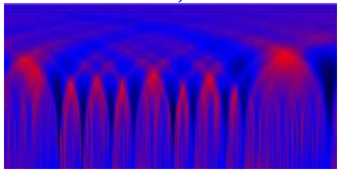
Unité d'Analyse, Analyse Fonctionnelle, Ondelettes



http://www.afaw.ulg.ac.be/

On generalized Hölder-Zygmund spaces

D. Kreit (joint work with S.Nicolay)
Université de Liège, Dpt. Mathematics, Liège, Belgium
(D.Kreit@alumni.ulg.ac.be)
(S.Nicolay@ulg.ac.be)

Université de Liège

Abstract. *The Besov spaces $B^{\alpha}_{p,q}(\mathbb{R}^d)$ and Hölder spaces $C^{\alpha}(\mathbb{R}^d)$ ($\alpha > 0,\ 1 < p,q \leq +\infty$) provide natural ways for measuring the smoothness of a function and are used in multiple areas from the solving of PDE to multifractal analysis. As such, it appears that generalizations of these spaces would prove themselves very useful in many domains and that is why a generalization of Besov Spaces has been extensively studied by many authors during those last 20 years. On the other hand, it has been proved that those spaces coincide with some kind of generalized Hölder spaces in particular cases ([4]). The purpose of this poster is to introduce those new Hölder spaces and to show that all main properties of the classical case are still true for the generalized ones.*

Notation $\Delta^1_h f(x) = f(x+h) - f(x), \quad \Delta^{n+1}_h(x) = \Delta^n_h f(x+h) - \Delta^n_h f(x)$

Definition of Hölder-Zygmund spaces $C^{\alpha}(\mathbb{R}^d)$

Let $f \in L^{\infty}(\mathbb{R}^d)$ and $\alpha > 0$; we say that f belongs to $C^{\alpha}(\mathbb{R}^d)$ if there exists $C, R > 0$ such that

$$\sup_{|h| \leq 2^{-j}} \|\Delta_h^{\lfloor \alpha \rfloor + 1} f\|_{L^\infty(\mathbb{R}^d)} \leq C 2^{-j\alpha}, \quad \forall j \in \mathbb{N}.$$

The Hölder exponent of f is $h_f = \sup\{\alpha : f \in C^{\alpha}(\mathbb{R}^d)\}$.

Definition of admissible sequences

A sequence $\sigma = (\sigma_j)_{j\in\mathbb{N}}$ of positive numbers is called *admissible* if there exists two positive constants d_0 and d_1 such that

$$d_0 \sigma_j \leq \sigma_{j+1} \leq d_1 \sigma_j, \qquad j \in \mathbb{N}.$$

Let

$$\underline{\sigma}_j := \inf_{k \geq 0} \frac{\sigma_{j+k}}{\sigma_k} \quad \text{and} \quad \overline{\sigma}_j := \sup_{k\geq 0} \frac{\sigma_{j+k}}{\sigma_k}, \qquad j \in \mathbb{N}.$$

The *lower and upper Boyd index* are respectively defined by

$$\underline{s}(\sigma) := \lim_{j \to +\infty} \frac{\log_2(\underline{\sigma}_j)}{j} \quad \text{and} \quad \overline{s}(\sigma) := \lim_{j\to+\infty} \frac{\log_2(\overline{\sigma}_j)}{j}.$$

Definition of generalized Hölder-Zygmund spaces $C^{\sigma,\alpha}(\mathbb{R}^d)$

Let $\alpha > 0$ and σ an admissible sequence. A function $f \in L^{\infty}(\mathbb{R}^d)$ belongs to the *generalized Hölder space* $C^{\sigma,\alpha}(\mathbb{R}^d)$ if there exists $C > 0$ such that

$$\sup_{x, |h| \leq 2^{-j}} |\Delta_h^{\lfloor \alpha \rfloor + 1} f(x)| \leq C\sigma_j \qquad \forall j \in \mathbb{N}_0.$$

Remark

The space $(C^{\sigma,\alpha}, \|.\|_{C^{\sigma,\alpha}})$ is a Banach space, with $\|.\|_{C^{\sigma,\alpha}} = \|.\|_{L^\infty} + |.|_{C^{\sigma,\alpha}}$

where $|f|_{C^{\sigma,\alpha}} = \sup_{j\in\mathbb{N}} \sigma_j^{-1} \sup_{|h|\leq 2^{-j}} \|\Delta_h^{\lfloor\alpha\rfloor+1} f\|_{L^\infty}$ is a seminorm.

Link with generalized Besov spaces

If $\underline{s}(\sigma^{-1}) > 0$, it can be shown that generalized Hölder spaces $C^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d)$ are indeed generalized Besov spaces $B^{\sigma^{-1}}_{\infty,\infty}$ (see [4]).

Example Let $\sigma_j := (2^{-j})^{\frac{1}{2}} |\log|\log(2^{-j})||^{\frac{1}{2}}$ for $j \in \mathbb{N}_0$. A. Khintchine proved that the trajectories of a Brownian Motion belong almost surely to $C^{\sigma,\alpha}(\mathbb{R})$ $(0 < \alpha < 1)$.

Proposition (link with classical regularity)

Let $K \in \mathbb{N}_0$ and $\sigma = (\sigma_j)_{j\in\mathbb{N}}$ an admissible sequence such that $K < \underline{s}(\sigma^{-1})$
If $f \in C^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d)$, then f is K-times continuously differentiable.

A characterization by the convolution

Let $\sigma = (\sigma_j)_{j\in\mathbb{N}}$ be an admissible sequence such that $\underline{s}(\sigma^{-1}) > 0$.
Then

$$C^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d) = \left\{ f \in L^\infty(\mathbb{R}^d) : \exists \Phi \in C_c^\infty(\mathbb{R}^d) \quad \sup_{j\in\mathbb{N}} \left(\sigma_j^{-1} \sup_{\delta \leq 2^{-j}} \|f \star \Phi_\delta - f\|_{L^\infty} \right) < \infty \right\}.$$

(where $\Phi_\delta = \delta^{-d}\Phi(x/\delta)$). Moreover, if $C \geq 1$, the norm

$$\|f\|_{L^\infty} + \inf \sup_{j\in\mathbb{N}} \left(\sigma_j^{-1} \sup_{\delta\leq 2^{-j}} \|f\star\Phi_\delta - f\|_{L^\infty} \right)$$

is equivalent to $\|f\|_{C^{\sigma,\lfloor \underline{s}(\sigma^{-1})\rfloor+1}}$ (where the infinum is taken on the set of functions $\Phi \in C_c^\infty(\mathbb{R}^d)$ that verify the inequality above and such that $\sup_{|\alpha| \in \{0, \overline{s}(\sigma^{-1})\rfloor + 1\}} \|D^\alpha \Phi\|_{L^1(\mathbb{R}^d)} \leq C$).

Hölder exponent

In [1], some sufficient conditions on admissible sequences σ^{α} $(\alpha > 0)$ are exposed so that we have $\alpha < \beta \Rightarrow C^{\sigma^\beta,\beta} \subseteq C^{\sigma^\alpha,\alpha}$. Those inclusions allow to define an Hölder exponent linked to these spaces by $H_f^\sigma = \sup\{\alpha > 0 : f \in C^{\sigma^\alpha,\alpha}\}$.

A characterization by polynomials

Let $\sigma = (\sigma_j)_{j\in\mathbb{N}}$ be an admissible sequence such that $\underline{s}(\sigma^{-1}) > 0$. Let $M \in \mathbb{N}$ such that $M > \overline{s}(\sigma)$, then

$$C^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d) = \left\{ f \in L^\infty(\mathbb{R}^d) : \sup_{x\in\mathbb{R}^d} \left(\sup_{j\in\mathbb{N}} \left(\sigma_j^{-1} \inf_{P \in \mathbb{P}_{M-1}} \|f - P\|_{L^\infty(B(x,2^{-j}))} \right) \right) < \infty \right\}.$$

Moreover, the norm $\|f\|_{L^\infty} + \sup_{x\in\mathbb{R}^d} \sup_{j\in\mathbb{N}} \sigma_j^{-1} \inf_{P\in\mathbb{P}_{M-1}} \|f - P\|_{L^\infty(B(x,2^{-j}))}$ is equivalent to $\|f\|_{C^{\sigma,\lfloor \underline{s}(\sigma^{-1})\rfloor+1}}$.

Taylor decomposition

Let σ be an admissible sequence such that $N - 1 < \underline{s}(\sigma^{-1}) \leq \overline{s}(\sigma^{-1}) < N$.
If $f \in C^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d)$, then we have

$$f(x+h) = \sum_{|\nu|\leq N-1} D^\nu f(x) \frac{h^\nu}{\nu!} + R_{N-1}(x,h) \frac{|h|^{N-1}}{(N-1)!}, \quad \forall x, h \in \mathbb{R}^d$$

where $\sup_{x,|h|\leq 2^{-j}} |R_{N-1}(x,h)| \leq C\sigma_j 2^{j(N-1)}$.
Conversely, if $f \in L^\infty(\mathbb{R}^d) \cap C^{N-1}(\mathbb{R}^d)$ satisfy the previous equality with $\sup_{x,|h|\leq 2^{-j}} |R_{N-1}(x,h)| \leq C\sigma_j 2^{j(N-1)}$, then $f \in C^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d)$.

Notation

- $J(t,a) = \max\{\|a\|_{A_0}, t\|a\|_{A_1}\} \quad (t > 0, a \in A_0 \cap A_1)$
- $a \in [A_0, A_1]_{\sigma,\psi,J}$ iff a can be written as $a = \sum_{j\in\mathbb{Z}} u_j$ where the convergence is in $A_0 + A_1$, $u_j \in A_0 \cap A_1$ and $(\sigma_j J(\psi_j, u_j))_{j\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$.
- $K(t,a) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1\} \quad (t > 0, a \in A_0 + A_1)$
- $a \in [A_0, A_1]_{\sigma,\psi,K}$ iff $a \in A_0 + A_1$ and $(\sigma_j K(\psi_j, a))_{j\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$.

Interpolation of Sobolev spaces

Let σ be an admissible sequence such that $N < \underline{s}(\sigma^{-1}) \leq \overline{s}(\sigma^{-1}) < M$ (where N, M are integers). Then we have

$$C^{\sigma,\overline{s}(\sigma^{-1})}(\mathbb{R}^d) = [W_N^\infty, W_M^\infty]_{\theta, 2^{j(M-N)}, J} = [W_N^\infty, W_M^\infty]_{\theta, 2^{j(M-N)}, K}$$

where θ is a new sequence defined by

$$\theta_j = \begin{cases} 2^{jN}\sigma_{-j}^{-1} & \forall j \in -\mathbb{N} \\ (\theta_{-j})^{-1} & \forall j \in \mathbb{N}_0. \end{cases}$$

A characterization by wavelet coefficients

Let $N \in \mathbb{N}_0$ and $\sigma = (\sigma_j)_{j\in\mathbb{N}}$ be a decreasing admissible sequence such that $N - 1 < \underline{s}(\sigma^{-1}) \leq \overline{s}(\sigma^{-1}) < N$.
For all $J \in \mathbb{N}$. Let be a multiresolution analysis of regularity $r > N$. Then the following are equivalent:

1. $f \in C^{\sigma,N-1}(\mathbb{R}^d)$;
2. $\exists C > 0 : \begin{cases} \sup_{k\in\mathbb{Z}^d} |C_k| \leq C \\ \sup_{k\in\mathbb{Z}^d} |c^i_{j,k}| \leq C\sigma_j, \quad \forall j \geq 0, \forall i \in \{1, \ldots, 2^d - 1\}, \end{cases}$

where C_k and $c^i_{j,k}$ are the classical wavelet coefficients associated with the multiresolution analysis (they correspond respectively to the father wavelet and the mother wavelet).

References.

- [1] D. Kreit, S. Nicolay, Some characterizations of generalized Hölder spaces, Math. Nachr. 285 (2012), 2157-2172.
- [2] D. Kreit, S. Nicolay, Characterizations of the elements of generalized Hölder-Zygmund spaces by means of their representation, J. Approx. Theory 172 (2013), 23-36.
- [3] D. Kreit, S. Nicolay, Generalized Pointwise Hölder Spaces, *submitted*.
- [4] S.D. Moura, On some characterizations of Besov spaces of generalized smoothness, *Mathematische Nachrichten*, 280 (2007).