

Uncoupled spectral analysis with non-proportional damping

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Abstract

The use of normal modes of vibration in the analysis of structures with non-proportional damping reduces the number of governing equations, but does not decouple them. A common practice consists in decoupling the equations by disregarding the off-diagonal elements of the modal damping matrix. This paper proposes a method based on the asymptotic expansion of the modal transfer matrix to take into account the modal coupling in Gaussian spectral analysis. First, the mathematical background is introduced, then the relevance of the method is illustrated within the context of the analysis of a large and real structure submitted to wind loadings.

1 Introduction

The structural damping matrix is often constructed with the assumption of proportionality. Consequently, the modal damping matrix and the modal transfer matrix are diagonal, and the equations of motion are decoupled. Therefore, frequency domain analysis consisting in the inversion of the transfer matrix for a sequence of frequencies, is deeply simplified.

Nevertheless, the consideration of aerodynamic damping forces or the presence of viscous dampers invalidate the assumption of proportionality. In these cases, the modal projection still enables to reduce the size of the system, but not to decouple the modal equations anymore.

In a deterministic context, some methods have already been proposed to deal with non-proportional damping in a modal basis. First, the decoupling approximation, proposed by Rayleigh (Rayleigh, 1945), consists in neglecting the off-diagonal elements of the modal damping matrix, because of their smallness compared with diagonal elements. However, Morzfeld (Morzfeld *et al.*, 2009) shows that this relative smallness is not sufficient to ensure small decoupling errors. More advanced methods have already been proposed, as the complex modal analysis (Foss, 1958) or the use of a truncated asymptotic expansion of the coupled modal transfer matrix around the decoupled one (Denoël & Degée, 2009).

This paper presents a method that preserves coupling in the system and avoids a sequence of full transfer matrix inversions. The concept of truncated asymptotic expansion of the modal transfer matrix is extended in a stochastic context. The method is then illustrated with the analysis of a large structure submitted to wind loadings.

2 Stochastic analysis with non-proportional damping

The equation of motion of an n -DOF linear system is

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f} \quad (1)$$

where \mathbf{M} , \mathbf{C} and \mathbf{K} are the mass, damping and stiffness matrices respectively, $\mathbf{f}(t)$ is the vector of random external forces, $\mathbf{x}(t)$ is the vector of nodal displacements. The response of the structure can

be computed with m normal modes of vibration ($m \ll n$). These modes gathered in a matrix Φ , are normalized to have unit generalized masses as $\Phi^T \mathbf{M} \Phi = \mathbf{I}$ and $\Phi^T \mathbf{K} \Phi = \Omega$ where Ω is a diagonal matrix containing squared circular frequencies. Using the modal superposition principle, the equation of motion Eqn.1 is written as

$$\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \Omega\mathbf{q} = \mathbf{g} \quad (2)$$

where $\mathbf{q}(t)$ is the vector of modal coordinates, $\mathbf{g}(t) = \Phi^T \mathbf{f}(t)$ is the vector of generalized forces and $\mathbf{D} = \Phi^T \mathbf{C} \Phi$ is the modal damping matrix.

For linear systems submitted to Gaussian loadings (wind actions), spectral analysis is performed to compute the probabilistic properties of any stationary structural responses (nodal/modal coordinates or internal forces). The power spectral density (psd) matrix of the modal coordinates $\mathbf{S}^{(q)}(\omega)$, is obtained by pre- and post-multiplication by the modal transfer matrix of the psd matrix of the generalized Gaussian forces $\mathbf{S}^{(g)}(\omega)$,

$$\mathbf{S}^{(q)} = \mathbf{H} \mathbf{S}^{(g)} \mathbf{H}^* \quad (3)$$

where the superscript $*$ denotes the conjugate transpose operator and $\mathbf{H}(\omega)$ is the modal transfer matrix defined as

$$\mathbf{H} = (\Omega - \omega^2 \mathbf{I} + i\omega \mathbf{D})^{-1} \quad (4)$$

with $i = \sqrt{-1}$ and ω the circular frequency. In the following developments, the damping matrix \mathbf{D} is decomposed into two terms as

$$\mathbf{D} = \mathbf{D}_d + \mathbf{D}_o \quad (5)$$

where \mathbf{D}_d and \mathbf{D}_o gather diagonal and off-diagonal elements, respectively. Substitution of Eqn.5 into Eqn.4 yields

$$\mathbf{H} = \left(\mathbf{I} + \mathbf{J}_d^{-1} \mathbf{J}_o \right)^{-1} \mathbf{H}_d \quad (6)$$

where $\mathbf{J}_d(\omega) = \Omega - \omega^2 \mathbf{I} + i\omega \mathbf{D}_d$, $\mathbf{J}_o(\omega) = i\omega \mathbf{D}_o$ and $\mathbf{H}_d(\omega) = \mathbf{J}_d^{-1}$. The decoupling approximation consists in neglecting \mathbf{D}_o in front of \mathbf{D}_d (so $\mathbf{H} \approx \mathbf{H}_d$) and thus the resulting approximate response is

$$\mathbf{S}^{(q_d)} = \mathbf{H}_d \mathbf{S}^{(g)} \mathbf{H}_d^* \quad (7)$$

In a deterministic context, Denoël *et al.* (Denoël & Degée, 2009) propose to preserve the influence of coupling terms using the asymptotic expansion of the transfer matrix, if the off-diagonal elements are small. Their smallness is quantified using the concept of diagonal dominance of \mathbf{D} defined as $\rho(\mathbf{D}) = \sigma(\mathbf{D}_d^{-1} \mathbf{D}_o)$ where σ is the spectral radius operator. Furthermore, if the index $\rho(\mathbf{D})$ is small, so is the index $\rho(\mathbf{J})$. This property allows to use a N -truncated asymptotic expansions \mathbf{H}_{c_N} of Eqn.6 in order to obtain a simplified and accurate estimation of the coupled transfer matrix, as

$$\mathbf{H}_{c_N} = \left(\mathbf{I} + \sum_{k=1}^N \left(-\mathbf{J}_d^{-1} \mathbf{J}_o \right)^k \right) \mathbf{H}_d \quad (8)$$

The computation of Eqn.8 does not require full matrix inversion contrary to Eqn.4. For large structures with many coupled modes, the inversion of the transfer matrix for each circular frequency is time consuming. With this approach, the inversion is bypassed and the accuracy improvement is achieved without any time-consuming operation compared with the decoupled approximation.

For deterministic problems, the approximation \mathbf{H}_{c_1} ($N=1$) is sufficient, because the transfer function appears as a single factor in the computation of the response. At the opposite, the spectral analysis requires the product of two factors involving the transfer function Eqn.3. For consistency, the use of the second order approximation for \mathbf{H}_{c_2} ($N=2$) is necessary. Therefore, the second order approximation of $\mathbf{S}^{(q)}$ is

$$\mathbf{S}^{(q_{c_2})} = \mathbf{S}^{(q_d)} + \Delta \mathbf{S}^{(q_1)} + \Delta \mathbf{S}^{(q_2)} \quad (9)$$

where $\Delta\mathbf{S}^{(q_1)}$ and $\Delta\mathbf{S}^{(q_2)}$ are two correction terms added to the decoupled approximation

$$\Delta\mathbf{S}^{(q_1)}(\omega) = - \left(\mathbf{H}_d \mathbf{J}_o \mathbf{S}^{(q_d)} + \mathbf{S}^{(q_d)} \mathbf{J}_o^* \mathbf{H}_d^* \right), \quad (10)$$

$$\Delta\mathbf{S}^{(q_2)}(\omega) = - \left(\mathbf{H}_d \mathbf{J}_o \Delta\mathbf{S}^{(q_1)} + \Delta\mathbf{S}^{(q_1)} \mathbf{J}_o^* \mathbf{H}_d^* \right) - \mathbf{H}_d \mathbf{J}_o \mathbf{S}^{(q_d)} \mathbf{J}_o^* \mathbf{H}_d^*. \quad (11)$$

If $\rho(\mathbf{D})$ is of order ε (a small parameter), so is the first term $\Delta\mathbf{S}^{(q_1)}$, whereas the second term $\Delta\mathbf{S}^{(q_2)}$ is of order ε^2 . However, the term $\Delta\mathbf{S}^{(q_2)}$ is not systematically negligible compared to $\Delta\mathbf{S}^{(q_1)}$, because not necessarily the same elements of $\mathbf{S}^{(q_d)}$ are concerned with the successive correction terms. More details will be given in a full length paper, but this fact is already shown in the following example.

3 Application to buffeting analysis of the Millau Viaduct

The proposed method is now applied to the buffeting analysis of Millau Viaduct shown in figure 1. The wind velocity field, supposed to be Gaussian, is characterized by on-site measurements summarized in (Canor *et al.*, 2012).

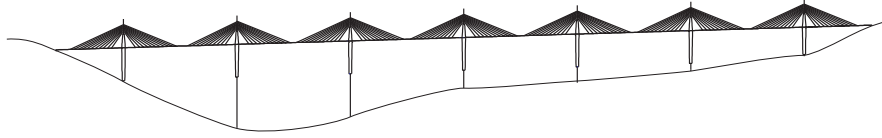


Figure 1: Shape of the Viaduct of Millau (Tarn-France).

The effects of the mean loading are determined by a static analysis, while a spectral analysis is required to evaluate the effects of turbulence. Similarly to Eqn.2, the modal equation of motion of the structure, disregarding the mean force, is

$$\ddot{\mathbf{q}} + (\mathbf{D}_s + \mathbf{D}_a)\dot{\mathbf{q}} + \Omega\mathbf{q} = \mathbf{g} \quad (12)$$

with \mathbf{D}_s the structural modal damping matrix built by setting a damping ratio of 0.3% for each of the 40 modes (those with natural frequency below 1Hz) kept in the analysis and \mathbf{D}_a the modal aerodynamic damping matrix which is not diagonal and generates modal coupling. In this application, drag, lift and moment coefficients contribute to the establishment of \mathbf{D}_a and \mathbf{g} (Simiu & Scanlan, 1978).

The exact variance vector $\Sigma^{(q)}$ (given by $\Sigma^{(q)} = \int_{-\infty}^{+\infty} \mathbf{S}^{(q)} d\omega$) and the exact correlation matrix $\rho^{(q)}$ for the first 40 modal coordinates are computed and shown in figures 2-a and 3-a, respectively.

Figure 2 also shows the relative errors on the variance of the modal coordinates obtained with the different approximations. Figure 2-b indicates that neglecting modal coupling induces a maximum error of +45% on the variance (mode 19). The proposed method does not neglect this coupling and reduces this error down to +10% (Fig.2-d). Figures 2-c and -d illustrate the influence of the first and second correction terms on the variances: the error decreases as an alternating series; the second term is necessary to reduce the error on the two groups of coupled modes (17 to 19 and 29 to 31).

Figure 3 shows the errors on the correlation coefficients obtained with the decoupling approximation $\rho^{(q_d)}$ and the proposed approximation $\rho^{(q_{e2})}$. The decoupling approximation (Fig.3-b) provides important differences (up to 0.2), especially for the groups of correlated modes. The proposed method reduces significantly these differences down to 0.06 (Fig.3-c), precisely because the correlation coming from non-proportional damping is integrated in this approximation.

To conclude, the developed method consists in solving the uncoupled equation of motion and to partially account for coupling with correction terms. Practitioners will surely consider this method as an efficient manner to progressively refine the response of structures submitted to wind loadings.

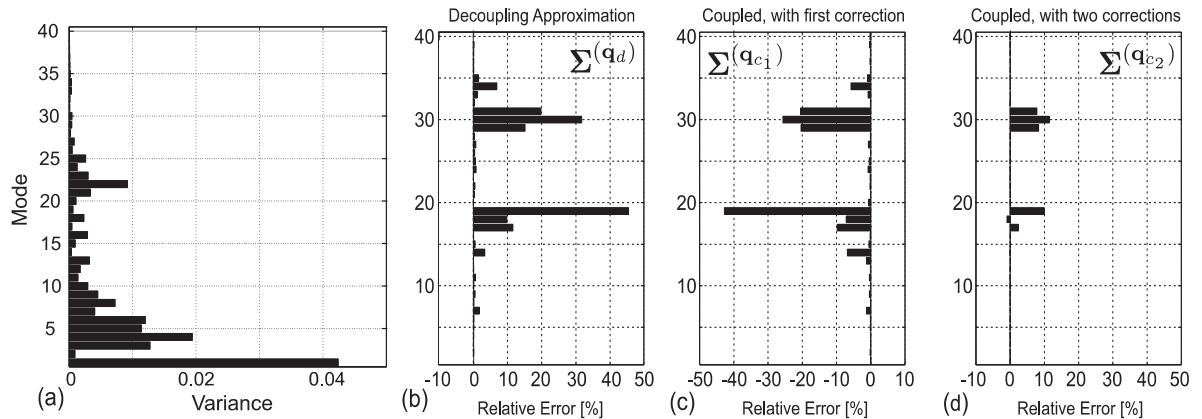


Figure 2: Exact variances of modal coordinates $\Sigma^{(q)}$ (a). Relative error on the variance of modal coordinates for different approximations $\Sigma^{(q_d)}$ (b), $\Sigma^{(q_{c1})}$ (c) and $\Sigma^{(q_{c2})}$ (d). Relative errors are expressed with respect to $\Sigma^{(q)}$. Positivity of the error means overestimation of $\Sigma^{(q)}$.

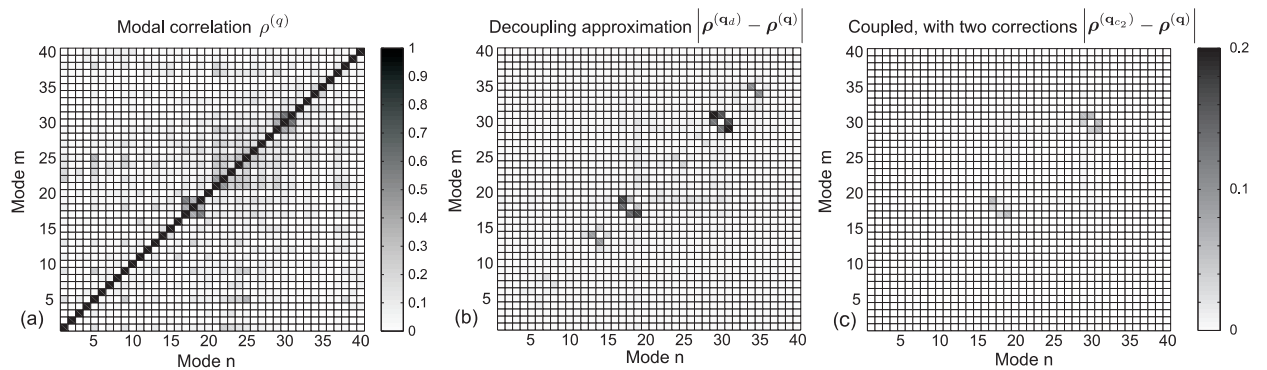


Figure 3: Exact correlation matrix of modal coordinates (a). Differences between exact correlation matrix of modal coordinates $\rho^{(q)}$ and approximations $\rho^{(q_d)}$ (b) and $\rho^{(q_{c2})}$ (c).

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