight function $\omega(t) = t$ (resp. $\omega(t) = t^\alpha$, $0 < \alpha < 1$), the space $\mathcal{E}_\omega(\Omega)$ corresponds to the space of real analytic functions on Ω (resp. the space of Gevrey differentiable functions of order $\frac{1}{\alpha}$ on Ω). However, in general, the definitions of ultradifferentiable functions using weight sequences or weight functions lead to different classes [11].

As done for ultradifferentiable classes defined with weight sequences, let us consider the following definition.

Definition 3.6. Given a weight sequence ω, we say that a function is *nowhere in $\mathcal{E}_{\{\omega\}}$* if its restriction to any open and non-empty subset Ω of $\mathbb{R}^n$ never belongs to $\mathcal{E}_{\{\omega\}}(\Omega)$.

In [12], the authors have also shown that if σ and ω are two weight functions such that $\sigma = o(\omega)$, then for any open set Ω, $\mathcal{E}_{\{\omega\}}(\Omega) \subset \mathcal{E}_{(\sigma)}(\Omega)$ and the inclusion is continuous. In this section, we will first show that in this case, the inclusion is even strict. We will then obtain generic results about those functions which are in $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$ but nowhere in $\mathcal{E}_{\{\omega\}}$. When dealing with ultradifferentiable classes defined using weight functions, it is generally difficult to construct an explicit function with some expected properties. That is the reason why, given a weight sequence ω, we will use the characterization of the strong dual spaces of $\mathcal{E}_{\{\omega\}}(\Omega)$ and $\mathcal{E}_{(\omega)}(\Omega)$, respectively denoted $\mathcal{E}'_{\{\omega\}}(\Omega)$ and $\mathcal{E}'_{(\omega)}(\Omega)$.

For this, let us introduce weighted spaces of entire functions, where we denote the space of entire functions on $\mathbb{C}^n$ by $\mathcal{H}(\mathbb{C}^n)$. For each compact set K of $\mathbb{R}^n$, the *support functional* of K is defined as

$$h_K : \mathbb{R}^n \to \mathbb{R} : x \mapsto h_K(x) := \sup_{y \in K} \langle x, y \rangle.$$

Then, for $\lambda > 0$, let

$$A(K, \lambda) := \left\{ f \in \mathcal{H}(\mathbb{C}^n)\colon |f|^\omega_{K,\lambda} := \sup_{z \in \mathbb{C}^n} \left| f(z) \right| \exp\left(-h_K(\Im z) - \lambda\omega\left(|z|\right)\right) < +\infty \right\}$$

endowed with its natural topology. We define

$$\mathcal{A}_{(\omega)}(\Omega) := \underset{K \subset \Omega}{\underrightarrow{\operatorname{ind}}}\ \underset{n \in \mathbb{N}}{\underrightarrow{\operatorname{ind}}}\ A(K, n)$$

and

$$\mathcal{A}_{\{\omega\}}(\Omega) := \underset{K \subset \Omega}{\underrightarrow{\operatorname{ind}}}\ \underset{n \in \mathbb{N}}{\underleftarrow{\operatorname{proj}}}\ A\left(K, \frac{1}{n} \right).$$

It is easy to check that $A(K, \lambda)$ is a Banach space, $\mathcal{A}_{(\omega)}(\Omega)$ is an (LB)-space and $\mathcal{A}_{\{\omega\}}(\Omega)$ is an (LF)-space.

Let us recall the following result from Heinrich and Meise [15, Theorems 3.6 and 3.7], where the Roumieu case was already proved by Rösner [22, Theorem 2.19].

Proposition 3.7. *For each weight function ω and each convex open set Ω in $\mathbb{R}^n$, the Fourier–Laplace transform*

$$\mathcal{F} : \mathcal{E}'_{\{\omega\}}(\Omega) \to \mathcal{A}_{\{\omega\}}(\Omega), \quad \mathcal{F}(u) : z \mapsto \underset{(x)}{u}\left(\exp\left(-i\langle x, z \rangle \right) \right)$$

is a linear topological isomorphism. The same holds for the Beurling type provided that $\omega(t) = o(t)$ as t tends to infinity.

Remark 3.8. If ω and σ are two weight functions such that $\sigma(t) = o(\omega(t))$ as t tends to infinity, then the condition $\sigma(t) = o(t)$ as t tends to infinity is automatically satisfied.

In what follows, we will also use the following results of [12] (Lemma 1.7).

Lemma 3.9. *Let ω be a weight function and assume that $g : [0,+\infty[\to [0,+\infty[$ satisfies $g(t) = o(\omega(t))$ as t tends to infinity. Then, there exists a weight function τ such that*

$$g(t) = o\big(\tau(t)\big) \quad \text{and} \quad \tau(t) = o\big(\omega(t)\big)$$

as t tends to infinity.

Let us finally recall the following proposition that follows from [16] (Theorem 4.4.2). See [10] (Proposition 12).

Proposition 3.10. *For each $n \in \mathbb{N}$, there exist $C_1, C_2 > 0$ such that for each plurisubharmonic function $u : \mathbb{C}^n \to \mathbb{R}$ and each $a \in \mathbb{C}^n$, there exists $f \in \mathcal{H}(\mathbb{C}^n)$ that satisfies*

$$f(a) = \exp\Big(\inf_{|v-a|\leqslant 1} u(v) - n\log\big(1+|a|^2\big)\Big)$$

and

$$\big|f(z)\big| \leqslant C_1 \exp\Big(\sup_{|v-z|\leqslant 1} u(v) + C_2\log\big(1+|z|^2\big)\Big), \quad \forall z \in \mathbb{C}^n.$$

The proof of our next result is inspired by the proofs of Propositions 13 and 18 in Bonet and Meise [10].

Proposition 3.11. *Let ω and σ be two weight functions such that $\sigma(t) = o(\omega(t))$ as t tends to infinity. If Ω is a convex open subset of $\mathbb{R}^n$, then $\mathcal{E}_{\{\omega\}}(\Omega)$ is strictly included in $\mathcal{E}_{(\sigma)}(\Omega)$.*

Proof. We can assume (up to a translation) that $0 \in \Omega$. Suppose that $\mathcal{E}_{\{\omega\}}(\Omega) = \mathcal{E}_{(\sigma)}(\Omega)$. Then, the continuity of the inclusion $\mathcal{E}_{\{\omega\}}(\Omega) \subset \mathcal{E}_{(\sigma)}(\Omega)$ and the closed graph theorem imply that $\mathcal{E}_{\{\omega\}}(\Omega) = \mathcal{E}_{(\sigma)}(\Omega)$ as locally convex spaces. Consequently they have the same dual spaces, i.e. by Proposition 3.7, the spaces $\mathcal{A}_{\{\omega\}}(\Omega)$ and $\mathcal{A}_{(\sigma)}(\Omega)$ coincide as locally convex spaces. In particular, the inclusion

$$\mathcal{A}_{\{\omega\}}(\Omega) \to \mathcal{A}_{(\sigma)}(\Omega)$$

is continuous. It follows that for every compact $K \subset \Omega$, the inclusion

$$\operatorname*{proj}_{\overleftarrow{m\in\mathbb{N}}} \mathcal{A}^{\omega}\left(K, \frac{1}{m}\right) \to \mathcal{A}_{(\sigma)}(\Omega)$$

is also continuous. Let us fix a compact subset K of Ω such that $0 \in K$. Now, we apply the localization theorem of De Wilde (see e.g. [18, Corollary 5.6.4]) to get a compact K' of Ω and a natural number m'_0 such that $\operatorname{proj}_{\overleftarrow{m\in\mathbb{N}}} \mathcal{A}^{\omega}(K, \frac{1}{m}) \subset \mathcal{A}^{\sigma}(K', m'_0)$ continuously. Therefore, there are $m_0 \in \mathbb{N}$ and $C > 0$ such that

$$|f|^{\sigma}_{K',m'_0} \leqslant C|f|^{\omega}_{K,\frac{1}{m_0}}, \quad \forall f \in \operatorname*{proj}_{\overleftarrow{m\in\mathbb{N}}} \mathcal{A}^{\omega}\left(K, \frac{1}{m}\right). \tag{1}$$

Since $\sigma(t) = o(\omega(t))$ as t tends to infinity, Lemma 3.9 gives a weight function τ such that $\sigma(t) = o(\tau(t))$ and $\tau(t) = o(\omega(t))$ as t tends to infinity. Next, we consider the radial extension $\tilde{\tau}$ of τ to $\mathbb{C}^n$ defined by

$$\tilde{\tau}(z) := \tau\big(|z|\big), \quad z \in \mathbb{C}^n.$$

This function is plurisubharmonic on $\mathbb{C}^n$ (see e.g. [20, Remark 1.6(b)]). For every $j \in \mathbb{N}$, we apply Proposition 3.10 with $a_j = (j, 0, \dots, 0)$ to get a function $f_j \in \mathcal{H}(\mathbb{C}^n)$ such that

$$f_j(a_j) = \exp\Big(\inf_{|v-a_j|\leqslant 1} \tilde{\tau}(v) - n\log\big(1+j^2\big)\Big) \tag{2}$$

and

$$\big|f_j(z)\big| \leqslant C_1 \exp\Big(\sup_{|v-z|\leqslant 1} \tilde{\tau}(v) + C_2\log\big(1+|z|^2\big)\Big), \quad \forall z \in \mathbb{C}^n. \tag{3}$$

Let us first show that for every $j \in \mathbb{N}$, the function f_j belongs to $\operatorname{proj}_{\overleftarrow{m\in\mathbb{N}}} \mathcal{A}^{\omega}(K, \frac{1}{m})$. We know from condition (α) that there is $L > 0$ such that

$$\tau\big(1+|z|\big) \leqslant \tau\big(2|z|\big) \leqslant L\tau\big(|z|\big) + L$$

for every $|z| > 1$ since τ is an increasing function. Moreover, using the continuity of τ, there is $D_1 > 0$ such that $\tau(1+|z|) \leqslant D_1$ if $|z| \leqslant 1$. So,

$$\tau\big(1+|z|\big) \leqslant L\tau\big(|z|\big) + L + D_1, \quad \forall z \in \mathbb{C}^n. \tag{4}$$

Consequently, using condition (γ), there exists $D_2 \geqslant 0$ such that

$$2C_2 \log\big(1+|z|\big) \leqslant L\tau\big(|z|\big) + L + D_2, \quad \forall z \in \mathbb{C}^n. \tag{5}$$

If we use (3), (4) and (5), we get

$$\begin{aligned}
\big|f_j(z)\big| &\leqslant C_1 \exp\Big(\sup_{|v-z|\leqslant 1} \tilde{\tau}(v) + C_2 \log\big(1+|z|^2\big)\Big) \\
&\leqslant C_1 \exp\big(\tau\big(1+|z|\big) + 2C_2 \log\big(1+|z|\big)\big) \\
&\leqslant C_1 \exp\big(L\tau\big(|z|\big) + L + D_1 + L\tau\big(|z|\big) + L + D_2\big) \\
&\leqslant D_3 \exp\big(2L\tau\big(|z|\big)\big)
\end{aligned}$$

for every $z \in \mathbb{C}^n$, where we have set $D_3 := C_1 \exp(2L + D_1 + D_2)$.

Moreover, since $0 \in K$, we have $h_K(x) \geqslant 0$ for every $x \in \mathbb{R}^n$. Therefore, for every $m \in \mathbb{N}$ fixed, we get

$$\begin{aligned}
|f_j|^{\omega}_{K,\frac{1}{m}} &= \sup_{z\in\mathbb{C}^n} \big|f_j(z)\big| \exp\left(-h_K(\Im z) - \frac{\omega(|z|)}{m}\right) \\
&\leqslant D_3 \sup_{z\in\mathbb{C}^n} \exp\left(2L\tau\big(|z|\big) - h_K(\Im z) - \frac{\omega(|z|)}{m}\right) \\
&\leqslant D_3 \sup_{z\in\mathbb{C}^n} \exp\left(2L\tau\big(|z|\big) - \frac{\omega(|z|)}{m}\right)
\end{aligned}$$

for every $j \in \mathbb{N}$. We know that $\tau(t) = o(\omega(t))$ as t tends to infinity and consequently, the function $x \in [0,+\infty[\mapsto 2L\tau(x) - \frac{\omega(x)}{m}$ is bounded from above. This implies that $f_j \in \operatorname{proj}_{\overleftarrow{m\in\mathbb{N}}} \mathcal{A}^{\omega}\big(K, \frac{1}{m}\big)$ for every $j \in \mathbb{N}$. In particular, we have also got the existence of a constant $D > 0$ such that

$$|f_j|^{\omega}_{K,\frac{1}{m_0}} \leqslant D \quad \forall j \in \mathbb{N}. \tag{6}$$

On the other hand, τ is increasing and consequently we have

$$\inf_{|v-a_j|\leqslant 1} \tilde{\tau}(v) \geqslant \tau(j-1)$$

for every $j \in \mathbb{N}$. Moreover, we have that $\Im a_j = 0$ for every j. Using (2), the condition (α) and the assumption that τ is increasing, we then get that for every $j \geqslant 2$,

$$\begin{aligned}
\|f_j\|^{\sigma}_{K',m_0'} &\geqslant \big|f_j(a_j)\big| \exp\big(-h_{K'}(\Im a_j) - m_0'\sigma(j)\big) \\
&\geqslant \exp\big(\tau(j-1) - n\log\big(1+j^2\big) - m_0'\sigma(j)\big) \\
&\geqslant \exp\left(\tau\left(\frac{j}{2}\right) - 2n\log(1+j) - m_0'\sigma(j)\right) \\
&\geqslant \exp\left(\frac{\tau(j)}{L} - 1 - 2n\log(1+j) - m_0'\sigma(j)\right) \\
&= \exp\left(\frac{\tau(j)}{L}\left(1 - \frac{L}{\tau(j)} - 2Ln\frac{\log(1+j)}{\tau(j)} - m_0'L\frac{\sigma(j)}{\tau(j)}\right)\right)
\end{aligned}$$

for every $j \geqslant 2$. Moreover, from the condition (γ) and the assumption $\sigma(t) = o(\tau(t))$, the term

$$\frac{L}{\tau(j)} + 2Ln\frac{\log(1+j)}{\tau(j)} + m_0'L\frac{\sigma(j)}{\tau(j)}$$

converges to 0 as j tends to infinity and therefore, there is $J \in \mathbb{N}$ such that

$$|f_j|^{\sigma}_{K',m'} \geqslant \exp\left(\frac{\tau(j)}{2L}\right)$$

for every $j \geqslant J$. Combining this with the relations (1) and (6), we finally get

$$\exp\left(\frac{\tau(j)}{2L}\right) \leqslant CD$$

for $j \geqslant J$. Taking $j \to +\infty$, we obtain a contradiction. □

Unlike the case of weight sequences, we have obtained the strict inclusion without exhibiting a particular function which is in $\mathcal{E}_{(\sigma)}(\Omega)$ but not in $\mathcal{E}_{\{\omega\}}(\Omega)$. The construction of a function of $\mathcal{E}_{(\sigma)}(\Omega)$ which is nowhere in $\mathcal{E}_{\{\omega\}}$ is therefore more complicated, but it will be obtained thanks to the following results.

Lemma 3.12. *Let ω and σ be two weight functions such that $\sigma(t) = o(\omega(t))$ as t tends to infinity. Fix $x \in \mathbb{R}^n$, $r, m \in \mathbb{N}$ and set $b_r := B(x, \frac{1}{r})$. Then the set*

$$E(x,r,m) = \left\{ f \in \mathcal{E}_{(\sigma)}\left(\mathbb{R}^n\right) \colon \sup_{\alpha \in \mathbb{N}_0^n} \sup_{x \in \overline{b_r}} \left| D^\alpha f(x) \right| \exp\left(-\frac{1}{m} \varphi_\omega^*\left(m|\alpha|\right) \right) < +\infty \right\}$$

is a proper linear subspace of $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$.

Proof. It is clear that the set $E(x,r,m)$ is a linear subspace of $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$. Moreover, Proposition 3.11 provides a function $f \in \mathcal{E}_{(\sigma)}(b_r) \setminus \mathcal{E}_{\{\omega\}}(b_r)$ so that there is a compact K included in b_r such that

$$\sup_{\alpha \in \mathbb{N}_0^n} \sup_{y \in K} \left| D^\alpha f(y) \right| \exp\left(-\frac{1}{m} \varphi_\omega^*\left(m|\alpha|\right) \right) = +\infty$$

for every $m \in \mathbb{N}$. Multiplying f by any function of $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$ with compact support and identically equal to 1 on K, we get a function of $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$ which does not belong to $E(x,r,m)$. This gives the conclusion. □

Proposition 3.13. *Let ω and σ be two weight functions such that $\sigma(t) = o(\omega(t))$ as t tends to infinity. The set of functions of $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$ which are nowhere in $\mathcal{E}_{\{\omega\}}$ is prevalent in $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$.*

Proof. Consider a countable dense subset $\{x_p \colon p \in \mathbb{N}\}$ in $\mathbb{R}^n$. The set of functions of $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$ which are somewhere in $\mathcal{E}_{\{\omega\}}$ is given by

$$\bigcup_{p \in \mathbb{N}} \bigcup_{r \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} E(x_p, r, m),$$

using the notation of Lemma 3.12. As done previously, since any countable union of shy sets is shy [17], it is enough to prove that $E(x_p, r, m)$ is shy for every $p, r, m \in \mathbb{N}$. Remark that $E(x_p, r, m)$ is a Borel subset of $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$. Indeed, we have

$$E(x_p, r, m) = \bigcup_{s \in \mathbb{N}} \left\{ f \in \mathcal{E}_{(\sigma)}\left(\mathbb{R}^n\right) \colon \sup_{\alpha \in \mathbb{N}_0^n} \sup_{x \in \overline{b_{p,r}}} \left| D^\alpha f(x) \right| \exp\left(-\frac{1}{m} \varphi_\omega^*\left(m|\alpha|\right) \right) \leqslant s \right\}$$

and an easy computation shows that every set of the countable union is closed in $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$. We get the conclusion using Lemmas 2.6 and 3.12. □

A prevalent subset is not empty (it is even dense in the considered space, see [17]) and therefore, we get the following corollary.

Corollary 3.14. *For every weight functions ω and σ such that $\sigma(t) = o(\omega(t))$ as t tends to infinity, there exists a function of $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$ which is nowhere in $\mathcal{E}_{\{\omega\}}$.*

Proposition 3.15. *Let ω and σ be two weight functions such that $\sigma(t) = o(\omega(t))$ as t tends to infinity. The set of functions of $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$ which are nowhere in $\mathcal{E}_{\{\omega\}}$ is residual in $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$.*

Proof. From the previous proof, we know that the set of functions of $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$ which are somewhere in $\mathcal{E}_{\{\omega\}}$ is a countable union of sets closed in $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$. Moreover, each closed set has empty interior since it is included in $E(x_p, r, m)$ which is a proper linear subspace of the locally convex space $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$. □

Proposition 3.16. *Let ω and σ be two weight functions such that $\sigma(t) = o(\omega(t))$ as t tends to infinity. The set of functions of $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$ which are nowhere in $\mathcal{E}_{\{\omega\}}$ is lineable.*

Proof. Since $\sigma(t) = o(\omega(t))$ as t tends to infinity, Lemma 3.9 gives a weight function $\omega^{(1)}$ such that $\sigma(t) = o(\omega^{(1)}(t))$ and $\omega^{(1)}(t) = o(\omega(t))$ as t tends to infinity. Repeating this procedure, we construct recursively a sequence $(\omega^{(p)})_{p \in \mathbb{N}}$ of weight functions such that

$$\sigma(t) = o\left(\omega^{(p)}(t)\right) \quad \text{and} \quad \omega^{(p)} = o\left(\omega^{(p-1)}(t)\right)$$

for every $p \geqslant 2$, when t tends to infinity. For every $p \in \mathbb{N}$, Corollary 3.14 gives a function $g_p \in \mathcal{E}_{(\omega^{(2p+1)})}(\mathbb{R}^n)$ which is nowhere in $\mathcal{E}_{\{\omega^{(2p)}\}}$. In particular, every g_p is in $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$. Moreover, the functions g_p, $p \in \mathbb{N}$, are linearly independent. Indeed, assume there exist $\alpha_1, \ldots, \alpha_N \in \mathbb{C}$ with $\alpha_N \neq 0$ and $p_1 < \cdots < p_N$ such that $\sum_{j=1}^N \alpha_j g_{p_j} = 0$. Then

$$g_{p_N} = \frac{-1}{\alpha_N} \sum_{j=1}^{N-1} \alpha_j g_{p_j}$$

so that $g_{p_N} \in \mathcal{E}_{\{\omega^{(2p_N)}\}}(\mathbb{R}^n)$ since $\mathcal{E}_{(\omega^{(2p_j+1)})}(\mathbb{R}^n) \subset \mathcal{E}_{\{\omega^{(2p_N)}\}}(\mathbb{R}^n)$ for every $j \leqslant N-1$, which is impossible.

With the same technique, let us also show that every non-zero linear combination of the functions g_p, $p \in \mathbb{N}$, is nowhere in $\mathcal{E}_{\{\omega\}}$. Let $\alpha_1, \ldots, \alpha_N \in \mathbb{C}$ with $\alpha_N \neq 0$, $p_1 < \cdots < p_N$ and

$$G = \sum_{j=1}^{N} \alpha_j g_{p_j}.$$

If there is an open set Ω such that G belongs to $\mathcal{E}_{\{\omega\}}(\Omega) \subset \mathcal{E}_{\{\omega^{(2p_N)}\}}(\Omega)$, then the function

$$g_{p_N} = \frac{1}{\alpha_N}\left(G - \sum_{j=1}^{N-1} \alpha_j g_{p_j}\right)$$

belongs to $\mathcal{E}_{\{\omega^{(2p_N)}\}}(\Omega)$, which is impossible. This concludes the proof. □

As for the case of classes of ultradifferentiable functions defined using weight sequences, we have the following result of density.

Lemma 3.17. *(See [15].) For each weight function ω such that $\omega(t) = o(t)$ as t tends to infinity and each open subset Ω of $\mathbb{R}^n$, the polynomials form a dense subset of $\mathcal{E}_{(\omega)}(\Omega)$.*

Using Theorem 2.2 and Remark 2.5 of [2] and Proposition 3.16 and Lemma 3.17, we directly get this last result.

Proposition 3.18. *Let ω and σ be two weight functions such that $\sigma(t) = o(\omega(t))$ as t tends to infinity. The set of functions of $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$ which are nowhere in $\mathcal{E}_{\{\omega\}}$ is dense-lineable in $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$.*

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