Projectively Invariant Quantization in Super Geometry

Dissertation originale présentée en vue de l'obtention du grade de Docteur en Sciences

Juin 2013

Par : Thomas LEUTHER

Sous la direction de :
Pierre LECOMTE (promoteur), Université de Liège
Fabian RADOUX (co-promoteur), Université de Liège

Un doctorat, c'est un chemin qui se dessine à mesure qu'on y avance. C'est un premier projet professionnel à l'intersection d'un plan de carrière et d'un plan de vie.

Un doctorat, c'est un apprentissage de la Recherche par la Recherche, au contact de personnes qui ont avant nous tracé leur propre chemin de thèse. Parmi ces personnes, je tiens à remercier mon promoteur, Pierre LECOMTE, pour m'avoir donné le goût de la géométrie ainsi que la possibilité d'y réaliser une thèse. Je remercie aussi Fabian RADOUX, mon co-promoteur, pour nos collaborations et pour sa disponibilité et son soutien tout au long de la rédaction de ce document. Ensuite, je remercie Pierre MATHONET et Gijs TUYNMAN avec qui j'ai eu la chance de pouvoir travailler : dans les deux cas, l'expérience fut pour moi immensément enrichissante. Enfin, je remercie Simone GUTT et Pierre BELIAVSKY d'avoir accepté de faire partie de mon jury.

Un doctorat, c'est une expérience humaine, partagée avec tout un tas de personnes qui font que ça vaut le détour (famille, amis, collègues, doctorants...). J'ai l'audace de penser que mon naturel démonstratif pendant les années de mon doctorat compensera l'absence d'énumération de ces personnes ici et que chacune de ces personnes sait en fait déjà combien j'ai apprécié les moments partagés avec elle.

Mon doctorat, je l'ai vécu aux côtés de Gaëlle. Je souhaite lui témoigner ma gratitude pour son soutien durant ces années et lui exprimer tout mon Amour.

Thomas

Mathematics is a collection of languages (Jet Nestruev)

Contents

Introduction

The classical setting

A vector field X on an ordinary smooth manifold M can be lifted in a natural way to a vector field on T^*M, thereby defining an action of the algebra of vector fields on M, $\mathrm{Vect}(M)$, on the subspace of functions on T^*M that are polynomial in the fibers. Those functions are called "symbols". It turns out that there is a unique (up to a normalization) $\mathrm{Vect}(M)$-equivariant map from the space of symbols of degree at most one to the space of differential operators on M, namely geometric quantization [?]. However, geometric quantization *cannot* be extended to the whole space of symbols if one requires equivariance with respect to the Lie algebra $\mathrm{Vect}(M)$, due to cohomological reasons [?].

One can ask whether there exists a Lie subalgebra $\mathfrak{g} \subset \mathrm{Vect}(M)$ for which geometric quantization can be extended as a $\mathfrak{g}$-equivariant quantization map. This $\mathfrak{g}$ should be "as big as possible" to attain the uniqueness, but "small enough" to acquire the extension of the geometric quantization to the whole space of symbols. When $M = \mathbb{R}^n$ with a $\mathrm{PGL}(n+1, \mathbb{R})$-structure, the quantization map has been investigated by P. Lecomte and V. Ovsienko [?]. They showed that there exists a unique quantization map that is $\mathfrak{pgl}(n+1, \mathbb{R})$-equivariant.

The concept of $\mathfrak{pgl}(n+1, \mathbb{R})$-equivariant quantization on $\mathbb{R}^n$ has a counterpart on an arbitrary smooth manifold M [?]. It aims at constructing, for any manifold M, a quantization map $Q_{M,\nabla}$ by means of a connection, depending only on its projective class (i.e. projectively invariant) and natural in all of its arguments. The existence of such a quantization procedure was proved by M. Bordemann [?].

This natural projectively invariant quantization (NPIQ) on smooth manifolds is a generalization of projectively equivariant quantization on $\mathbb{R}^n$. Indeed, if Q is a natural projectively invariant quantization, then $Q_{\mathbb{R}^n,\nabla_0}$ (where ∇_0 stands for the canonical flat connection on $\mathbb{R}^n$) is a projectively equivariant quantization. The idea of the proof is as follows: naturality means that Q is equivariant with respect to all vector fields on $\mathbb{R}^n$, but only vector fields in $\mathfrak{pgl}(n+1, \mathbb{R})$ (seen as a subalgebra of $\mathrm{Vect}(M)$) preserve the projective class of ∇_0, so that $Q_{\mathbb{R}^n,\nabla_0}$ is projectively equivariant only with respect to those vector fields.

The super setting

In 2011, P. Mathonet and F. Radoux [?] extended the problem of projectively equivariant quantization (PEQ) from ordinary smooth manifolds to supermanifolds. As a starting point, they managed to embed the Lie superalgebra $\mathfrak{pgl}(p+1|q,\mathbb{R})$ in the Lie superalgebra of vector fields on the flat supermanifold $\mathbb{R}^{p|q}$. Their embedding provides formulas that superize the classical ones, and so does their construction of a PEQ.

In the same way as in the classical case, one can wonder if projectively equivariant quantization on $\mathbb{R}^{p|q}$ has a counterpart on arbitrary supermanifolds. A partial positive answer to this question has first been given in [?], where a projectively invariant quantization on supermanifolds has been constructed for symbols of degree less than or equal to two. Then F. Radoux and the author [?] showed that both the problem of NPIQ and M. Bordemann's method can be extended to the super setting (except for some peculiar values of the superdimension $p-q$).

At this stage, one could think that the super picture was complete since it encompassed both projectively equivariant quantization and natural projectively invariant quantization. Nevertheless, some important pieces of the puzzle were missing.

(i) The super *projective embedding* of $\mathfrak{pgl}(p+1|q,\mathbb{R})$ found by P. Mathonet and F. Radoux was not constructed in terms of fundamental vector fields associated with an action of a projective supergroup on a supermanifold (as in the classical case). Indeed, their construction goes as follows: with each element of $\mathfrak{pgl}(p+1|q,\mathbb{R})$, they first associate a linear vector field on $\mathbb{R}^{p+1|q}$; this linear vector field, restricted to some homogeneous superfunctions, then amounts to a vector field on $\mathbb{R}^{p|q}$.

(ii) Although the explicit formula found by P. Mathonet and F. Radoux for $\mathfrak{pgl}(p+1|q,\mathbb{R})$-equivariant quantization could be recovered by means of M. Bordemann's construction, no proof could be given that the problem of NPIQ was *a priori* a generalization of projectively equivariant quantization on the flat superspace.

(iii) No geometric interpretation in terms of supergeodesics could be given for the condition of projective invariance imposed on super NPIQ. In the classical setting, thanks to a result due to H. Weyl [?], it is known that the algebraic condition used to define projective equivalence between two torsion-free connections (i.e., their difference tensor can be expressed in terms of 1-form) means that the connections have the same geodesics up to reparametrization. As geodesics on smooth manifolds generalize the notion of straight lines in $\mathbb{R}^n$, H. Weyl's result relates somehow the projective invariance condition imposed on NPIQ to the projective origin of PEQ.

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We focus mainly on answering the following questions:

(i) Does the projective embedding of $\mathfrak{pgl}(p+1|q,\mathbb{R})$ as a subalgebra of $\mathrm{Vect}(\mathbb{R}^{p|q})$ arise from the action of a supergroup on a supermanifold of graded dimension $p|q$?

(ii) Does the problem of NPIQ on supermanifolds in graded dimension $p|q$ generalize *a priori* the problem of PEQ on $\mathbb{R}^{p|q}$?

(iii) Does the algebraic condition of projective equivalence of torsion-free connections have a geometric counterpart in terms of supergeodesics ?

The text is divided into four chapters.

In the first chapter, we answer question (i): we recover the realization of $\mathfrak{pgl}(p+1|q,\mathbb{R})$ as a subalgebra of vector fields starting from an action of a supergroup on a supermanifold. Moreover, we describe the problem of projectively equivariant quantization and we recall the results of F. Radoux and P. Mathonet about existence and uniqueness of such a quantization.

In the second chapter, we prepare the study of question (ii). More precisely, we set up a geometric definition for the Lie derivative of geometric objects (as a derivative along the flow of a vector field) and we establish a Peetre-like result for local linear operators between vector geometric objects. To this aim, we propose a definition of super natural bundles over $\mathcal{A}$-manifolds, a superization of natural bundles (over smooth manifolds) in the sense of A. Nijenhuis [?, ?].

In the third chapter, we answer question (iii): we show that projectively equivalence can be equivalently described in algebraic terms or in terms of super geodesics. This is based on a joint paper [?] with F. Radoux and G. Tuynman. Moreover, in the perspective of the study of question (ii), we show that the vector fields obtained in Chapter 1 by means of the projective embedding preserve the projective class of the canonical flat connection on the flat superspace.

In the fourth chapter, we finally answer question (ii): we prove that the problem of natural projectively invariant quantization on supermanifolds is *a priori* a generalization of the problem of projectively equivariant on the flat superspace. Moreover, we describe how the superization of M. Bordemann's procedure allows one to construct a NPIQ for most values of the graded dimension and we discuss the situation in the peculiar cases. This is based on a joint paper [?] with F. Radoux.

The language of $\mathcal{A}$-manifolds

We will work with the geometric H^∞ version of DeWitt supermanifolds, which is equivalent to the theory of graded manifolds of Berezin, Leites and Kostant (see [?, ?, ?, ?, ?]). More precisely, we will use the language of $\mathcal{A}$-manifolds introduced by G. Tuynman [?]. The choice of this language is motivated by the fact that it is well-suited to dealing with geometric notion like smooth supergroup actions, supercurves, etc.

For the reader who is not familiar with $\mathcal{A}$-manifolds, we provide a quick introduction to $\mathcal{A}$-manifolds (Appendix A) and fiber bundles over them (Appendix B). The presentation is very incomplete and covers only some basic ingredients of the formalism. For a more comprehensive presentation, the reading of G. Tuynman's book [?] is highly suggested.

Chapter 1

Projectively Equivariant Quantization in Super Geometry

In this chapter, we first aim to recover a geometric origin for the "super projective embedding" found by P. Mathonet and F. Radoux [**?**]. In the language of $\mathcal{A}$-manifolds, the superization of the classical construction turns out to be pretty straightforward and we do recover the formulas of P. Mathonet and F. Radoux for the embedding of $\mathfrak{pgl}(p+1|q, \mathbb{R})$ into the Lie algebra of vector fields on the flat superspace of dimension $p|q$.

Remark. Actually, the construction yields a bit more: a smooth family of (not necessarily smooth) vector fields indexed by an $\mathcal{A}$-Lie algebra, with respect to which the projective embedding of $\mathfrak{pgl}(p+1|q, \mathbb{R})$ is just the restriction to the smooth elements. This family will play a major role in Chapter 4, where we prove that Natural Projectively Invariant Quantization (NPIQ) is a generalization of Projectively Equivariant Quantization (PEQ).

Having recovered the geometry of the projective embedding, we describe the problem of Projectively Equivariant Quantization on $E_0^{p|q}$, the flat superspace of dimension $p|q$. An important point there is that all objects involved in the quantization problem are obtained in an algebraic way from the *real* superalgebra $\mathrm{C}^\infty(E_0^{p|q})$. Consequently, because of the canonical isomorphism $\mathrm{C}^\infty(E_0^{p|q}) \cong \mathrm{C}^\infty(\mathbb{R}^{p|q})$, PEQ on $E_0^{p|q}$ in our sense is *the same* as $\mathfrak{pgl}(p+1|q, \mathbb{R})$-equivariant quantization on $\mathbb{R}^{p|q}$ in the sense of P. Mathonet and F. Radoux. In particular, their main result about existence and uniqueness of $\mathfrak{pgl}(p+1|q, \mathbb{R})$-equivariant quantization rules existence and uniqueness of PEQ here.

Contents

1.1 There is a local action of the projective supergroup on the flat superspace.

Preliminary Remark

In the framework of $\mathbb{R}$-manifolds, matrix mutliplication provides a smooth (left) action of the Lie group $\mathrm{GL}(p+1,\mathbb{R})$ on $\mathbb{R}^{p+1}$. This action corresponds to the evaluation of the elements of the group $\mathrm{Aut}(\mathbb{R}^{p+1})$ through the canonical matrix representation associated with any automorphism of $\mathbb{R}^{p+1}$. In the framework of $\mathcal{A}$-manifolds, remember [**?**]:

> "Special attention has to be paid to matrices associated to linear maps. Whereas in usual linear algebra there is a single natural way to associate a matrix to a linear map when a basis has been given, in graded linear algebra there are three natural ways to do so. Each of these three ways has its own advantages and disadvantages."

The different matrix representations for elements of $\mathrm{End}_R(E) \cong E \otimes E^*$ correspond to the choice of either the left-, right- or middle-coordinates associated with the basis vectors $e_i \otimes e^j$.

In view of this subtlety, we shall mainly speak in terms of automorphisms. When using matrices (i.e., when performing computations in coordinates), we shall always specify which coordinates are understood.

1.1.1 The flat superspace

By definition, the $\mathcal{A}$-vector space $E^{p+1|q}$ is the $\mathcal{A}$-module $\mathcal{A}^{p+1} \times (\Pi\mathcal{A})^q$, where Π is the parity reversal operation (see [**?**, pg 102] or Appendix B), together with the equivalence class of bases of the canonical basis

$$\left\{ e_i = (\overset{(0)}{0}, \ldots, 0, \overset{(i)}{1}, 0, \ldots, \overset{(p+q)}{0}) : i = 0, \ldots, p+q \right\} .$$

The elements of the even part $E_0^{p+1|q}$ are thus represented by $p+1$ even (commuting) coordinates and q odd (anti-commuting) coordinates in any basis of E.

Definition. The *flat supermanifold* of dimension $p+1|q$ is $E_0^{p+1|q} \cong \mathcal{A}_0^{p+1} \times \mathcal{A}_1^q$.

By definition, $E_0^{p+1|q}$ is an $\mathcal{A}$-manifold covered by the chart $\mathrm{id}_{E_0^{p+1|q}} : E_0^{p+1|q} \to E_0^{p+1|q}$. Moreover, the topology on $E_0^{p+1|q}$ is the De witt topology: a subset U is *open* in $E_0^{p+1|q}$ if $\mathbf{B}U$ is open in $\mathbf{B}E_0^{p+1|q} \simeq \mathbb{R}^{p+1}$ and $U = \mathbf{B}^{-1}(\mathbf{B}U)$.

1.1.2 The projective superspace

Roughly speaking, the projective superspace of dimension $p|q$ is the space of (even) straight lines in the flat superspace $E_0^{p+1|q}$.

Definition. Any two points $x, y \in E_0^{p+1|q} \setminus \mathbf{B}^{-1}(\{0\})$ are *projectively equivalent* if there is an element $a \in \mathcal{A}_0 \setminus \mathbf{B}^{-1}(\{0\})$ such that $y = a \cdot x$. The corresponding quotient space

$$\mathrm{P}(E_0^{p+1|q}) = (E_0^{p+1|q} \setminus \mathbf{B}^{-1}(\{0\})) \;/\; (\mathcal{A}_0 \setminus \mathbf{B}^{-1}(\{0\})) ,$$

endowed with the quotient topology, is called the *projective superspace* of dimension $p|q$.

Local charts

The space $\mathrm{P}(E_0^{p+1|q})$ is an $\mathcal{A}$-manifold. For any $i = 0, ..., p$, we set

$$V_i = \{[(x_0, ..., x_p, \xi_1, ..., \xi_q)] \in \mathrm{P}(E_0^{p+1|q}) : \mathbf{B}x_i \neq 0\} .$$

Obviously, we have $\mathrm{P}(E_0^{p+1|q}) = \bigcup_{i=0}^{p} V_i$. We also define

$$\varphi_i : V_i \to E_0^{p|q} = (\mathcal{A}^p \times \Pi\mathcal{A}^q)_0 \;,\; [(x_0, ..., x_p, \xi_1, ..., \xi_q)] \mapsto x_i^{-1} \cdot (x_0, , ...\widehat{i}..., x_p, \xi_1, ..., \xi_q) ,$$

where $\widehat{i}$ means that x_i is omitted. Each map φ_i is a homeomorphism with inverse

$$\varphi_i^{-1} : E_0^{p|q} \to V_i \;,\; (x_1, ..., x_p, \xi_1, ..., \xi_q) \mapsto [(x_1, ..., x_i, 1, x_{i+1}, ..., x_p, \xi_1, ..., \xi_q)] .$$

The change of charts $\varphi_{ji} = \varphi_j \circ \varphi_i^{-1}$ are smooth because multiplying and inverting elements in $\mathcal{A}$ are smooth operations. Thus, we have endowed $\mathrm{P}(E_0^{p+1|q})$ with an atlas.

1.1.3 The projective supergroup

Definition. We say that any two automorphisms $g, h \in \mathrm{Aut}(E^{p+1|q})$ are *projectively equivalent* if there is an element $a \in \mathcal{A}_0 \setminus \mathbf{B}^{-1}(\{0\})$ such that $g = \lambda_a \circ h$, where λ_a stands for the left multiplication by a (i.e., $\lambda_a(x) = a \cdot x$). The corresponding quotient space

$$\mathrm{PAut}(p+1|q, \mathcal{A}) = \mathrm{Aut}(E^{p+1|q}) \;/\; (\mathcal{A}_0 \setminus \mathbf{B}^{-1}(\{0\})) ,$$

endowed with the quotient topology, is called the *projective supergroup* in dimension $p|q$.

An $\mathcal{A}$-Lie group

The space $\mathrm{PAut}(p+1|q,\mathcal{A})$ is an $\mathcal{A}$-manifold. Indeed, $\mathrm{PAut}(p+1|q,\mathcal{A})$ can easily be covered with charts similar to those of the projective space, but valued in $(\mathcal{A}^{(p+1)^2+q^2-1} \times \Pi\mathcal{A}^{2pq})_0$: for any $(i,j) \in I = \{0,...,p\}^2 \cup \{p+1,...,p+q\}^2$, we set

$$V_{i,j} = \left\{[g] \in \mathrm{PAut}(p+1|q,\mathcal{A}) : (\mathbf{B}g)^i_j \neq 0\right\} ,$$

where $\mathbf{B}g$ is seen as an element of $\mathrm{GL}(p+1|q,\mathbb{R})$. Then we define

$$\varphi_{i,j} : V_{i,j} \to E_0^{(p+1)^2+q^2-1|2(p+1)q} , \left[g = \sum_{k,l=0}^{p+q} y^k_l \cdot e_k \otimes e^l\right] \mapsto (y^i_j)^{-1} \cdot \left(y^0_0, ..., y^0_{p+q}, y^1_0, ...\widehat{(i,j)}..., y^{p+q}_{p+q}\right) .$$

The $\mathcal{A}$-manifold $\mathrm{PAut}(p+1|q,\mathcal{A})$ is an $\mathcal{A}$-Lie group. Indeed, for any $g, h \in \mathrm{Aut}(E^{p+1|q})$, we have a smooth multiplication (smoothness is easily shown using charts):

$$[g] \cdot [h] = [g \circ h] .$$

Action on the projective superspace

The $\mathcal{A}$-Lie group $\mathrm{PAut}(p+1|q,\mathcal{A})$ acts (smoothly, on the left) on the projective superspace of dimension $p|q$ by means of the map

$$\widetilde{\Phi} : \mathrm{PAut}(p+1|q,\mathcal{A}) \times \mathrm{P}(E_0^{p+1|q}) \to \mathrm{P}(E_0^{p+1|q}) , \ ([g],[x]) \mapsto [g(x)] .$$

The body of this action recovers the usual action of $\mathrm{PGL}(p+1,\mathbb{R}) = \mathrm{GL}(p+1,\mathbb{R})/\mathbb{R}_0.\mathrm{Id}$ on the projective space $\mathrm{P}^p\mathbb{R} = (\mathbb{R}^{p+1}\backslash\{0\})/\mathbb{R}_0$, i.e., we have a commutative diagram

$$\begin{tikzcd}
\mathrm{PAut}(p+1|q,\mathcal{A}) \times \mathrm{P}(E_0^{p+1|q}) \arrow[r, "\widetilde{\Phi}"] \arrow[d, "\mathbf{B}"] & \mathrm{P}(E_0^{p+1|q}) \arrow[d, "\mathbf{B}"] \\
\mathrm{PGL}(p+1,\mathbb{R}) \times \mathrm{P}^p\mathbb{R} \arrow[r, "\mathbf{B}\widetilde{\Phi}"] & \mathrm{P}^p\mathbb{R}
\end{tikzcd}$$

with

$$\mathbf{B}\widetilde{\Phi} : \mathrm{PGL}(p+1,\mathbb{R}) \times \mathrm{P}^p\mathbb{R} \to \mathrm{P}^p\mathbb{R} , \ ([A],[x]) \mapsto [Ax] .$$

1.1.4 The projective superalgebra

Definition. We denote by $\mathfrak{paut}(p+1|q,\mathcal{A})$ the $\mathcal{A}$-Lie algebra of the $\mathcal{A}$-Lie group $\mathrm{PAut}(p+1|q,\mathcal{A})$.

We have

$$\mathfrak{paut}(p+1|q,\mathcal{A}) = \mathrm{End}_R(E^{p+1|q})/\mathcal{A}.\mathrm{Id}\ .$$

Indeed, the $\mathcal{A}$-Lie algebra of $\mathrm{Aut}(E^{p+1|q})$ can be interpreted as the whole $\mathcal{A}$-vector space $\mathrm{End}_R(E^{p+1|q})$, endowed with the usual (graded) commutator of endomorphisms (see [?]). Moreover, since $\mathrm{Lie}((\mathcal{A}_0 \setminus \mathbf{B}^{-1}(\{0\}).\mathrm{Id}) = \mathrm{Lie}(\mathrm{Aut}(\mathcal{A})) = \mathrm{End}_R(\mathcal{A}) = \mathcal{A}.\mathrm{Id}$, it follows from [?, Theorem VI.5.9] that the $\mathcal{A}$-Lie algebra of $\mathrm{PAut}(p+1|q,\mathcal{A})$ is nothing but $\mathrm{End}_R(E^{p+1|q})/\mathcal{A}.\mathrm{Id}$, where two automorphisms $g,h \in \mathrm{End}_R(E^{p+1|q})$ belong to the same coset if there is an element $a \in \mathcal{A}$ such that $g - h = \lambda_a : x \mapsto a\cdot x$.

The 3-grading of $\mathfrak{pgl}(p+1|q,\mathbb{R})$

In the classical context, the Lie algebra $\mathfrak{pgl}(p+1,\mathbb{R})$ has a natural decomposition into a direct sum of 3 Lie algebras. This decomposition was extended to the super context in [?] as follows: any element g of $\mathfrak{pgl}(p+1|q,\mathbb{R})$ can be represented in the canonical coordinates of $\mathbb{R}^{p+1|q}$ by a matrix([1])

$$\begin{pmatrix} a_g & v_g \\ \xi_g & A_g \end{pmatrix} , \tag{1.1}$$

where $a_g \in \mathbb{R}$, $v_g \in \mathbb{R}^{p|q}$, $\xi_g \in (\mathbb{R}^{p|q})^*$ and $A_g \in \mathfrak{gl}(p|q,\mathbb{R})$; this decomposition of matrices defines an even $\mathbb{R}$-linear bijection

$$\mathrm{j} : \mathfrak{pgl}(p+1|q,\mathbb{R}) \to \mathbb{R}^{p|q} \oplus \mathfrak{gl}(p|q,\mathbb{R}) \oplus (\mathbb{R}^{p|q})^* : [g] \mapsto (v_g, (A_g - a_g\cdot \mathrm{Id}_{p|q}), \xi_g)\ .$$

Using this bijection, the Lie algebra structure of $\mathfrak{pgl}(p+1|q,\mathbb{R})$ can be transported to $\mathbb{R}^{p|q} \oplus \mathfrak{gl}(p|q,\mathbb{R}) \oplus (\mathbb{R}^{p|q})^*$: the transported Lie bracket reads

$$\left\{ \begin{array}{rclrcl} [v,w] &=& 0\ , & [A,v] &=& A(v)\ , \\ [A,B] &=& A\circ B - (-1)^{\varepsilon(A).\varepsilon(B)} B\circ A\ , & [\xi,v] &=& -\xi(v)\cdot \mathrm{Id}_{p|q} - (-1)^{\varepsilon(\xi).\varepsilon(v)}\cdot v\otimes\xi\ , \\ [\xi,\zeta] &=& 0\ , & [\xi,A] &=& \xi\circ A\ , \end{array}\right. \tag{1.2}$$

where v,w (resp. A,B, resp. ξ,ζ) stand for elements of $\mathbb{R}^{p|q}$ (resp. $\mathfrak{gl}(p|q,\mathbb{R})$, resp. $(\mathbb{R}^{p|q})^*$).

[1] Note that in the context of real super vector spaces, the entries of matrices are real numbers and the matrix representation of a linear map is unique.

The 3-grading of $\mathfrak{paut}(p+1|q,\mathcal{A})$

In turn, the $\mathcal{A}$-Lie algebra $\mathfrak{paut}(p+1|q,\mathcal{A})$ inherits a similar decomposition through the body map. Indeed, there is a unique even $\mathcal{A}$-linear bijection

$$\mathbf{G}\mathrm{j} : \mathfrak{paut}(p+1|q,\mathcal{A}) \to \mathfrak{g}_{(-1)} \oplus \mathfrak{g}_{(0)} \oplus \mathfrak{g}_{(1)} \ ,$$

where $\mathfrak{g}_{(-1)} = E^{p|q}$, $\mathfrak{g}_{(0)} = \mathrm{End}_{\mathrm{R}}(E^{p|q})$ and $\mathfrak{g}_{(1)} = (E^{p|q})^*$, such that $\mathbf{BG}\mathrm{j} = \mathrm{j}$. More explicitly, we have

$$\begin{aligned}
\mathbf{G}\mathrm{j}\left(\left[\sum_{k=1}^{p+q} y_0^k \cdot e_k \otimes e^0\right]\right) &= \sum_{k,l=1}^{p+q} y_l^k \cdot e_k \in E^{p|q} \\
\mathbf{G}\mathrm{j}\left(\left[y_0^0 \cdot e_0 \otimes e^0 + \sum_{k,l=1}^{p+q} y_l^k \cdot e_k \otimes e^l\right]\right) &= \sum_{k,l=1}^{p+q} (y_l^k - \delta_l^k \cdot y_0^0) \cdot e_k \otimes e^l \in \mathrm{End}_{\mathrm{R}}(E^{p|q}) \\
\mathbf{G}\mathrm{j}\left(\left[\sum_{l=1}^{p+q} y_l^0 \cdot e_0 \otimes e^l\right]\right) &= \sum_{l=1}^{p+q} y_l^0 \cdot e^l \in (E^{p|q})^*
\end{aligned}$$

Using $\mathbf{G}\mathrm{j}$, we can transport the $\mathcal{A}$-Lie algebra structure from $\mathfrak{paut}(p|q,\mathcal{A}) = \mathrm{End}_R(E^{p+1|q})/\mathcal{A}.\mathrm{Id}$ to $E^{p|q} \oplus \mathrm{End}_{\mathrm{R}}(E^{p|q}) \oplus (E^{p|q})^*$:

$$[h_1, h_2] = \mathbf{G}\mathrm{j}\left(\left[A \circ B - (-1)^{\varepsilon(A).\varepsilon(B)} \cdot B \circ A\right]\right) \ ,$$

if $\mathbf{G}\mathrm{j}^{-1}(h_1) = [A]$ and $\mathbf{G}\mathrm{j}^{-1}(h_2) = [B]$. Doing so, we recover formulas (1.2) but now with v, w (resp. A, B, resp. ξ, ζ) in $E^{p|q}$ (resp. $\mathfrak{gl}(p|q,\mathcal{A})$, resp. $(E^{p|q})^*$).

1.1.5 The projective embedding

Identifying $E_0^{p|q}$ with the open subset $\varphi_0^{-1}(E_0^{p|q}) \subset \mathrm{P}(E_0^{p+1|q})$, we can associate with each element $h \in \mathfrak{g}_{(-1)} \oplus \mathfrak{g}_{(0)} \oplus \mathfrak{g}_{(1)}$ a vector field X^h on $E_0^{p|q}$, namely (the local expression of) the fundamental vector field corresponding to $\mathbf{G}\mathrm{j}^{-1}(h) \in \mathfrak{paut}(p+1|q,\mathcal{A})$.

PROPOSITION 1. *In terms of the* Euler vector field $\mathcal{E} = \sum_k y^k \cdot \partial_{y^k}$*, we have*

$$\left\{\begin{array}{lcll}
X^v &=& -\sum_{i=1}^{p+q} v^i \cdot \partial_{y^i} \ , & \textit{if } v = \sum_{i=1}^{p+q} v^i \cdot e_i \in \mathfrak{g}_{(-1)} \ , \\
X^A &=& -\sum_{i,j=1}^{p+q} (-1)^{\varepsilon_j \cdot (\varepsilon_i + \varepsilon_j)} \cdot A_j^i \cdot \partial_{y^i} \ , & \textit{if } A = \sum_{i,j=1}^{p+q} A_j^i \cdot e_i \otimes e^j \in \mathfrak{g}_{(0)} \ , \\
X^\xi &=& \sum_{j=1}^{p+q} (-1)^{\varepsilon_j} \cdot \xi_j \cdot y^j \cdot \mathcal{E} \ , & \textit{if } v = \sum_{j=1}^{p+q} \xi_j \cdot e^j \in \mathfrak{g}_{(1)} \ .
\end{array}\right. \tag{1.3}$$

In particular, the vector field X^h is smooth if and only if h lies in the body of $E^{p|q} \oplus \mathrm{End}_{\mathrm{R}}(E^{p|q}) \oplus (E^{p|q})^*$, i.e., the components of h in any basis are real numbers.

Proof. By definition of the fundamental vector fields [**?**, VI.5.1], the value of X^h at $x \in E_0^{p|q}$ is given by the local expression of the generalized tangent map $-T\widetilde{\Phi}_{\varphi_0^{-1}(x)}$, evaluated at $\mathbf{G}\mathrm{j}^{-1}(h) \in \mathfrak{paut}(p+1|q,\mathcal{A})$ (seen as a tangent vector at the identity of $\mathrm{PAut}(p+1|q,\mathcal{A})$). In practice, we then have

$$X^h(x) = h^i \cdot \Big((\partial_{x^i}\widetilde{\Phi}^j)(x) \Big) \cdot \partial_{x^j}|_x \ ,$$

where the h^i are the left coordinates of $h \in \mathfrak{g}_{(-1)} \oplus \mathfrak{g}_{(0)} \oplus \mathfrak{g}_{(1)}$ and where the $\widetilde{\Phi}^j$ are the $p+q$ components of the local expression

$$\varphi_0 \circ \widetilde{\Phi}_{\varphi_0^{-1}(x)} \circ \varphi_{0,0}^{-1}(x) : \varphi_{0,0}(V_{0,0}) \subset E_0^{p^2+q^2-1|2pq} \to E_0^{p|q},$$

Computations are straightforward. □

COROLLARY 2. *If for any two $h_1, h_2 \in \mathfrak{g}$, we define the Lie bracket of X^{h_1} and X^{h_2} by*

$$[X^{h_1}, X^{h_2}](x) = \mathrm{T}\pi_3([Z_1, Z_2](h_1, h_2, x)) \ ,$$

where each Z_i is the smooth vector field on $\mathfrak{g} \times \mathfrak{g} \times E_0^{p|q}$ defined by $Z_i(h_1, h_2, x) = \underline{0}_{h_1} + \underline{0}_{h_2} + X_x^{h_i}$, then the map $h \mapsto X^h$ becomes a morphism of $\mathcal{A}$-Lie algebras. In particular, the set of smooth vector fields $\{X^h : h \in \mathbf{B}\mathfrak{g}\}$ is canonically isomorphic to $\mathfrak{pgl}(p+1|q,\mathbb{R})$ as a real superalgebra .

Proof. Computations are straightforward from the local expression of the vector fields X^h and the local formula for the graded bracket of vector fields [**?**, V.1.21]. Note that for $h_1, h_2 \in \mathbf{B}\mathfrak{g}$, we deal with matrices having real entries and our computations coincide with those of [**?**] since they used formulas (1.3) too.

For arbitrary $h_1, h_2 \in \mathfrak{g}$, you basically have to use the usual local formula for the graded bracket of super vector fields, taking care to treat the coefficients of h as constant with parities. Doing so, you can check that the local formulas for $X^{[h_1,h_2]}$ and $[X^{h_1}, X^{h_2}]$ coincide. □

1.2 A PEQ is a quantization that is equivariant with respect to the projective superalgebra of vector fields.

With $\{X^h : h \in \mathbf{B}\mathfrak{paut}(p+1|q,\mathcal{A})\}$, we have somehow recovered the realization found by F. Radoux and P. Mathonet [?] of the real superalgebra $\mathfrak{pgl}(p+1|q,\mathbb{R})$ as a subalgebra of vector fields on the flat superspace of graded dimension $p|q$. We are now in position to describe the problem of Projectively Equivariant Quantization (or PEQ, for short) in the language of $\mathcal{A}$-manifolds.

1.2.1 Tensor densities and weighted symmetric tensors

Definition. The space $\mathcal{F}_\lambda$ of densities of weight λ over $E_0^{p|q}$ is an $\mathbb{R}$-linear representation of $\mathrm{Vect}(E_0^{p|q})$ on the (infinite-dimensional, $\mathbb{Z}_2$-graded) $\mathbb{R}$-vector space $\mathcal{F} = \mathrm{C}^\infty(E_0^{p|q})$. The action of $\mathrm{Vect}(E_0^{p|q}) \cong \mathrm{Der}\mathrm{C}^\infty(E_0^{p|q})$ is defined by the Lie derivative

$$\mathrm{L}^\lambda_X f = D_X(f) + \lambda \cdot \mathrm{div}(X) \cdot f, \tag{1.4}$$

where D_X is the derivation associated with X and where div, the *divergence*, is the even $\mathbb{R}$-linear operator whose value on a homogeneous vector field $X = \sum_{i=1}^{p+q} X^i \cdot \partial_{y^i}$ is given by

$$\mathrm{div}(X) = \sum_{i=1}^{p+q} (-1)^{\varepsilon_i \cdot (\varepsilon(X)+\varepsilon_i)} \cdot (\partial_{y^i} X^i) \in \mathcal{F}\ . \tag{1.5}$$

Definition. The space of symmetric tensor fields of weight δ and degree k is an $\mathbb{R}$-linear representation of $\mathrm{Vect}(E_0^{p|q})$ on the (infinite-dimensional, $\mathbb{Z}_2$-graded) $\mathbb{R}$-vector space

$$\mathcal{S}^k_\delta = \mathcal{F}_\delta \otimes_{\mathrm{C}^\infty(E_0^{p|q})} \vee^k \mathrm{Vect}(E_0^{p|q})\ .$$

The action of $\mathrm{Vect}(E_0^{p|q})$ is defined by the Lie derivative

$$\begin{aligned}\mathrm{L}^{\delta,k}_X (f \otimes X_1 \vee \cdots \vee X_k) &= \mathrm{L}^\delta_X f \otimes X_1 \vee \cdots \vee X_k \\ &\quad + \sum_{i=1}^{k} (-1)^{\varepsilon(X)\cdot(\sum_{l=1}^{i-1}\varepsilon(X_l))} \cdot X_1 \vee \cdots \vee [X, X_i] \vee \cdots \vee X_k\ .\end{aligned} \tag{1.6}$$

Remark. Because of the definition of the partial derivatives (see Subsection A.2.2), the canonical isomorphism of real superalgebras $\Phi : \mathrm{C}^\infty(E_0^{p|q}) \cong \mathrm{C}^\infty(\mathbb{R}^{p|q})$ (see formula (A.2)) induces an isomorphism Φ_* between the representation of $\mathrm{Vect}(E_0^{p|q})$ on $\mathcal{S}^k_\delta$ and the representation of $\mathrm{Vect}(\mathbb{R}^{p|q})$ on the space of weighted symmetric tensor fields considered by F. Radoux and P. Mathonet [?]: $\Phi_*(f \otimes \partial_{y^1} \vee \cdots \partial_{y^{p+q}}) = \Phi(f) \otimes \partial_{y^1} \vee \cdots \partial_{y^{p+q}}$.

1.2.2 Differential operators and symbols

Let $k \in \mathbb{N}$. We denote by $\mathcal{D}^k_{\lambda,\mu}$ the (infinite-dimensional, $\mathbb{Z}_2$-graded) $\mathbb{R}$-vector space of $\mathbb{R}$-linear differential operators $D : \mathcal{F}_\lambda \to \mathcal{F}_\mu$ of order at most k. Any differential operator $D \in \mathcal{D}^k_{\lambda,\mu}$ reads in coordinates as

$$D(f)(y^1, \ldots, y^{p+q}) = \sum_{|\alpha| \leqslant k} D_\alpha(y^1, \ldots, y^{p+q}) \cdot \left((\partial_{y^\alpha} f)(y^1, \ldots, y^{p+q})\right) \ , \tag{1.7}$$

where α is a multi-index, each D_α is a local δ-density ($\delta = \mu - \lambda$), $|\alpha| = \sum_{i=1}^{p+q} \alpha_i$, $\alpha_{p+1}, \ldots, \alpha_{p+q}$ are either 0 or 1 and ∂_{y^α} stands for $\partial^{\alpha_1}_{y^1} \cdots \partial^{\alpha_{p+q}}_{y^{p+q}}$.

The natural (left) linear representation of $\mathrm{Vect}(E_0^{p|q})$ on $\mathcal{D}^k_{\lambda,\mu}$ is given by the graded commutator: for any homogeneous $D \in \mathcal{D}_{\lambda,\mu}$ and $X \in \mathrm{Vect}(E_0^{p|q})$, we set

$$\mathrm{L}^{k,\lambda,\mu}_X D = \mathrm{L}^\mu_X \circ D - (-1)^{\varepsilon(X).\varepsilon(D)} \cdot D \circ \mathrm{L}^\lambda_X. \tag{1.8}$$

The $\mathrm{Vect}(E_0^{p|q})$-module of *symbols* is then the graded space associated with the filtered space $\mathcal{D}_{\lambda,\mu} = \bigcup_{k \in \mathbb{N}} \mathcal{D}^k_{\lambda,\mu}$. It is isomorphic (as a representation of $\mathrm{Vect}(E_0^{p|q})$) to the space of weighted symmetric tensor fields

$$\mathcal{S}_\delta = \bigoplus_{k \in \mathbb{N}} \mathcal{S}^k_\delta, \quad \delta = \mu - \lambda \ .$$

Indeed, the isomorphism comes from the *principal symbol operator*, $\sigma_k : \mathcal{D}^k_{\lambda,\mu} \to \mathcal{S}^k_\delta$, whose value on an element D which reads as (1.7), is given by

$$\sigma_k(D) = \sum_{|\alpha| = k} D_\alpha \otimes \partial^{\alpha_1}_{y^1} \vee \cdots \vee \partial^{\alpha_{p+q}}_{y^{p+q}} \ . \tag{1.9}$$

This operator commutes with the action of smooth vector fields and induces an $\mathbb{R}$-linear even bijection from $\mathcal{D}^k_{\lambda,\mu}/\mathcal{D}^{k-1}_{\lambda,\mu}$ to $\mathcal{S}^k_\delta$.

Remark. Let $\Phi : \mathrm{C}^\infty(E_0^{p|q}) \cong \mathrm{C}^\infty(\mathbb{R}^{p|q})$ be the canonical isomorphism of real superalgebras. The map $D \mapsto \Phi^*(D) = \Phi \circ D \circ \Phi^{-1}$ is an isomorphism between the representation of $\mathrm{Vect}(E_0^{p|q})$ on $\mathcal{D}^k_{\lambda,\mu}$ and the representation of $\mathrm{Vect}(\mathbb{R}^{p|q})$ on the space of differential operators from λ-densities to μ-densities considered by F. Radoux and P. Mathonet [?].

1.2.3 Projectively Equivariant Quantization

Let $\lambda, \mu \in \mathbb{R}$ and $\delta = \lambda - \mu$. By a *quantization* on $E_0^{p|q}$, we mean an even $\mathbb{R}$-linear bijection

$$Q : \mathcal{S}_\delta \to \mathcal{D}_{\lambda,\mu}$$

that preserves the principal symbol, i.e., for any $k \in \mathbb{N}$ and any $T \in \mathcal{S}_\delta^k$, Q must satisfy

$$\sigma_k(Q(T)) = T \ . \tag{1.10}$$

We say that a quantization Q is *projectively equivariant* when we have

$$\mathrm{L}_{X^h}^{k,\lambda,\mu} \circ Q = Q \circ \mathrm{L}_{X^h}^{k,\delta} \quad \text{for all } h \in \mathfrak{pgl}(p+1|q,\mathbb{R}) \ ,$$

where $\mathfrak{pgl}(p+1|q,\mathbb{R})$ is identified to $\mathbf{B}\mathfrak{paut}(p+1|q,\mathcal{A})$.

Existence and uniqueness

Through the canonical isomorphism of real superalgebras $\Phi : \mathrm{C}^\infty(E_0^{p|q}) \cong \mathrm{C}^\infty(\mathbb{R}^p) \otimes_\mathbb{R} \bigwedge \mathbb{R}^q = \mathrm{C}^\infty(\mathbb{R}^{p|q})$, the problem of PEQ described above is just a rewording of the problem of $\mathfrak{pgl}(p+1|q,\mathbb{R})$-equivariant quantization studied by P. Mathonet and F. Radoux in [?]. In particular, their main result about existence and uniqueness rules existence and uniqueness of PEQ here.([2])

Definition. When $n - m \neq -1$, we define the numbers

$$\gamma_{2k-l} = \frac{(n - m + 2k - l - (n - m + 1)\delta)}{n - m + 1}.$$

A value of δ is said to be *critical* if there exist $k, l \in \mathbb{N}$ such that $1 \leq l \leq k$ and $\gamma_{2k-l} = 0$.

Theorem 3 (Mathonet-Radoux).

(i) *When $p - q \neq -1$ and $\delta = \mu - \lambda$ is not critical, there is a unique $\mathfrak{pgl}(p+1|q,\mathbb{R})$-equivariant quantization $Q : \mathcal{S}_\delta \to \mathcal{D}_{\lambda,\mu}$.*

(ii) *When $p - q = -1$, there is a 1-parameter family of $\mathfrak{pgl}(p+1|q,\mathbb{R})$-equivariant quantizations $Q : \mathcal{S}_\delta \to \mathcal{D}_{\lambda,\mu}$ (without any restriction on the values of λ and μ).*

[2] If Q is a $\mathfrak{pgl}(p+1|q,\mathbb{R})$-equivariant quantization on $\mathbb{R}^{p|q}$, then $\Phi^* \circ Q \circ \Phi_*$ is a PEQ on $E_0^{p|q}$.

1.2.4 About the equivariance condition

So far, the condition of projective equivariance composed on PEQ is mainly algebraic: it is expressed in terms of algebraic Lie derivatives, not in terms of Lie derivatives corresponding to derivatives along the flow of vector fields. Moreover, since our algebraic Lie derivatives exist only for smooth vector fields, the equivariance condition is limited to asking for equivariance with respect to the fundamental vector fields associated with elements in the body of the $\mathcal{A}$-Lie algebra $\mathfrak{paut}(p+1|q,\mathcal{A})$.

We shall see that the equivariance condition can be stated equivalently in terms of the fundamental vector fields associated with the whole even part of $\mathfrak{paut}(p+1|q,\mathcal{A})$, this even part being known to capture the whole information about the Lie group action (see [**?**, Chapter VI, paragraph 5]). But before being in position to see it, we shall need to develop a geometric language for symbols, differential operators and their Lie derivatives in the direction of (not necessarily smooth) vector fields. The development of this language will be the core of Chapter 2.

As often, the "semantic limitation" (here, the fact that algebraic Lie derivatives make sense only for smooth vector fields) will be removed by means of a change of viewpoint: rather then restricting the *projective embedding* $h \mapsto X^h$ to $\mathfrak{pgl}(p+1|q,\mathbb{R})$ to avoid non-smooth vector fields, we can see it as a **smooth family** of (possibly non-smooth) vector fields, i.e., we can consider h as a variable and look at the smooth map

$$Z : \mathfrak{paut}(p+1|q,\mathcal{A})_0 \times E_0^{p+q} \to TE_0^{p+q},\ (h,x) \mapsto X_x^h$$

The geometric equivariance condition will consist in asking for equivariance with respect to the (yet to define) Lie derivative in the direction of this smooth family of even vector fields.

Chapter 2

Natural Bundles over $\mathcal{A}$-Manifolds

The concept of a natural bundle over a smooth manifold was introduced in the 1970's by A. Nijenhuis [?] in order to formalize in modern terms the idea of a *geometric* object on a smooth manifold.

In this chapter, we first aim at extending the concept of a natural bundle from ordinary smooth manifolds to $\mathcal{A}$-manifolds. Following A. Nijenhuis, we shall define geometric objects on manifolds by means of natural bundle *functors*, i.e., functors that associate with an $\mathcal{A}$-manifold a fiber bundle over it.

As often in supergeometry, it will be useful to perform a slight change of viewpoint in order to circumvent some *semantic* obstructions. For instance, remember that although the flow of a vector field $X \in \Gamma(TM)$ is smooth as a map $\mathcal{A}_0 \times M \to M$, the induced maps $M \to M$ corresponding to a fixed $t \in \mathcal{A}_0$ are not smooth in general. Since natural bundle functors should be able to lift the flow of vector fields in order to define Lie derivatives, natural bundle functors in the context of $\mathcal{A}$-manifolds need to be defined not only on local diffeomorphisms $M \to N$, but on all smooth families $P \times M \to N$ (where P is an $\mathcal{A}$-manifold of *parameters*) of locally invertible maps $M \to N$.

By the way, also our spaces of geometric objects will be larger than what a straightforward superization would suggest: given a natural bundle functor $\mathcal{F}$, a *geometric object of type* $\mathcal{F}$ on a $\mathcal{A}$-manifold is any smooth family of local sections, i.e., smooth maps $P \times M \to \mathcal{F}M$ such that for any $p \in P$, the induced map $M \to \mathcal{F}M$ is a section (not necessarily a smooth one) of the bundle $\pi : \mathcal{F}M \to M$ built by $\mathcal{F}$ over M.

In terms of these "extended" geometric objects, we will then give a definition of natural operators between natural bundle functors. At the end of this chapter, we will then show that natural linear operators are differential operators.

Contents

2.1 Natural bundle functors on $\mathcal{A}$-manifolds can lift *smooth families* of local homeomorphisms.

2.1.1 From local diffeomorphisms to smooth families

The classical setting

In the classical setting, natural bundle functors operate from the category of smooth manifolds and local diffeomorphisms between them to the category of fiber bundles and fibered smooth maps between them. They are defined as follows (see [?]):

A *natural bundle functor* in dimension n is a covariant functor

$$\mathcal{F} : \mathrm{Man}_m \to \mathrm{Fib}_n$$

possessing the following three properties.

(P) Prolongation:

(i) Each $\mathcal{F}M$ is a fiber bundle $\pi_{\mathcal{F}M} : \mathcal{F}M \to M$ over M.

(ii) The image of a morphism $\Phi : M \to N$ is a morphism

$$\mathcal{F}\Phi : \mathcal{F}M \to \mathcal{F}N$$

such that the following diagram commutes.

$$\begin{tikzcd} \mathcal{F}M \arrow[r, "\mathcal{F}\Phi"] \arrow[d, "\pi_{\mathcal{F}M}"] & \mathcal{F}N \arrow[d, "\pi_{\mathcal{F}N}"] \\ M \arrow[r, "\Phi"] & N \end{tikzcd}$$

(R) Regularity: if $\Phi : P \times M \to N$ is a smooth map such that all $\Phi_p = \Phi(p, \cdot)$ are local diffeomorphisms, then $\widetilde{\mathcal{F}}\Phi : P \times \mathcal{F}M \to \mathcal{F}N$, defined by $(\widetilde{\mathcal{F}}\Phi)_p = \mathcal{F}\Phi_p$, is also smooth.

(L) Locality: If $\iota : U \to M$ is the inclusion of an open submanifold, then $\mathcal{F}U = \pi^{-1}_{\mathcal{F}M}(U)$ and $\mathcal{F}\iota : \mathcal{F}U \to \mathcal{F}M$ is the inclusion of $\pi^{-1}_{\mathcal{F}M}(U)$ in $\mathcal{F}M$.

The regularity property was shown to be a consequence of the other two requirements (prolongation and locality). However, it is very useful in the theory of natural bundles, especially for defining Lie derivatives of geometric objects (the regularity condition ensures that the pullback of a geometric object by the flow of a vector field is smooth with respect to the time parameter). Therefore, one usually continues to include it in the definition of (classical) natural vector bundle functors.

The super setting

Passing from ordinary manifolds to $\mathcal{A}$-manifolds, a quick look at how the regularity condition should be superized suggests that a change of viewpoint could be necessary. Indeed, if super natural bundle functors could only lift local diffeomorphisms between $\mathcal{A}$-manifolds, then the regularity condition would not be applicable to the flow of super vector fields (the flow is not made of local diffeomorphisms because if we fix the time parameter, the resulting local homeomorphism is in general not smooth). Therefore, if we want super natural bundles to be able to lift the flow of vector fields (and we do want it in order to define Lie derivatives), we need to enlarge the space of morphisms.

Definition. We denote by $\widetilde{\mathrm{Man}}(\mathcal{A})$ the category whose objects are $\mathcal{A}$-manifolds. The space $\mathrm{Hom}_{\widetilde{\mathrm{Man}}(\mathcal{A})}(M,N)$, also denoted by $\widetilde{\mathrm{C}}^\infty(M,N)$, is made of all smooth maps $\Phi : W \subset P\times M \to N$, where W is an open subset in $P\times M$ (the $\mathcal{A}$-manifold P is called the *parameter space* of Φ). Morphisms are thus of the form

$$\Phi(p,x) = \Phi_p(x)$$

for some (not necessarily smooth) maps $\Phi_p : M \to N$. By definition, the composition of two morphisms $\Phi : W \subset P \times M \to M'$ and $\Phi' : W' \subset P' \times M' \to N$ is

$$\Phi'\circ\Phi : \{(p',p,x) : (p,x) \in W \text{ and } (p',\Phi_p(x)) \in W'\} \subset P'\times P\times M \to N,\ (p',p,x) \mapsto \Phi'_{p'}\circ\Phi_p(x)\ .$$

In particular, we have $(\Phi' \circ \Phi)_{(p',p)} = \Phi'_{p'} \circ \Phi_p$.

Definition.

- We denote by $\widetilde{\mathrm{Man}}_{n|m}(\mathcal{A})$ the subcategory of $\widetilde{\mathrm{Man}}(\mathcal{A})$ whose objects are $\mathcal{A}$-manifolds of graded dimension $n|m$. The space $\mathrm{Hom}_{\widetilde{\mathrm{Man}}_{n|m}(\mathcal{A})}(M,N)$ is the subset of all elements $\Phi : W \subset P \times M \to N$ in $\widetilde{\mathrm{C}}^\infty(M;N)$ for which the map

  $$\tilde{\Phi} : W \subset P \times M \to P \times N\ ,\ (p,x) \mapsto (p,\Phi_p(x))$$

  is a local diffeomorphism.([1])
- We denote by $\widetilde{\mathrm{Fib}}_{n|m}(\mathcal{A})$ the category whose objects are fiber bundles $\pi : E_\pi \to M$ over $\mathcal{A}$-manifolds of graded dimension $n|m$. The space $\mathrm{Hom}_{n|m}(\pi,\eta)$ is the subset of all *fiber-preserving* elements in $\widetilde{\mathrm{C}}^\infty(E_\pi,E_\eta)$, i.e., smooth maps $\Psi : W \subset P \times E_\pi \to E_\eta$ such that $\psi_p(E_{\pi(e)}) \subset E_{\eta(\psi_p(e))}$ for all $(p,e) \in W$.

[1] In view of the inverse function theorem for $\widetilde{\Phi}$ and of the definition of the generalized tangent map (see [?]), asking for $\tilde{\Phi}$ to be a local diffeomorphism amounts to asking for all the maps $T\Phi_p|_{T_xM} : T_xM \to T_{\Phi_p(x)}N$ to be (even, left) $\mathcal{A}$-linear bijections.

2.1.2 Natural bundle functors on $\mathcal{A}$-manifolds

Definition. A *natural bundle functor* in graded dimension $n|m$ is a covariant functor

$$\mathcal{F} : \widetilde{\mathrm{Man}}_{n|m}(\mathcal{A}) \to \widetilde{\mathrm{Fib}}_{n|m}(\mathcal{A})$$

possessing the following three properties.

(P1) Prolongation: Each $\mathcal{F}M$ is a fiber bundle $\pi_{\mathcal{F}M} : \mathcal{F}M \to M$ over M.

(R) Regularity: The image of a map $\Phi : W \subset P \times M \to N$ is a map

$$\mathcal{F}\Phi : \{(p,e) \in P \times \mathcal{F}M : (p, \pi_{\mathcal{F}M}(e)) \in W\} \subset P \times \mathcal{F}M \to \mathcal{F}N$$

Moreover, each $(\mathcal{F}\Phi)_p$ depends only of Φ_p in the sense that

$$\Phi'_{p'} = \Phi_p \;\Rightarrow\; (\mathcal{F}\Phi')_{p'} = (\mathcal{F}\Phi)_p$$

In particular, if $p \in \mathbf{B}P$, then $(\mathcal{F}\Phi)_p = \mathcal{F}\Phi_p$.([2])

(P2) Prolongation: Each $(\mathcal{F}\Phi)_p$ is over the corresponding Φ_p, i.e., the following diagram commutes.

$$\begin{tikzcd}
\mathcal{F}M \arrow[r, "(\mathcal{F}\Phi)_p"] \arrow[d, "\pi_{\mathcal{F}M}"] & \mathcal{F}N \arrow[d, "\pi_N"] \\
M \arrow[r, "\Phi_p"] & N
\end{tikzcd}$$

(L) Locality: If $\iota : U \to M$ is the inclusion of an open submanifold, then $\mathcal{F}U = \pi^{-1}_{\mathcal{F}M}(U)$ and $\mathcal{F}\iota : \mathcal{F}U \to \mathcal{F}M$ is the inclusion of $\pi^{-1}_{\mathcal{F}M}(U)$ in $\mathcal{F}M$.

Remark. The locality property **(L)** and the regularity property **(R)** ensure together that a natural bundle functor $\mathcal{F}$ is local on morphisms in the sense that

$$(\Phi'_{p'})|_U = (\Phi_p)|_U \;\Rightarrow\; (\mathcal{F}\Phi')_{p'}|_{\pi^{-1}(U)} = (\mathcal{F}\Phi)_p|_{\pi^{-1}(U)} \;.$$

Indeed, we have $\Phi_p|_U = (\Phi \circ \iota)_p$ and $\Phi'_{p'}|_U = (\Phi' \circ \iota)_{p'}$, where $\iota : U \to M$ is the inclusion. Using this and both the locality and the regularity property, we obtain $(\mathcal{F}\Phi')_{p'}|_{\pi^{-1}(U)} = (\mathcal{F}\Phi')_{p'} \circ \mathcal{F}\iota = (\mathcal{F}\Phi' \circ \mathcal{F}\iota)_{p'} = (\mathcal{F}(\Phi' \circ \iota))_{p'} = (\mathcal{F}(\Phi \circ \iota))_p = (\mathcal{F}\Phi \circ \mathcal{F}\iota)_p = (\mathcal{F}\Phi)_p|_{\pi^{-1}(U)}$.

[2] Now that we consider smooth maps with a parameter space, regularity of an operator means that this operator somehow leaves the parameter untouched and that it is compatible with reparametrizations. In the sequel, all operators acting on smooth families will be assumed to have this property.

Geometric objects on $\mathcal{A}$-manifolds

Definition.

- A *natural bundle* is a fiber bundle $\pi_{\mathcal{F}M} : \mathcal{F}M \to M$ built from the data of an $\mathcal{A}$-manifold M by means of a natural bundle functor $\mathcal{F}$.
- A *geometric object of type* $\mathcal{F}$ on M is a smooth family of (not necessarily) smooth sections of $\pi_{\mathcal{F}M} : \mathcal{F}M \to M$, i.e., a smooth map

  $$\sigma : W \subset P \times M \to \mathcal{F}M \ ,$$

  such that for any $(p, x) \in W$, we have $\pi_{\mathcal{F}M} \circ \sigma(p, x) = x$. The space of geometric objects of type $\mathcal{F}$ on M is denoted by $\widetilde{\Gamma}(\mathcal{F}M)$.

Remark. Let us stress that the space $\widetilde{\Gamma}(\mathcal{F}M)$ contains smooth families of sections with all possible parameter $\mathcal{A}$-manifolds P. In particular, the space $\Gamma(\mathcal{F}M)$ of (unparametrized) smooth sections is a subset of $\widetilde{\Gamma}(\mathcal{F}M)$ (corresponding somehow to $P = \{0\}$).

The typical fiber of a natural bundle functor

If $\mathcal{F}$ is a natural bundle functor, all fiber bundles $\mathcal{F}M$ share the same typical fiber. Indeed, note first that $\mathcal{F}E_0^{n|m}$ is trivial: a global trivialization is given by the map

$$\mathcal{F}E_0^{n|m} \to E_0^{n|m} \times F \ , \ e \mapsto (\pi(e), \mathcal{F}(\mathrm{t})(-\pi(e), e)) \ ,$$

where $F = \mathcal{F}_0 E_0^{n|m}$ is the fiber at 0([3]) and $\mathrm{t} : E_0^{n|m} \times E_0^{n|m} \to E_0^{n|m}$ is the smooth family of all translations in $E_0^{n|m}$. By the locality property, it follows that

$$\mathcal{F}O \cong O \times F$$

for all open subsets $O \subset E_0^{n|m}$. Then, if $(U_a, \varphi_a : U_a \to O_a)$ is a chart of M, the locality property gives

$$\mathcal{F}M|_{U_a} = \mathcal{F}U_a \overset{\mathcal{F}\varphi_a}{\cong} \mathcal{F}O_a \cong O_a \times F \cong U_a \times F \ ,$$

showing that F is also the typical fiber of $\mathcal{F}M$.

Remark. In the paragraph above, we constructed local trivializations to show that all fiber bundles $\mathcal{F}M$ share the same typical fiber. Note that these local trivializations are completely determined by the lifts $\mathcal{F}\varphi_a$ of local charts (U_a, φ_a) of M.

[3] Remember that when we have a fiber bundle, the fibers above the base points with real coordinates are diffeomorphic to the typical fiber. Therefore we can assume here that the fiber at 0 is the typical fiber.

2.1.3 Natural vector bundles

Definition. A *natural vector bundle functor* in graded dimension $n|m$ is a natural bundle functor $\mathcal{F} : \mathrm{Man}_{n|m}(\mathcal{A}) \to \mathrm{Fib}_{n|m}(\mathcal{A})$ such that

- all bundles $\mathcal{F}M$ are vector bundles;
- all maps $\mathcal{F}\Phi$ are smooth families of fiberwise even $\mathcal{A}$-linear maps.

Transition functions on a natural vector bundles

When $\mathcal{F}$ is a natural vector bundle functor, the typical fiber $F = \mathcal{F}_0 E_0^{n|m}$ is an $\mathcal{A}$-vector space. Moreover, the local trivializations of $\mathcal{F}M$ that we constructed in 2.1.2 from local charts (U_a, φ_a) of M, namely

$$\Psi_a : \mathcal{F}M|_U \to U \times F \ , \ e \mapsto (\pi(e), \mathcal{F}\mathrm{t}(-\varphi_a(\pi(e)), \mathcal{F}\varphi_a(e))) \ ,$$

are fiberwise even $\mathcal{A}$-linear because both $\mathcal{F}\mathrm{t}$ and $\mathcal{F}\varphi_a$ are. In other words, the maps Ψ_a are local trivializations of $\mathcal{F}M$ as a vector bundle.

Given an atlas $\{(U_a, \varphi_a)\}$ of M, we claim that the morphisms $\mathcal{F}\varphi_{ba}$ completely determine the vector bundle structure of $\mathcal{F}M$. On the one hand, we have

$$\Psi_b \circ \Psi_a^{-1}(x, f) = (x, \mathcal{F}\mathrm{t}(-\varphi_b(x), \mathcal{F}\varphi_{ba} \circ \mathcal{F}\mathrm{t}(\varphi_a(x), f))) \ ,$$

showing that the transition functions are the maps

$$\Psi_{ba}(x) = (\mathcal{F}\mathrm{t})_{(-\varphi_b(x))} \circ \mathcal{F}\varphi_{ba} \circ (\mathcal{F}\mathrm{t})_{(\varphi_a(x))} \ . \tag{2.1}$$

On the other hand, these maps are completely determined by the knowledge of the atlas $\{(U_a, \varphi_a\}$ and the collection of all maps $\{\mathcal{F}\varphi_{ba}\}$. Hence the claim.

The space of geometric objects

If $\mathcal{F}$ is a natural vector bundle, the space $\widetilde{\Gamma}(\mathcal{F}M)$ of geometric object of type $\mathcal{F}$ on M is a $\mathrm{C}^\infty(M)$-module. Indeed, the zero section $\mathbf{0} : M \to \mathcal{F}M$ is an element of $\widetilde{\Gamma}(\mathcal{F}M)$. Moreover, for any $\sigma, \sigma' \in \widetilde{\Gamma}(\mathcal{F}M)$, we can define

$$(\sigma + \sigma')(p, p', x) = \sigma(p, x) + \sigma'(p', x) \in \mathcal{F}_x M \ .$$

for all (p, p', x) such that $(p, x) \in W$ and $(p', x) \in W'$. Finally, given a smooth function $f \in \mathrm{C}^\infty(M)$, we form

$$(f \cdot \sigma) : W \subset P \times M \to \mathcal{F}M \ , \ (p, x) \mapsto f(x) \cdot \sigma(p, x) \in \mathcal{F}_x M \ .$$

This being said, the space $\widetilde{\Gamma}(\mathcal{F}M)$ also has a linear structure (over $\mathcal{A}$). Indeed, for an arbitrary $a \in \mathcal{A}$, the maps $(p, x) \mapsto a \cdot \sigma(p, x)$ and $(p, x) \mapsto \sigma(p, x) \cdot a$ are in general not smooth because a is not considered as a variable. However, nothing prevents us from considering a as an additional parameter, i.e., we can define smooth families of sections

$$\mathcal{A} \cdot \sigma : \mathcal{A} \times W \subset \mathcal{A} \times P \times U \to \mathcal{F}M \ , \ (a, p, x) \mapsto a \cdot (\sigma(p, x)) \in \mathcal{F}_x M$$

and

$$\sigma \cdot \mathcal{A} : \mathcal{A} \times W \subset \mathcal{A} \times P \times U \to \mathcal{F}M \ , \ (a, p, x) \mapsto (\sigma(p, x)) \cdot a \in \mathcal{F}_x M \ .$$

With these new operations at hand, we introduce what is $\mathcal{A}$-linearity for an operator acting on geometric objects over an $\mathcal{A}$-manifold.

Definition. A regular map $T : \tilde{\Gamma}(\mathcal{F}M) \to \tilde{\Gamma}(\mathcal{G}M)$ is said to be *left $\mathcal{A}$-linear* (resp. *right $\mathcal{A}$-linear*) if

$$\left\{ \begin{array}{rcl} T(\sigma + \sigma') & = & T(\sigma) + T(\sigma') \\ T(\mathcal{A} \cdot \sigma) & = & \mathcal{A} \cdot (T(\sigma)) \quad \text{(resp. } T(\mathcal{A} \cdot \sigma) = \mathcal{A} \cdot (T(\sigma))\text{)} \ . \end{array} \right.$$

for all $\sigma, \sigma' \in \tilde{\Gamma}(\mathcal{F}M)$.

Example 4. From any left linear vector bundle morphism $\Phi : \mathcal{F}M \to \mathcal{G}M$ over $\phi = \mathrm{id}_M$, we can define a map $T_\Phi : \tilde{\Gamma}(\mathcal{F}M) \to \tilde{\Gamma}(\mathcal{G}M)$ by setting

$$T_\Phi(\sigma)(p, x) = \Phi(\sigma(p, x)) \in \mathcal{G}_x M$$

Because of the fiberwise left $\mathcal{A}$-linearity of Φ, T_Φ is both $\mathrm{C}^\infty(M)$-linear and left $\mathcal{A}$-linear.

Example : the tangent bundle

Let M be an $\mathcal{A}$-manifold of dimension $n|m$ and let $\{(U_a \subset M, \varphi_a : U_a \to O_a \subset E_0^{n|m})\}$ be an atlas of M. The *tangent bundle* of M is the vector bundle TM with typical fiber $E^{n|m}$ and structure group $\mathrm{Aut}(E^{n|m})$ determined by the transition functions

$$\Psi_{ba} = \mathrm{Jac}(\varphi_{ba}) \circ \varphi_a : U_b \cap U_a \to \mathrm{Aut}(E^{n|m}) \ .$$

Remember that in terms of a basis $\{e_i\}$ of $E^{n|m}$, the Jacobian $\mathrm{Jac}(\varphi_{ba}) \in \mathrm{C}^\infty(\varphi_a(U_a \cap U_b), \mathrm{Hom}_L(E^{n|m}, E^{n|m}))$ is given by

$$\iota(h^k \cdot e_k)(\mathrm{Jac}(\varphi_{ba})(x)) = \tilde{h}^l \cdot e_l \ ,$$

where

$$\tilde{h}^l = h^k \cdot (\partial_{x^k} \varphi_{ba}^l(x)) \ . \tag{2.2}$$

Remark. The fact that the functions $\mathrm{Jac}(\varphi_{ba}) \circ \varphi_a$ satisfy the cocycle conditions (B.1) follows from the chain rule:

$$\begin{aligned}
\Psi_{aa}(x) &= \mathrm{Jac}(\mathrm{id}_{O_a})(\varphi_a(x)) = \mathrm{id}_{E^{n|m}} \\
\Psi_{cb}(x) \circ \Psi_{ba}(x) &= \mathrm{Jac}(\varphi_{cb})(\varphi_b(x)) \circ \mathrm{Jac}(\varphi_{ba})(\varphi_a(x)) \\
&= \mathrm{Jac}(\varphi_{cb} \circ \varphi_{ba})(\varphi_a(x)) = \mathrm{Jac}(\varphi_{ca})(\varphi_a(x)) \ .
\end{aligned}$$

Remark. Let U be an open subset of M. Since $\{(U \cap U_a, \varphi_a|_{U_a}\}$ is an atlas of U, we have the first part of the locality condition: $TU = TM|_U = \pi^{-1}(U)$.

Definition. The tangent bundle functor associates with an $\mathcal{A}$-manifold M its tangent bundle TM while the image of a morphism $\Phi : W \subset P \times M \to N$ is the smooth family of all *generalized tangent maps*, i.e., $T\Phi : \{(p, h) : (p, \pi(h)) \in W\} \subset P \times TM \to TN$ is given in fibered coordinates by

$$\iota\left(h^i \cdot \partial_{x^i}|_x\right)(T\Phi)_p = h^i \cdot \left((\partial_{x^i}\Phi^j)(p, x)\right) \cdot \partial_{y^j}\big|_{\Phi(p,x)} \ .$$

The fact that T is a functor is an immediate consequence of the chain rule. The regularity and locality conditions are obvious from the local expression of $T\Phi$ because there is no derivative in the direction of the parameter p.

Remark. Comparing the local expression of $T\Phi$ with the definition of the maps Ψ_{ba} shows that the transition functions of the tangent bundle correspond to the tangent maps $T\varphi_{ba}$ of the transition functions between charts.

Smooth families of vector fields

Smooth families of sections of the tangent bundle, are called (smooth families of) *vector fields*. As shown in [?, Chapter V], there is a one-to-one correspondence between (unparametrized) smooth vector fields and $\mathbb{R}$-linear graded derivations of the algebra $\mathrm{C}^\infty(M)$.

Now from a smooth family $X \in \widetilde{\Gamma}(TM)$ of vector fields, we can define a map $D_X : \widetilde{\mathrm{C}}^\infty(M) \to \widetilde{\mathrm{C}}^\infty(M)$ by setting, in local coordinates,

$$D_X(f)(p,p',x) = \sum_i X^i(p,x) \cdot (\partial_{x^i} f)(p',x) \ ,$$

if $X^i(p,x) = X^i(p,x) \cdot \partial_{x^i}|_x$. If $X \in \widetilde{\Gamma}(TM)$ and $f, g \in \widetilde{\mathrm{C}}^\infty(M)$ are such that all X_p and all f_p are homogeneous, we have

$$D_X(f \cdot g)(p,p',p'',x) = D_X(f)(p,p',x) \cdot g(p'',x) + (-1)^{\varepsilon(X_p).\varepsilon(f_{p'})} f(p',x) \cdot D_X(g)(p,p'',x) \ ,$$

so that a smooth family of even vector fields can be seen as a smooth family of “derivations”.

2.1.4 Natural affine bundle functors

Definition. A *natural affine bundle functor* in graded dimension $n|m$ is a natural bundle functor $\mathcal{F} : \mathrm{Man}_{n|m}(\mathcal{A}) \to \mathrm{Fib}_{n|m}(\mathcal{A})$ such that all bundles $\mathcal{F}M$ are affine bundles while all maps $\mathcal{F}\Phi$ are morphisms of affine bundles.

The space of geometric objects

Definition. A *geometric object of affine type* (or simply, a *affine geometric object*) in graded dimension $n|m$ is an element $\sigma \in \widetilde{\Gamma}(\mathcal{F}M)$ with $\mathcal{F}$ a natural affine bundle functor.

If $\mathcal{F}$ is a natural affine bundle functor, the space $\widetilde{\Gamma}(\mathcal{F}M)$ of geometric object of type $\mathcal{F}$ on M is an affine space modeled on the $\mathrm{C}^\infty(M)$-module $\widetilde{\Gamma}(\vec{\mathcal{F}}M)$ of smooth families of sections of the underlying vector bundle. The affine space structure is defined fiberwise, i.e., we set

$$(\sigma + s)(p,p',x) = \sigma(p,x) + s(p',x) \in \pi_x \ .$$

for all $\sigma \in \Gamma(\mathcal{F}M)$ and all $s \in \Gamma(\vec{\mathcal{F}}M)$.

Remark. Since the fibers π_x of an affine bundle do not come with an origin (because affine transition functions do not preserve the origin of the typical fiber), spaces of affine geometric objects do not come with a canonical element as it was the case for vector geometric objects.

Example: the bundle of connections

Definition. Let M be an $\mathcal{A}$-manifold modeled on an $\mathcal{A}$-vector space E and let $\{(U_a \subset M, \varphi_a : U_a \to E_0)\}$ be the atlas of all charts for M (i.e., the differentiable structure). We define maps $\mathcal{C}(\varphi_{ba}) : \varphi_a(U_a \cap U_b) \to \mathrm{Aff}(E^* \otimes \mathrm{End}_R E)$ by analogy with the transformation law of the Christoffel symbols of a linear connection on M under a change of chart from (U_a, φ_a) to (U_b, φ_b): in terms of a basis $(e_1, ..., e_{m+n})$ of E, we set

$$\iota\left(\sum {}^s e \otimes {}^t e \cdot \Gamma^r_{st} \otimes e_r\right)(\mathcal{C}(\varphi_{ba}))(x) \quad = \quad \sum {}^v e \otimes {}^w e \cdot \bar{\Gamma}^u_{vw} \otimes e_u \ ,$$

with

$$\bar{\Gamma}^u_{vw} = (-1)^{\varepsilon_v(\varepsilon_t+\varepsilon_w)} \cdot (\partial_{y^w}\varphi^t_{ab}(\varphi_{ba}(x)))) \cdot (\partial_{y^v}\varphi^s_{ab}(\varphi_{ba}(x)))) \cdot \Gamma^r_{st} \cdot (\partial_{x^r}\varphi^u_{ba}(x)) \\ + (\partial^2_{y^v y^w}\varphi^k_{ab}(\varphi_{ba}(x))) \cdot (\partial_{x^k}\varphi^u_{ba}(x)) \ . \quad (2.3)$$

Now we can define the functions $\psi_{ba} : U_b \cap U_a \to \mathrm{Aff}(E^* \otimes \mathrm{End}_R(E))$ by

$$\psi_{ba} = \mathcal{C}(\varphi_{ba}) \circ \varphi_a \ .$$

It can be checked using the chain rule that these functions satisfy the cocycle conditions, but this is actually an immediate consequence of the fact that the transformation law is that of Christoffel symbols under the effect of a coordinate change (see formula 3.11). The affine bundle $\pi : \mathcal{C}M \to M$ so obtained is called the *bundle of connections* of M.

The bundle functor $\mathcal{C}$ associates with an $\mathcal{A}$-manifold M its bundle of connections while the image of a morphism $\Phi : W \subset P \times M \to N$ is the smooth collection $\mathcal{C}\Phi : \{(p, \Gamma) : (p, \pi(\Gamma)) \in W\} \subset P \times \mathcal{C}M \to \mathcal{C}N$ defined in fibered coordinates as

$$\iota\left(\mathrm{d}x^k\big|_x \otimes \mathrm{d}x^j\big|_x \cdot \Gamma^i_{jk} \otimes \partial_{x^k}|_x\right)(\mathcal{C}\Phi)_p = \sum \mathrm{d}y^w|_x \otimes \mathrm{d}y^v|_x \cdot \bar{\Gamma}^u_{vw} \otimes \partial_{y^u}|_x \quad (2.4)$$

with

$$\bar{\Gamma}^u_{vw} = (-1)^{\varepsilon_v(\varepsilon_r+\varepsilon_u)} \cdot (\partial_{y^w}\widetilde{\Phi}^{-1,k}(\tilde{\Phi}(p,x)))(\partial_{y^v}\widetilde{\Phi}^{-1,j}(\tilde{\Phi}(p,x))) \cdot \Gamma^i_{jk} \cdot (\partial_{x^i}\Phi^u(p,x)) \\ + (\partial^2_{y^v y^v}\widetilde{\Phi}^{-1,i}(\tilde{\Phi}(p,x))) \cdot (\partial_{x^i}\Phi^u(p,x)) \ , \quad (2.5)$$

where $\widetilde{\Phi}^{-1}$ stands for a local inverse of $\tilde{\Phi} : (p, x) \mapsto (p, \Phi_p(x))$.

Remark. Comparing the definition of the lifted map $\mathcal{C}\Phi$ with the definition of the transition functions for the bundle $\mathcal{C}M$ shows that the latter correspond to the lifted maps of the transition functions between the charts.

Smooth families of covariant derivatives

From a smooth family $\sigma : P \times M \to (\mathcal{C}M)^{(0)}$ of even sections of $\mathcal{C}M$, we can define a map $\nabla : \Gamma(TM) \times \Gamma(TM) \to \widetilde{\Gamma}(TM)$ by setting, in local coordinates,

$$(\iota(X,Y)\nabla^\sigma)(p,x) = \sum_{ij} X^j(x) \cdot \frac{\partial Y^i}{\partial x^j}(x) \cdot \partial_{x^i}|_x + \sum_{i,j,k} (-1)^{\varepsilon_j(\varepsilon(Y)+\varepsilon_k)} \cdot X^j(x) \cdot Y^k(x) \cdot \Gamma^i_{jk}(p,x) \cdot \partial_{x^i}|_x \ . \tag{2.6}$$

In other words, a smooth family of even connections defines a smooth family of covariant derivatives.

Proposition 5. *There is a one-to-one correspondence between even sections of $\mathcal{C}M$ and covariant derivatives on M.*

Proof. With an even section σ of $\mathcal{C}M$, we associate the covariant derivative whose Christoffel symbols in a chart (U_a, φ_a) of M are the local components Γ^i_{jk} of the local expression of σ in the local adapted coordinates on $\mathcal{C}M$ associated with (U_a, φ_a).

This correspondence is well-defined because the transformation law (2.3) of the local components of sections is the same as the transformation law of the Christoffel symbols of a covariant derivative under a change of local coordinates.

This correspondence is also bijective because a covariant derivative on M is completely determined by its Christoffel symbols in an atlas of M. □

Remark. If $\Phi : M \to N$ is a diffeomorphism, then for any smooth section $\sigma \in \Gamma(\mathcal{C}M)$, the covariant derivative corresponding to $\mathcal{C}\Phi \circ \sigma \in \Gamma(\mathcal{C}N)$ is given by

$$\iota(X,Y)\nabla^{\mathcal{C}\Phi\circ\sigma} = T\Phi\left(\iota(T\Phi^{-1} \circ X, T\Phi^{-1} \circ Y)\nabla^\sigma\right).$$

In other words, the action of the functor $\mathcal{C}$ on morphisms correspond to the push-forward of covariant derivatives along (local) diffeomorphisms.

2.2 The Lie derivative of vector/affine geometric objects is a derivation along the flow of a vector field.

2.2.1 The flow of a smooth family of even vector fields

Let $X : P \times M \to TM$ be a smooth family of even vector fields on M. From X, we can define an even vector field $\hat{X} \in \Gamma(T(P \times M))$ by setting $\hat{X}(p, x) = \mathbf{0}_p + X_p(x)$. The flow

$$\Phi_{\hat{X}} : W_{\hat{X}} \subset (\mathcal{A}_0 \times P) \times M \to P \times M$$

satisfies $T\Phi_{\hat{X}} \circ \partial_t = \hat{X} \circ \Phi_{\hat{X}}$ and $\Phi_{\hat{X}}(0, \cdot, \cdot) = \mathrm{id}_{P\times M}$. It is of the form

$$\Phi_{\hat{X}}(p, t, x) = (p, \pi_M \circ \Phi_{\hat{X}}(p, t, x)) \ .$$

By definition , the *flow of the smooth family* X is the map

$$\Phi_X = \pi_M \circ \Phi_{\hat{X}} : W_{\hat{X}} \subset (\mathcal{A}_0 \times P) \times M \to M \ .$$

It is an element of $\widetilde{\mathrm{Hom}}_{n|m}(M, M)$.

2.2.2 The Lie derivative of vector geometric objects

Differentiating with respect to the time parameter

Let $\pi : E_\pi \to M$ be a vector bundle. Given a smooth family of sections $\sigma : W \subset (\mathcal{A}_0 \times P) \times M \to E_\pi$, it is possible (see [**?**, V.3.6]) to define the smooth family of sections

$$\partial_t \cdot \sigma : W \subset (\mathcal{A}_0 \times P) \times M \to E_\pi \ .$$

Locally, in terms of a set $\{\mathbf{e}_j \in \Gamma_U(E_\pi)\}$ of local trivializing sections, $\partial_t \cdot \sigma$ is given by

$$(\partial_t \cdot \sigma)|_U \,(t, p, x) = \sum_j \partial_t(\sigma^j)(t, p, x) \cdot \mathbf{e}_j(x) \ ,$$

if $\sigma|_U = \sum_j \sigma^j \cdot \mathbf{e}_j$.

Remark. Remember that the function $\partial_t(\sigma^j)$ is obtained by differentiating with respect to the time coordinate each of the ordinary smooth functions on $\mathbb{R} \times \mathbf{B}P \times \mathbf{B}M$ appearing in the Taylor expansion of the local expressions of σ^j in charts (see Subsection A.2.2).

Differentiating along the flow

Let $\mathcal{F}$ be a natural vector bundle functor. Given a smooth family of even vector fields, $X : P \times M \to TM^{(0)}$, and a smooth family of sections $\sigma : P' \times M \to \mathcal{F}M$, we can form the smooth family of local sections

$$\Phi_X^* \sigma : W_{\hat{X}} \subset (\mathcal{A}_0 \times P \times P') \times M \to \mathcal{F}M \ , \ (t, p, p', x) \mapsto (\mathcal{F}\Phi_X)_{(-t,p)} \circ \sigma_{p'} \circ \Phi_{X,(t,p)}(x) \ .$$

Definition. The *Lie derivative* of a smooth section $\sigma \in \Gamma(\mathcal{F}M)$ in the direction of a smooth family of even vector fields, $X : P \times M \to TM^{(0)}$, is the smooth family $\mathrm{L}_X \sigma \in \tilde{\Gamma}(\mathcal{F}M)$ whose value at $(p, x) \in W_{\hat{X}}$ is given by

$$(\mathrm{L}_X \sigma)(p, x) = (\partial_t \cdot (\Phi_X^* \sigma))\,(0, p, x) \in \mathcal{F}_x M \ .$$

Example : the Lie derivative of smooth functions

If $X : P \times M \to TM^{(0)}$ is a smooth family of even vector fields on M, then for any $f \in \mathrm{C}^\infty(M)$, we have $\mathrm{L}_X f = D_X(f)$. Indeed, it follows from the definitions and the chain rule that

$$\begin{aligned}
(\mathrm{L}_X f)(p, x) &= (\partial_t \cdot (f \circ \Phi_X))\,(0, p, x) \\
&= \sum_i \left(\frac{\partial \Phi_X^i}{\partial t}(0, p, x) \right) \cdot \left(\frac{\partial f}{\partial x^i}(\Phi_X(0, p, x)) \right) \\
&= \sum_i X^i(p, x) \cdot \frac{\partial f}{\partial x^i}(x) \ ,
\end{aligned}$$

if X reads as $X(p, x) = \sum_i X^i(p, x) \cdot \partial_{x^i}|_x$.

Example : the Lie derivative of smooth vector fields

If $X : P \times M \to TM$ is a smooth family of even vector fields on M, then for any $Y \in \Gamma(TM)$, we can show using the chain rule that

$$(\mathrm{L}_X Y)(p, x) = \sum_{i=1}^{n+m} \left(X^j(p, x) \cdot \frac{\partial Y^i}{\partial x^j}(x) - Y^j(x) \cdot \frac{\partial X^i}{\partial x^j}(p, x) \right) \cdot \partial_{x^i}|_x \ .$$

In particular, $(\mathrm{L}_X Y)_p = [X_p, Y]$ for all $p \in \mathbf{B}P$.

2.2.3 The Lie derivative of affine geometric objects

Differentiating with respect to the time parameter

Let $\pi : Z_\pi \to M$ be an affine bundle. Given a smooth family of sections, $\sigma : W \subset (\mathcal{A}_0 \times P) \times M \to Z_\pi$, it is possible to define the smooth family of sections (of $\vec{\pi}$)

$$\partial_t \cdot \sigma : W \subset (\mathcal{A}_0 \times P) \times M \to E_{\vec{\pi}} \ .$$

Given a local section $\mathbf{a_0} \in \Gamma_U(Z_\pi)$ and a set $\{\mathbf{e}_j \in \Gamma_U(E_{\vec{\pi}})\}$ of local trivializing sections, if $\sigma|_U = \mathbf{a}_0 + \sum_j \sigma^j \cdot \mathbf{e}_j$, then $\partial_t \cdot \sigma$ is given by

$$(\partial_t \cdot \sigma)|_U \, (t,p,x) = \sum_j \partial_t \sigma^j(t,p,x) \cdot \mathbf{e}_j(x) \ ,$$

Differentiating along the flow

Definition. Let $\mathcal{F}$ be a natural affine bundle functor. The *Lie derivative* of $\sigma \in \Gamma(\mathcal{F}M)$ in the direction of the smooth family $X \in \tilde{\Gamma}(TM)$ is the smooth family $\mathrm{L}_X\sigma \in \tilde{\Gamma}(\vec{\mathcal{F}}M)$ whose value at $(p,x) \in W_{\hat{X}}$ is given by

$$(\mathrm{L}_X\sigma)(p,x) = \left(\partial_t \cdot (\Phi_X^*\sigma)\right)(0,p,x) \in \mathcal{F}_x M \ ,$$

where $\Phi_X^*\sigma : W_{\hat{X}} \subset (\mathcal{A}_0 \times P) \times M \to \mathcal{F}M$ is defined as in the vector case.

Example : the Lie derivative of covariant derivatives

If $X : P \times M \to TM$ is a smooth family of even vector fields on M, then for any $Y \in \Gamma(TM)$,

$$(\mathrm{L}_X\nabla)(p,x) = \sum_{i,j,k=1}^{n+m} \mathrm{d}x^k\big|_x \otimes \mathrm{d}x^j\big|_x \cdot S^i_{jk}(p,x) \otimes \partial_{x^i}|_x$$

with

$$S^i_{jk}(p,x) = X^l(p,x) \cdot \frac{\partial \Gamma^i_{jk}}{\partial x^l}(x) - \Gamma^l_{jk}(x) \cdot \frac{\partial X^i}{\partial x^l}(p,x) + (-1)^{j(l+k)} \left(\frac{\partial X^l}{\partial x^k}(p,x)\right) \cdot \Gamma^i_{jl}(x) + \left(\frac{\partial X^l}{\partial x^j}(p,x)\right) \cdot \Gamma^i_{lk}(x) + \frac{\partial^2 X^i}{\partial x^j \partial x^k}(p,x) \quad (2.7)$$

In particular, $(\mathrm{L}_X\nabla)_p(Y,Z) = [X_p, \nabla_Y Z] - \nabla_{[X_p,Y]}Z - \nabla_Y[X_p,Z]$ for all $p \in \mathbf{B}P$.

2.3 Natural operators on $\mathcal{A}$-manifolds transform *smooth families of* sections.

2.3.1 Natural operators on $\mathcal{A}$-manifolds

By definition, *natural operators* from $\mathcal{F}$ to $\mathcal{G}$ transform geometric objects of type $\mathcal{F}$ into geometric objects of type $\mathcal{G}$. In our context, natural operators are thus operators acting between *smooth families of* sections of natural bundles.

DEFINITION. A *natural operator* from $\mathcal{F}$ to $\mathcal{G}$ is a collection of operators

$$D = \{D_M : \widetilde{\Gamma}(\mathcal{F}M) \to \widetilde{\Gamma}(\mathcal{G}M)\}_{M\in \mathrm{Ob}(\mathrm{Man}_{n|m}(\mathcal{A}))}$$

with the following properties.

(R) Regularity: The image of a smooth family

$$\sigma : W \subset P \times M \to \mathcal{F}M$$

is a smooth family

$$D_M(\sigma) : W \subset P \times M \to \mathcal{G}M \ .$$

Moreover, $D_M(\sigma)_p$ only depends on σ_p in the sense that if $\sigma' : W' \subset P' \times M \to \mathcal{F}M$ is such that $\sigma_p = \sigma'_{p'}$ for some $p \in P$ and some $p' \in P$, then we must have

$$D_M(\sigma)_p = D_M(\sigma')_{p'} \ .$$

In particular, (unparametrized) smooth sections are transformed into (unparametrized) smooth sections and for any $p \in \mathbf{B}P$, we have $D(\sigma)_p = D(\sigma_p) \in \Gamma(\mathcal{F}M)$.

(L) Locality: For any $\sigma : W \subset P \times M \to \mathcal{F}M$ and any open subset U of M, we have

$$D_U(\sigma|_U) = (D_M\sigma)|_U \ ,$$

where $\sigma|_U$ stands for the restriction of X to $W \cap (P \times U)$.

(N) Naturality: For any $\Phi \in \widetilde{\mathrm{Hom}}_{n|m}(M, N)$, $\sigma \in \widetilde{\Gamma}(\mathcal{F}M)$ and $\sigma' \in \widetilde{\Gamma}(\mathcal{F}N)$, we have

$$\sigma'_{p'} \circ \Phi_q = \mathcal{F}\Phi_q \circ \sigma_p \quad \Rightarrow \quad D_N(\sigma')_{p'} \circ \Phi_q = (\mathcal{G}\Phi)_q \circ D_M(\sigma)_p \ .$$

We say that D sends Φ-related objects of type $\mathcal{F}$ to Φ-related objects of type $\mathcal{G}$.

2.4 Natural linear operators are differential operators.

We aim to obtain a Peetre-like theorem for linear (super) natural operators. Let us first recall Peetre theorem for local linear operators over classical smooth manifolds.

THEOREM 6 (Classical Peetre theorem). *Let $\pi : E_\pi \to M$ and $\pi' = E_{\pi'} \to M$ be vector bundles. If $D : \Gamma(E_\pi) \to \Gamma(E_{\pi'})$ is a local $\mathbb{R}$-linear operator, then D reads in local adapted coordinates as*

$$D(\sigma)^i(x) = \sum_{|\alpha|\leqslant k} D^i_{\alpha,j}(x) \cdot \left(\frac{\partial^{|\alpha|}\sigma^j}{\partial x^\alpha}(x) \right) ,$$

where α is a multi-index, $|\alpha| = \sum_r \alpha_r$ and each $D^i_{\alpha,j}$ is a local smooth function on M.

2.4.1 Peetre theorem on $\mathcal{A}$-manifolds

Linear operators between functions

THEOREM 7. *Let M be an $\mathcal{A}$-manifold. If $D : \mathrm{C}^\infty(M) \to \mathrm{C}^\infty(M)$ is a local $\mathbb{R}$-linear operator, then D reads in local graded coordinates as*

$$D(f)(y) = \sum_{|\alpha|\leqslant k} D_\alpha(y) \cdot \left(\frac{\partial^{|\alpha|} f}{\partial y^\alpha}(y) \right) ,$$

where α is a multi-index with $\alpha_{n+1}, \dots, \alpha_{n+m} \in \{0, 1\}$, $|\alpha| = \sum_{i=1}^{n+m} \alpha_i$ and each D_α is a local smooth function.

Proof. Given a chart $(U_a, \varphi_a : U_a \to O_a)$ of M, we have an isomorphism of real superalgebras

$$\{ f|_{U_a} : f \in \mathrm{C}^\infty(M)\} \simeq \Gamma(\Lambda(\mathrm{pr}_1 : \mathbf{B}O_a \times \mathbb{R}^m \to \mathbf{B}O_a)) .$$

If $f \in \mathrm{C}^\infty(M)$ reads in $(U_a, \varphi_a : U_a \to O_a)$ as

$$f(x, \xi) = \sum_{r=0}^{m} \sum_{1\leqslant i_1<\dots<i_r\leqslant m} \xi^{i_1} \cdot \ \cdots \ \cdot \xi^{i_r} \cdot \widetilde{f_{i_1\dots i_r}}(x) ,$$

the corresponding form $\alpha_f \in \Gamma(\Lambda(\mathrm{pr}_1 : \mathbf{B}O_a \times \mathbb{R}^m \to \mathbf{B}O_a))$ is given by

$$\alpha_f(x) = \sum_{r=0}^{m} \sum_{1\leqslant i_1<\dots<i_r\leqslant m} \xi^{i_1} \wedge \dots \wedge \xi^{i_r} \cdot f_{i_1\dots i_r}(x) ,$$

where the ξ^{i_j} now stand for a basis of $\mathbb{R}^m$.

Through this correspondence, a local $\mathbb{R}$-linear operator $D : \mathrm{C}^\infty(M) \to \mathrm{C}^\infty(M)$ induces a local $\mathbb{R}$-linear operator $\widehat{D} : \Gamma(\Lambda(\mathrm{pr}_1 : \mathbf{B}O_a \times \mathbb{R}^m \to \mathbf{B}O_a)) \to \Gamma(\Lambda(\mathrm{pr}_1 : \mathbf{B}O_a \times \mathbb{R}^m \to \mathbf{B}O_a))$ defined by

$$\widehat{D}(\sigma_f) = \sigma_{D(f)} \ .$$

Note that $\widehat{D}$ is well-defined thanks to the locality of D. Moreover, $\widehat{D}$ is local and $\mathbb{R}$-linear because both D and the correspondence $f|_{U_a} \leftrightarrow \sigma_f$ are.

In view of (the classical) Peetre theorem, $\widehat{D}(\sigma_f)$ is locally of the form

$$\widehat{D}(\sigma_f)(x) = \sum_{|\beta|=0}^{k'} \sum_{r,r'=0}^{m} \sum_{\substack{1\leqslant i_1<\cdots<i_r\leqslant m \\ 1\leqslant j_1<\cdots<j_{r'}\leqslant m}} \xi^{j_1}\wedge\cdots\wedge\xi^{j_{r'}} \cdot \widehat{D}^{i_1\ldots i_r}_{\beta,j_1\ldots j_{r'}}(x)\cdot\left(\frac{\partial^{|\beta|} f_{i_1\ldots i_r}}{\partial x^\beta}(x)\right) \ . \quad (2.8)$$

In order to make f appear in the right-hand side of 2.8, we use the identity

$$\left(\frac{\partial^{|\beta|} f_{i_1\ldots i_r}}{\partial x^\beta}(x)\right)^{\sim}(x) = (-1)^{\frac{1}{2}r(r-1)} \cdot \left(\frac{\partial^{|\beta|}}{\partial x^\beta}\cdot\frac{\partial}{\partial \xi^{i_1}}\cdot\ \cdots\ \cdot\frac{\partial}{\partial \xi^{i_r}}\cdot f\right)(x,\xi) \ .$$

For any $\alpha \in \mathbb{N}^{n+m}$ with $|(\alpha_1,\ldots,\alpha_n)| \leqslant k'$ and $\alpha_{n+1},\ldots,\alpha_{n+m} \in \{0,1\}$, we set

$$D_\alpha(x,\xi) = \sum_{r'=0}^{m} \sum_{1\leqslant j_1<\cdots<j_{r'}\leqslant m} (-1)^{\frac{1}{2}r(r-1)}\cdot\xi^{j_1}\cdot\ \cdots\ \cdot\xi^{j_{r'}}\cdot\left(\widehat{D}^{i_1\ldots i_r}_{(\alpha_1\ldots\alpha_n),j_{i_1}\cdots j_{i_{r'}}}\right)^{\sim}(x) \ ,$$

where $i_1,\ldots,i_r$ are the indices $i_j \in \{1,\ldots,m\}$ for which $\alpha_{n+i_j} \neq 0$ (i.e. $\alpha_{n+i_j} = 1$).

Now if for any $\alpha \in \mathbb{N}^{n+m}$ with $|(\alpha_1,\ldots,\alpha_n)| \geq k'$, we set $D_\alpha = 0$, then the result follows from formula (2.8) through the correspondence $\hat{D}(\sigma_f) \leftrightarrow \sigma_{D(f)}$:

$$\begin{aligned} D(f)(x,\xi) &= \sum_{|\beta|=0}^{k'} \sum_{r,r'=0}^{m} \sum_{\substack{1\leqslant i_1<\cdots<i_r\leqslant m \\ 1\leqslant j_1<\cdots<j_{r'}\leqslant m}} \xi^{j_1}\cdot\ \cdots\ \cdot\xi^{j_{r'}}\cdot\left(\widehat{D}^{i_1\ldots i_r}_{\beta,j_1\ldots j_{r'}}\right)^{\sim}(x)\cdot\left(\left(\frac{\partial^{|\beta|} f_{i_1\ldots i_r}}{\partial x^\beta}\right)^{\sim}(x)\right) \\ &= \sum_{|\alpha|\leqslant k'+m} D_\alpha(x,\xi)\cdot\left(\frac{\partial^{|\alpha|} f}{\partial(x,\xi)^\alpha}(x,\xi)\right) \ . \end{aligned}$$

□

Corollary 8. *If $D : \widetilde{\mathrm{C}}^\infty(M) \to \widetilde{\mathrm{C}}^\infty(M)$ is a regular local (i.e., $f_p|_U = 0 \Rightarrow D(f)_p|_U = 0$) even (i.e. $\epsilon(D(f)_p) = \epsilon(f_p)$ whenever f_p is homogeneous) left $\mathcal{A}$-linear operator, then D reads in local graded coordinates as*

$$D(f)(y) = \sum_{|\alpha| \leqslant k} D_\alpha(y) \cdot \left(\frac{\partial^{|\alpha|} f}{\partial y^\alpha}(y) \right)$$

where α is a multi-index with $\alpha_{n+1}, \ldots, \alpha_{n+m} \in \{0, 1\}$, $|\alpha| = \sum_{r=1}^{n+m} \alpha_r$ and each $D^i_{\alpha,j}$ is a local smooth function.

Proof. Let $f : W \subset P \times M \to \mathcal{A}$ be a smooth family of functions. In a local chart $(V_a \times U_a, \psi_a \times \varphi_a)$ of $P \times M$, f can be written as

$$\begin{aligned} f(p, \eta; x, \xi) &= \sum_{I,J} \eta^J \cdot \widetilde{f_{J,I}}(p, x) \cdot \xi^I \\ &= \sum_{I,J,K} \eta^J \cdot \frac{(p - \mathbf{B}p)^K}{K!} \cdot \left(\frac{\partial^{|K|} f_{J,I}}{\partial p^K}(\mathbf{B}p, \cdot) \right)^{\sim} (x) \cdot \xi^I \end{aligned}$$

for some local smooth functions $f_{J,I}$ on $\mathbf{B}P \times \mathbf{B}M$. For a fixed parameter in P with coordinates (p, η), the map $f_{(p,\eta)}$ is, in general, not smooth, but it can be written locally as

$$f_{(p,\eta)}\big|_{U_a} = \sum_{J,K} (\mathcal{A} \cdot f_{K,J})_{\left(\eta^J, \frac{(p-\mathbf{B}p)^K}{K!}\right)}$$

where the local smooth functions $f_{K,J} \in \mathrm{C}^\infty(U_a)$ are given by

$$f_{K,J}(x, \xi) = \sum_I \left(\frac{\partial^{|K|} f_{J,I}}{\partial p^K}(\mathbf{B}p, \cdot) \right)^{\sim} (x) \cdot \xi^I$$

Using the regularity, the left $\mathcal{A}$-linearity and the locality of D, we obtain

$$D(f)_{(p,\eta)}\big|_{U_a} = \sum_{J,K} \eta^J \cdot \frac{(p - \mathbf{B}p)^K}{K!} \cdot D(\underline{f}_{K,J})\Big|_{U_a},$$

where $\underline{f}_{K,J}$ stands for a global smooth function such that $\underline{f}_{K,J}|_{U_a} = f_{K,J}$ (multiply $f_{K,J}$ by a plateau function and restricts U_a if necessary). Applying theorem 7, we then get

$$D(f)_{(p,\eta)}(x, \xi) = \sum_{|\alpha| \leqslant k} \sum_{J,K} \eta^J \cdot \frac{(p - \mathbf{B}p)^K}{K!} \cdot D_\alpha(x, \xi) \cdot \left(\frac{\partial^{|\alpha|} f_{K,J}}{\partial (x, \xi)^\alpha}(x, \xi) \right).$$

The result follows from the definition of the smooth function $f_{K,J}$ using the linearity of the even operators $D_\alpha \cdot \left(\frac{\partial^{|\alpha|}}{\partial (x,\xi)^\alpha} \right)$ to reconstruct f in the right-hand side. □

Linear operators between sections of vector bundles

Corollary 9. *Let $\pi : E_\pi \to M$ and $\pi' : E_{\pi'} \to M$ be vector bundles. If D is a regular local (i.e., $\sigma_p|_u = 0 \Rightarrow D(\sigma)_p|_U = 0$) even left $\mathcal{A}$-linear operator from $\widetilde{\Gamma}(E_\pi)$ to $\widetilde{\Gamma}(E_{\pi'})$, then each D_M reads in local adapted coordinates as*

$$D(\sigma)^i(y) = \sum_{|\alpha|\leqslant k} D^i_{\alpha,j}(y)\cdot\left(\frac{\partial^{|\alpha|}\sigma^j}{\partial y^\alpha}(y)\right) ,$$

where α is a multi-index with $\alpha_{n+1},\ldots,\alpha_{n+m}\in\{0,1\}$, $|\alpha| = \sum_{r=1}^{n+m}\alpha_r$ and each $D^i_{\alpha,j}$ is a local smooth function.

Proof. Let $\{\mathbf{e}_j : U_a \subset M \to E_\pi\}$ (resp. $\{\mathbf{f}_i : U_a \subset M \to E_{\pi'}$) be a (finite) set of local trivializing sections of E_π (resp. $E_{\pi'}$) above the domain of a chart (U_a,φ_a) of M.

For each (i,j), we have an even local left $\mathcal{A}$-linear operator

$$D^i_j : \widetilde{\mathrm{C}}^\infty(U_a)\to\widetilde{\mathrm{C}}^\infty(U_a)\ ,\ f\mapsto D_M(f\cdot\mathbf{e}_j)^i\ ,$$

where $D(f\cdot\mathbf{e}_j)^i$ stands for the component of $D(f\cdot\mathbf{e}_j)$ along $\mathbf{f}_i$.

The conclusion follows immediately from Corollary 8:

$$D(\sigma)^i(y) = D^i_j(\sigma^j) = \sum_{|\alpha|\leqslant k} D^i_{j,\alpha}(y)\cdot\left(\frac{\partial^{|\alpha|}\sigma^j}{\partial y^\alpha}(y)\right) .$$

□

2.4.2 Peetre theorem for natural linear operators

Locality

Lemma 10. *Let $\mathcal{F}$ and $\mathcal{G}$ be two natural bundle functors. If D is natural $\mathbb{R}$-linear operator from $\mathcal{F}$ to $\mathcal{G}$, then for any open subset $U\subset M$ such that $\{p\}\times U\subset W$, we have*

$$\sigma_p|_U = 0 \Rightarrow D_M(\sigma)_p|_U = 0\ .$$

Proof. It is a consequence of conditions **(R)** and **(L)**:

$$\sigma_p|_U = 0 \Rightarrow \left(\sigma|_U\right)_p = 0 \Rightarrow \left((D_U(\sigma|_U)\right)_p = 0 \Rightarrow \left(D_M(\sigma)|_U\right)_p = 0 \Rightarrow D_M(\sigma)_p|_U = 0\ .$$

□

Local expression of natural linear operators

PROPOSITION 11. *Let $\mathcal{F}$ and $\mathcal{G}$ be natural vector bundle functors. If D is an even left $\mathcal{A}$-linear natural operator from $\mathcal{F}$ to $\mathcal{G}$, then each D_M reads in local adapted coordinates as*

$$D_M(\sigma)^i(p,y) = \sum_{|\alpha| \leqslant k} D^i_{\alpha,j}(y) \cdot \left(\frac{\partial^{|\alpha|} \sigma^j}{\partial y^\alpha}(p,y) \right) ,$$

where α is a multi-index with $\alpha_{n+1}, \ldots, \alpha_{n+m} \in \{0,1\}$, $|\alpha| = \sum_{r=1}^{n+m} \alpha_r$ and each $D^i_{\alpha,j}$ is a local smooth function.

REMARK. Note that the property (**N**) of natural operators does not play any role for the above proposition to be true. Natural operators are just particular cases of local regular operators. Moreover, it follows from the proof of Corollary 11 that an even left $\mathcal{A}$-linear natural operator is completely determined by its value on (unparametrized) smooth sections.

Chapter 3

Projective Equivalence of Torsion-free Connections in Super Geometry

The concept of projective equivalence of connections goes back to the 1920's, with the study of the so-called "geometry of paths" (see [?, ?, ?] or [?, ?, ?] for a modern formulation).

By definition, two connections are called *projectively equivalent* if they have the same geodesics, up to parametrization. In other words, the geodesics of two equivalent connections are the same, provided that we see them as sets of points, rather than as maps from an open interval of $\mathbf{R}$ into the manifold. In [?], H. Weyl showed that projective equivalence can be rephrased in an algebraic way: two connections are projectively equivalent if and only if the symmetric tensor which measures the difference between them can be expressed by means of a 1-form.

H. Weyl's algebraic characterization of projective equivalence provides a convenient way to transport projective equivalence to the framework of supergeometry: two superconnections are said to be *projectively equivalent* if the (super)symmetric tensor which measures the difference between them can be expressed by means of a (super)1-form.

Remembering the classical picture, it is natural to ask whether it is possible to find a geometric counterpart to the algebraic definition of projective equivalence of superconnections, i.e., a characterization in terms of supergeodesics. In this chapter, we first answer this question in the affirmative (cf. [?]). Then, in the perspective of Chapter 4, we show that the vector fields obtained in Chapter 1 by means of the projective embedding preserve the projective class of the canonical flat connection on the flat superspace.

About supergeodesics

As in the classical case, we define, in section 3.1, supergeodesics associated with a superconnection ∇ on a supermanifold M as being the projections onto M of the integral curves of a vector field G^{∇} on the tangent bundle TM: the *geodesic vector field* of ∇. In section 3.2 we then define the notion of reparametrization of a geodesic and establish that two connections ∇ and $\widehat{\nabla}$ on a supermanifold M have the same geodesics up to parametrization if and only if there is an even 1-form α such that

$$\widehat{\nabla}_X Y = \nabla_X Y + X \cdot \iota(Y)\alpha + (-1)^{\varepsilon(X)\cdot\varepsilon(Y)} \cdot Y \cdot \iota(X)\alpha \quad \forall X, Y \in \Gamma(TM),$$

thus showing that Weyl's characterization also holds in supergeometry.

Our approach to supergeodesics differs from that of O. Goertsches [?]. In particular, our equations for supergeodesics are the natural generalization of the classical ones. Actually, our approach is nearly identical to that recently proposed by S. Garnier and T. Wurzbacher in [?], where they consider supergeodesics associated with a Levi-Civita superconnection.

In fact, beyond the fact that they restrict to the Riemannian setting where we consider arbitrary connections, the main difference between Garnier-Wurzbacher's supergeodesics and ours lies in the way we interpret geodesics. In [?], geodesics are seen as individual supercurves on M (which obliges them to add sometimes an arbitrary additional supermanifold S, in particular to specifiy intial conditions), whereas we focus on the geodesic flow as a whole, seen as the projection on M of the flow of an even vector field on the tangent bundle TM.

Supercurves should be images of 1-dimensional manifolds, but as it is well-known, the theory of supercurves with a single parameter turns out to be very shallow: supercurves in a single even parameter are reduced to ordinary curves in the body of the manifold while supercurves in a single odd parameter are simply odd straight lines. In order to overcome these limitations, we choose to change the viewpoint. Usually curves do not come singly, they appear in families. And in particular the integral curves of a vector field on a supermanifold N should not be seen as a simplistic collection of curves, but as a map (the flow) defined on (an open subset of) $\mathbf{R} \times N$ ([1]), incorporating the initial condition in the domain of the map. And indeed, the flow of a vector field is jointly smooth in the time parameter t and the initial condition $n \in N$. In the simplistic viewpoint one writes $\gamma_n(t)$ for an integral curve with initial condition $n \in N$, whereas in the viewpoint of a flow one rather writes $\varphi_t(n)$ or $\varphi(t, n)$. Roughly speaking, we could say that our change of viewpoint enlarges in a natural way (we do not add an arbitrary manifold S as in [?]) the domain of supercurves so that it is now possible to get supercurves with desirable properties.

[1] In fact, rather $\mathcal{A}_0 \times N$ than $\mathbb{R} \times N$ since maps defined on $\mathcal{A}_0 \times N$ live in the category of supermanifolds while containing the same information as maps defined on $\mathbb{R} \times N$.

Contents

3.1 Supergeodesics of a torsion free connection on an $\mathcal{A}$-manifold are projections of the flow of a vector field on the tangent bundle.

Before dealing with the specific problem of geodesics on a supermanifold, we first recall some general definitions and facts about connections on $\mathcal{A}$-manifolds. Then we attack the problem of defining super geodesics: we associate with any connection a so-called *geodesic vector field* on the tangent bundle, whose flow equations are the straightforward super analogs of the classical geodesic equations.

3.1.1 Connections on $\mathcal{A}$-manifolds

Definition ([**?**, VII§6]). A *connection* (or *covariant derivative*) on an $\mathcal{A}$-manifold M is a map $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ such that

(i) ∇ is bi-additive (in $\Gamma(TM)$ and $\Gamma(TM)$) and even;

(ii) for $X \in \Gamma(TM)$, $s \in \Gamma(TM)$ and $f \in C^\infty(M)$ we have

$$\nabla_{fX} s = f \cdot \nabla_X s\ ; \tag{3.1}$$

(iii) for homogeneous $X \in \Gamma(TM)$, $s \in \Gamma(TM)$ and $f \in C^\infty(M)$ we have

$$\nabla_X(fs) = D_X(f) \cdot s + (-1)^{\varepsilon(X)\cdot\varepsilon(f)} f \cdot \nabla_X s\ . \tag{3.2}$$

Lemma 12. *If ∇ and $\widehat{\nabla}$ are connections in TM, the map $S : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ defined by*

$$S(X,s) = \nabla_X s - \widehat{\nabla}_X s \tag{3.3}$$

*is even and bilinear over $C^\infty(M)$. In other words, S is a "tensor", i.e., can be seen as a section of the bundle $TM^* \otimes \mathrm{End}(TM)$ [**?**, IV§5].*

Lemma 13. *If ∇ is a connection on M, then the map $T : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ defined on homogeneous $X, Y \in \Gamma(TM)$ by*

$$T(X,Y) = \nabla_X Y - (-1)^{\varepsilon(X)\cdot\varepsilon(Y)} \cdot \nabla_Y X - [X,Y] \tag{3.4}$$

*is even, graded anti-symmetric and bilinear over $C^\infty(M)$. In other words, T is a "tensor", i.e., can be seen as a section of the bundle $\bigwedge^2 TM^* \otimes TM$, i.e., as a 2-form on M with values in TM [**?**, IV§5].*

Torsion-free connections

Definition. A connection ∇ in TM is said to be *torsion-free* if the tensor T is identically zero.

Corollary 14. *If ∇ and $\widehat{\nabla}$ are torsion-free connections in TM, the tensor $S = \nabla - \widehat{\nabla} : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ is graded symmetric.*

Let ∇ be a connection in TM (we also say a connection *on* M). On a local chart for M with coordinates $x = (x^1, \dots, x^{n+m})$ we define the Christoffel symbols Γ^i_{jk} of ∇ by

$$\Gamma^i_{jk}(x) = \iota(\nabla_{\partial_{x^j}} \partial_{x^k}) \ \mathrm{d}x^i\big|_x \tag{3.5}$$

with parity $\varepsilon\left(\Gamma^i_{jk}\right) = \varepsilon_i + \varepsilon_j + \varepsilon_k$.([2]) It follows that for homogeneous vector fields $X = \sum_i X^i \cdot \partial_{x^i}$ and $Y = \sum_i Y^i \cdot \partial_{x^i}$, we have

$$\nabla_X Y = \sum_{ij} X^j \cdot \frac{\partial Y^i}{\partial x^j} \cdot \partial_{x^i} + \sum_{i,j,k} (-1)^{\varepsilon_j(\varepsilon(Y)+\varepsilon_k)} \cdot X^j \cdot Y^k \cdot \Gamma^i_{jk} \cdot \partial_{x^i} \ . \tag{3.6}$$

When the vector field X is even, we have $(-1)^{\varepsilon_j(\varepsilon(Y)+\varepsilon_k)} = (-1)^{(\varepsilon(X)+\varepsilon_j)(\varepsilon(Y)+\varepsilon_k)}$ and in that case the above formula can be written without signs as

$$\nabla_X Y = \sum_{ij} X^j \cdot \frac{\partial Y^i}{\partial x^j} \cdot \partial_{x^i} + \sum_{i,j,k} Y^k \cdot X^j \cdot \Gamma^i_{jk} \cdot \partial_{x^i} \ . \tag{3.7}$$

Corollary 15. *If ∇ and $\widehat{\nabla}$ are connections on M with Christoffel symbols Γ^i_{jk} and $\widehat{\Gamma}^i_{jk}$ respectively, the tensor S reads locally as*

$$S = \sum_{i,j,k} \mathrm{d}x^k \otimes \mathrm{d}x^j \cdot \left((\Gamma^i_{jk} - \widehat{\Gamma}^i_{jk}\right)) \otimes \partial_{x^i} \tag{3.8}$$

while the tensor T is given by

$$\begin{aligned} T &= \sum_{i,j,k} \mathrm{d}x^k \wedge \mathrm{d}x^j \cdot \Gamma^i_{jk} \otimes \partial_{x^i} \\ &= \tfrac{1}{2} \cdot \sum_{i,j,k} \mathrm{d}x^k \wedge \mathrm{d}x^j \cdot \big((\Gamma^i_{jk} - (-1)^{\varepsilon_j \varepsilon_k} \cdot \Gamma^i_{kj})\big) \otimes \partial_{x^i} \end{aligned}$$

In particular ∇ is torsion-free if and only if the Christoffel symbols are graded symmetric in the lower indices, i.e.,

$$\Gamma^i_{jk} = (-1)^{\varepsilon_j \varepsilon_k} \cdot \Gamma^i_{kj} \ .$$

[2] Remember that, by definition, we have $\iota(\partial_{x^j})\mathrm{d}x^i = \delta^i_j$.

Transformation law of Christoffel symbols

If $y = (y^1, \dots, y^{n+m})$ is another local system of coordinates, we can consider the Christoffel symbols $\widetilde{\Gamma}^i_{jk}$ in terms of these coordinates:

$$\widetilde{\Gamma}^i_{jk}(y) = \iota(\nabla_{\partial_{y^j}} \partial_{y^k})\ \mathrm{d}y^i\Big|_y \tag{3.9}$$

Now let $x_0 \in M$ be the point in M whose coordinates are x or y depending upon the choice of local coordinate system. As tangent vectors transform as $\partial_{x^i}|_{x_0} = \sum_p (\partial_{x^i} y^p)(x) \cdot \partial_{y^p}|_{x_0}$, it follows that the relation between Γ and $\widetilde{\Gamma}$ is given by

$$\begin{aligned}
\sum_i \Gamma^i_{jk}(x) \cdot \partial_{x^i}|_{x_0} &= \left((\nabla_{\partial_{x^j}} \left((\sum_r (\partial_{x^k} y^r)(x) \cdot \partial_{y^r} \right)) \right))\Bigg|_{x_0} \\
&= \sum_r (\partial_{x^j} \partial_{x^k} y^r)(x) \cdot \partial_{y^r}|_{x_0} \\
&\quad + \sum_r (-1)^{\varepsilon_j(\varepsilon_r + \varepsilon_k)} \cdot (\partial_{x^k} y^r)(x) \cdot \big((\nabla_{\partial_{x^j}} \partial_{y^r})\big)\Big|_{x_0} \\
&= \sum_p (\partial_{x^j} \partial_{x^k} y^p)(x) \cdot \partial_{y^p}|_{x_0} \\
&\quad + \sum_{qr} (-1)^{\varepsilon_j(\varepsilon_r + \varepsilon_k)} \cdot (\partial_{x^k} y^r)(x) \cdot (\partial_{x^j} y^q)(x) \cdot \big((\nabla_{\partial_{y^q}} \partial_{y^r})\big)\Big|_{x_0} \\
&= \sum_p (\partial_{x^j} \partial_{x^k} y^p)(x) \cdot \partial_{y^p}|_{x_0} \\
&\quad + \sum_{qr} (-1)^{\varepsilon_j(\varepsilon_r + \varepsilon_k)} \cdot (\partial_{x^k} y^r)(x) \cdot (\partial_{x^j} y^q)(x) \cdot \widetilde{\Gamma}_q{}^p{}_r(y) \cdot \partial_{y^p}|_{x_0}
\end{aligned}$$

which gives us the relations

$$\begin{aligned}
&\sum_i \Gamma^i_{jk}(x) \cdot (\partial_{x^i} y^r)(x) \\
&\qquad = (\partial_{x^j} \partial_{x^k} y^r)(x) + \sum_{s,t} (-1)^{\varepsilon_j(\varepsilon_t + \varepsilon_k)} \cdot (\partial_{x^k} y^t)(x) \cdot (\partial_{x^j} y^s)(x) \cdot \widetilde{\Gamma}^r_{st}(y)
\end{aligned} \tag{3.10}$$

Equivalently, we can write

$$\begin{aligned}
\widetilde{\Gamma}^r_{st}(y) &= (\partial_{y^s} \partial_{y^t} x^i)(y) \cdot (\partial_{x^i} y^r)(x) \\
&\quad + \sum_{i,j,k} (-1)^{s(k+t)} \cdot (\partial_{y^t} x^k)(y) \cdot (\partial_{y^s} x^j)(y) \cdot \Gamma^i_{jk}(x) \cdot (\partial_{x^i} y^r)(x)
\end{aligned} \tag{3.11}$$

3.1.2 Supergeodesics of a torsion-free connection

We start very naively and copy the classical case: a geodesic is a map $\gamma : \mathcal{A}_0 \to M$ given in local coordinates by $\gamma(t) = (\gamma^1(t), \ldots, \gamma^{n+m}(t))$ satisfying the equations

$$\frac{\partial^2 \gamma^i}{\partial t^2}(t) = -\sum_{j,k} \frac{\partial \gamma^k}{\partial t}(t) \cdot \frac{\partial \gamma^j}{\partial t}(t) \cdot \Gamma^i_{jk}(\gamma(t)) \tag{3.12}$$

Since any system of second order differential equations on a manifold can be expressed as a system of first order differential equations on the tangent bundle, we can equivalently look at curves $\widetilde{\gamma} : \mathcal{A}_0 \to TM^{(0)}$ given in local coordinates by

$$\widetilde{\gamma}(t) = (\gamma^1(t), \ldots, \gamma^{n+m}(t), \bar{\gamma}^1(t), \ldots, \bar{\gamma}^{n+m}(t)) \tag{3.13}$$

and satisfying the local equations

$$\begin{cases} \frac{\partial \gamma^i}{\partial t}(t) &= \bar{\gamma}^i_{(x,v)}(t) \\ \frac{\partial \bar{\gamma}^i}{\partial t}(t) &= -\sum_{j,k} \bar{\gamma}^k(t) \cdot \bar{\gamma}^j(t) \cdot \Gamma^i_{jk}(\gamma(t)) \; . \end{cases}$$

Initial conditions

In order to solve second order differential equations one needs initial conditions, which in our case are a starting point x and an initial velocity v. A geodesic γ depends upon these initial conditions (forcing us to write $\gamma_{(x,v)}$ instead of simply γ) through the equations

$$\gamma^i_{(x,v)}(0) = x^i \qquad \text{and} \qquad \frac{\partial \gamma^i_{(x,v)}}{\partial t}(0) = v^i \; . \tag{3.14}$$

Passing from second order differential equations to first order differential equations (i.e., from γ to $\widetilde{\gamma}$), we thus end up looking at families of curves

$$\widetilde{\gamma} : TM^{(0)} \times \mathcal{A}_0 \to TM^{(0)} \; , \; (x, v, t) \mapsto \widetilde{\gamma}_{(x,v)}(t) \; ,$$

satisfying the local equations

$$\begin{cases} \frac{\partial \gamma^i_{(x,v)}}{\partial t}(t) &= \bar{\gamma}^i_{(x,v)}(t) \\ \frac{\partial \bar{\gamma}^i_{(x,v)}}{\partial t}(t) &= -\sum_{j,k} \bar{\gamma}^k_{(x,v)}(t) \cdot \bar{\gamma}^j_{(x,v)}(t) \cdot \Gamma^i_{jk}(\gamma(t)) \; . \end{cases}$$

together with the initial conditions

$$\gamma^i_{(x,v)}(0) = x^i \qquad \text{and} \qquad \bar{\gamma}^i_{(x,v)}(0) = v^i \; .$$

The geodesic vector field and its flow

The above equations for $\widetilde{\gamma}$ are exactly the equations of the integral curves of a vector field on $TM^{(0)}$. Indeed, using the Christoffel symbols we can define a vector field G^{∇} on $TM^{(0)}$ in local coordinates (x, v) by

$$G|_{\vec{v}} = \sum_i v^i \cdot \partial_{x^i}|_{\vec{v}} - \sum_{i,j,k} v^k \cdot v^j \cdot \Gamma^i_{jk}(x) \cdot \partial_{v^i}|_{\vec{v}} \tag{3.15}$$

These local expressions glue together to form a well-defined global vector field G^{∇} on $TM^{(0)}$. As it is an even vector field, it has a flow Ψ defined in an open subset W_G of $\mathcal{A}_0 \times TM^{(0)}$ containing $\{0\} \times TM^{(0)}$ and with values in $TM^{(0)}$ [**?**, V.4.9]. In local coordinates we will write $\Psi(t, x, v) = (\Psi_1(t, x, v), \Psi_2(t, x, v))$, where $\Psi_1 = (\Psi_1^1, \ldots, \Psi_1^{n+m})$ represents the base point while $\Psi_2 = (\Psi_2^1, \ldots, \Psi_2^{n+m})$ represents the tangent vector. By definition of a flow, these functions thus satisfy the equations

$$\begin{cases} \frac{\partial \Psi_1^i}{\partial t}(t, x, v) &= \Psi_2^i(t, x, v) \\ \frac{\partial \Psi_2^i}{\partial t}(t, x, v) &= -\sum_{j,k} \Psi_2^k(t, x, v) \cdot \Psi_2^j(t, x, v) \cdot \Gamma^i_{j,k}(\Psi_1(t, x, v)) \end{cases}$$

together with the initial conditions

$$\Psi_1(0, x, v) = x \qquad \text{and} \qquad \Psi_2(0, x, v) = v \tag{3.16}$$

With the global vector field G^{∇} we thus have found an intrinsic coordinate free description of the equations we wrote for the geodesic curves $\widetilde{\gamma}_{(x,v)}(t)$ and we are now in position to state a definition.

Definition. Let ∇ be a connection in TM, let $\pi : TM^{(0)} \to M$ denote the canonical projection, let G^{∇} be the even vector field 3.15 and let $\Psi : W_G \to TM^{(0)}$ be its flow. For a fixed $\vec{v} \cong (x, v) \in TM^{(0)}$ we will call the map $\gamma : \mathcal{A}_0 \to M$ defined by

$$\gamma(t) = \pi\left((\Psi(t, \vec{v}))\right) \cong \Psi_1(t, x, v) \tag{3.17}$$

the geodesic through $x \in M$ with initial velocity $\vec{v}$. Note that if $\vec{v}$ is not in the body of $TM^{(0)}$, this curve is not necessarily smooth (see [**?**, III.1.23g, V.3.19]).

3.2 Weyl's algebraic characterization of projective equivalence can be extended to $\mathcal{A}$-manifolds.

3.2.1 Projective equivalence in terms of super geodesics

We now consider the situation in which we have two connections $\nabla, \widehat{\nabla}$ on M and we wonder under what conditions these two connections have "the same" geodesics as trajectories on M: if $\Psi(t, \vec{v})$ and $\widehat{\Psi}(t, \vec{v})$ are the geodesic flows for ∇ and $\widehat{\nabla}$ respectively, the naive question is under what conditions we have

$$\{ \Psi_1(t, x, v) : t \in \mathcal{A}_0 \} = \{ \widehat{\Psi}_1(t, x, v) : t \in \mathcal{A}_0 \} \tag{3.18}$$

A more precise question is under what conditions we can find a reparametrization function, $r : \mathcal{A}_0 \times TM^{(0)} \to \mathcal{A}_0$, such that for any $\vec{v} \in TM^{(0)}$, we would have

$$\forall t \in \mathcal{A}_0 \quad : \quad \Psi_1(r(t, x, v), x, v) = \widehat{\Psi}_1(t, x, v) \ . \tag{3.19}$$

Note that we added an explicit dependence on the initial condition $\vec{v}$ in the reparametrization function r, as there is no reason that geodesics through different points should be reparametrized in the same way: a reparametrization is a *smooth family* of maps $\mathcal{A}_0 \to \mathcal{A}_0$.

Definition. We say that ∇ and $\widehat{\nabla}$ have *the same geodesics up to reparametrization* if there exists a function $r : \mathcal{A}_0 \times TM^{(0)} \to \mathcal{A}_0$ such that $r(0, \vec{v}) = 0$, $(\partial r/\partial t)(0, \vec{v}) = 1$ and for which equation (3.19) holds.[3]

3.2.2 Algebraic characterization of projective equivalence

First, we show that (3.19) holds if and only if the geodesic flow Ψ of G^{∇}, the (difference) tensor $S = \nabla - \widehat{\nabla}$ and the reparametrization function r are related through a certain differential equation.

Proposition 16. *The connections ∇ and $\widehat{\nabla}$ have the same geodesics up to reparametrization if and only if there exists a function $r : \mathcal{A}_0 \times TM^{(0)} \to \mathcal{A}_0$ such that $r(0, \vec{v}) = 0$, $(\partial r/\partial t)(0, \vec{v}) = 1$ and for which the following differential equation holds:*

$$\begin{aligned} &\frac{\partial^2 r}{\partial t^2}(t, x, v) \cdot \frac{\partial \Psi_1}{\partial t}(r(t, x, v), x, v) \\ &= \Big(\frac{\partial r}{\partial t}(t, x, v)\Big)^2 \cdot S_{\Psi_1(r(t,x,v),x,v)}\Big(\frac{\partial \Psi_1}{\partial t}(r(t, x, v), x, v) \, , \, \frac{\partial \Psi_1}{\partial t}(r(t, x, v), x, v) \Big) \end{aligned} \tag{3.20}$$

[3] The additional conditions $r(0, \vec{v}) = 0$ and $(\partial r/\partial t)(0, \vec{v}) = 1$ ensure that the reparametrization transforms each geodesic of ∇ into the geodesic of $\widehat{\nabla}$ with the same initial conditions.

Proof. Let us show that the condition is necessary. In view of (3.12), if $\Psi_1(r(t,x,v),x,v)$ is a geodesic for $\widehat{\nabla}$, then

$$0 = \frac{\partial^2 \Psi_1^i(r(t,x,v),x,v)}{\partial t^2} + \sum_{j,k} \frac{\partial \Psi_1^k(r(t,x,v),x,v)}{\partial t} \cdot \frac{\partial \Psi_1^j(r(t,x,v),x,v)}{\partial t} \cdot \hat{\Gamma}^i_{jk}(\Psi_1(r(t,x,v),x,v)) \tag{3.21}$$

Let us replace in this equation $\hat{\Gamma}^i_{jk}$ by $\Gamma^i_{jk} - S^i_{jk}$ and let us apply the chain rule to compute the derivatives of the functions $\Psi_1^i(r(t,x,v),x,v)$. Doing so, we obtain

$$\begin{aligned}
0 \quad = \quad & \frac{\partial^2 r}{\partial t^2}(t,x,v) \cdot \frac{\partial \Psi_1}{\partial t}(r(t,x,v),x,v) + \left(\frac{\partial r}{\partial t}(t,x,v)\right)^2 \left(\frac{\partial^2 \Psi_1^i}{\partial t^2}(r(t,x,v),x,v)\right) \\
& + \left(\frac{\partial r}{\partial t}(t,x,v)\right)^2 \left(\sum_{j,k} \frac{\partial \Psi_1^k}{\partial t}(r(t,x,v),x,v) \cdot \frac{\partial \Psi_1^j}{\partial t}(r(t,x,v),x,v) \cdot \Gamma^i_{jk}(\Psi_1(r(t,x,v),x,v))\right) \\
& - \Big(\frac{\partial r}{\partial t}(t,x,v)\Big)^2 \left(\sum_{j,k} \frac{\partial \Psi_1^k}{\partial t}(r(t,x,v),x,v) \cdot \frac{\partial \Psi_1^j}{\partial t}(r(t,x,v),x,v) \cdot S^i_{jk}(\Psi_1(r(t,x,v),x,v))\right)
\end{aligned}$$

Using the fact that Ψ_1 is a geodesic for ∇, the second and third term on the right hand side cancel and hence this equation reduces to (3.20).

In order to show the converse, it suffices to note that the above computations also show that if (3.20) is satisfied, then the curve

$$\left(\Psi_1(r(t,x,v),x,v), \frac{\partial r}{\partial t}(t,x,v) \cdot \frac{\partial \Psi_1}{\partial t}(r(t,x,v),x,v)\right)$$

satisfies the equation of the flow $(\widehat{\Psi}_1(t,x,v), \widehat{\Psi}_2(t,x,v))$ of $\hat{G}$, the geodesic vector field corresponding to $\widehat{\nabla}$. As it satisfies the same initial conditions as $(\widehat{\Psi}_1(t,x,v), \widehat{\Psi}_2(t,x,v))$ at $t = 0$, these two curves have to coincide, and in particular $\Psi_1(r(t,x,v),x,v) = \widehat{\Psi}_1(t,x,v)$. $\square$

3.2.3 Weyl's characterization on $\mathcal{A}$-manifolds

It remains to show that condition (3.20) amounts to imposing that S can be expressed by means of an even (super) 1-form. As for the previous Proposition, the proof of the theorem follows the lines of the classical case. It invokes a technical Lemma which roughly says that if we have a bilinear function $S(v,w)$ such that $S(v,v) = h(v) \cdot v$ for some function h, then h must be linear in v. The proof of this technical Lemma is elementary but long, simply because we have to be careful with the odd coordinates and moreover, everything depends upon additional parameters (the local coordinates x and ξ on M). The proof of the lemma can be found in [**?**].

LEMMA 17. *Let E be a graded vector space of graded dimension $p|q$ with even basis vectors $e_1, \ldots, e_p$ and odd basis vectors $f_1, \ldots, f_q$, let U be an open coordinate subset of a manifold M with local even coordinates x and local odd coordinates ξ. Suppose that $S : U \times E \times E \to E$ is a smooth function which is left-bilinear, graded symmetric in the product $E \times E$ and for which there is a smooth function $h : U \times E_0 \to \mathcal{A}$ such that*

$$\forall (x, \xi) \in U \ \forall v \in E_0 : S(x, \xi, v, v) = h(x, \xi, v) \cdot v \tag{3.22}$$

Then there exists a unique smooth function $\alpha : U \to E^*$ such that $h(x, \xi, v) = \iota(v)\alpha(x, \xi)$ and

$$S(x, \xi, v, w) = \tfrac{1}{2} \cdot \big(v \cdot \iota(w)\alpha(x, \xi) + (-1)^{\varepsilon(v)\cdot\varepsilon(w)} \cdot w \cdot \iota(v)\alpha(x, \xi) \big) \tag{3.23}$$

THEOREM 18. *Two torsion-free connections ∇ and $\widehat{\nabla}$ on M have the same geodesics up to reparametrization if and only if there exists a smooth even 1-form α on M such that the tensor $S = \nabla - \widehat{\nabla}$ is given by*

$$S_x(v, w) = \tfrac{1}{2} \cdot (v \cdot \iota(w)\alpha_x + (-1)^{\varepsilon(v)\cdot\varepsilon(w)} \cdot w \cdot \iota(v)\alpha_x) \tag{3.24}$$

for any $x \in M$ and any homogeneous $v, w \in T_xM$.

Proof. We first assume that we have a reparametrization r that transforms the geodesics of ∇ into those of $\widehat{\nabla}$. Taking $t = 0$ in (3.20) and using the initial conditions for Ψ and r, we get the following (vector) equation in local coordinates:

$$v \cdot \frac{\partial^2 r}{\partial t^2}(0, x, v) = S_x(v, v) \tag{3.25}$$

Lemma 17, with h being here the function $h(x, v) = \frac{\partial^2 r}{\partial t^2}(0, x, v)$, gives us a (local) smooth 1-form α, which must be even by parity considerations. But (3.25) is an intrinsic equation which does not depend upon the choice of local coordinates (because (3.20) is intrinsic). As the 1-form α is unique, the local 1-forms α given by Lemma 17 glue together to form a global smooth even 1-form α satisfying (3.29).

To show the converse, let us now assume that we have an even 1-form α on M such that the tensor S is given by (3.29). Then (3.20) reduces to the (vector) equation

$$\begin{aligned} &\frac{\partial^2 r}{\partial t^2}(t, x, v) \cdot \frac{\partial \Psi_1}{\partial t}(r(t, x, v), x, v) \\ &\quad = \Big(\frac{\partial r}{\partial t}(t, x, v) \Big)^2 \cdot \iota \left(\frac{\partial \Psi_1}{\partial t}(r(t, x, v), x, v) \right) \alpha_{\Psi_1(r(t,x,v),x,v)} \cdot \frac{\partial \Psi_1}{\partial t}(r(t, x, v), x, v) \end{aligned} \tag{3.26}$$

For this to be true for all geodesics of ∇, the function r thus has to satisfy the second order differential equation

$$\frac{\partial^2 r}{\partial t^2}(t,x,v) = \left(\frac{\partial r}{\partial t}(t,x,v)\right)^2 \cdot \iota\left(\frac{\partial \Psi_1}{\partial t}(r(t,x,v),x,v)\right) \alpha_{\Psi_1(r(t,x,v),x,v)}$$

As for the geodesic equations, we translate this into a system of first order differential equations by introducing a second function $s : \mathcal{A}_0 \times TM^{(0)} \to \mathcal{A}_0$ and we obtain

$$\begin{cases} \frac{\partial r}{\partial t}(t,x,v) &= s(t,x,v) \\ \frac{\partial s}{\partial t}(t,x,v) &= s(t,x,v)^2 \cdot \iota\left(\frac{\partial \Psi_1}{\partial t}(r(t,x,v),x,v)\right) \alpha_{\Psi_1(r(t,x,v),x,v)} \end{cases}$$

while the initial conditions for r yield $r(0,x,v) = 0$ and $s(0,x,v) = 1$. To show that these equations always have a (unique) solution, we just note that these equations determine the flow of the even vector field R on $(\mathcal{A}_0)^2 \times TM^{(0)}$ given by

$$R|_{(r,s,x,v)} = s \cdot \frac{\partial}{\partial r} + s^2 \cdot \iota\left(\frac{\partial \Psi_1}{\partial t}(r,x,v)\right) \alpha_{\Psi_1(r,x,v)} \cdot \frac{\partial}{\partial s} \tag{3.27}$$

And indeed, the equations for the flow $\Phi = (\Phi_r, \Phi_s, \Phi_1, \Phi_2)$ of R are given by

$$\begin{cases} \frac{\partial \Phi_r}{\partial t}(t,r_o,s_o,x,v) &= \Phi_s(t,r_o,s_o,x,v) \\ \frac{\partial \Phi_s}{\partial t}(t,r_o,s_o,x,v) &= (\Phi_s(t,r_o,s_o,x,v))^2 \\ & \quad \cdot \iota\left(\frac{\partial \Psi_1}{\partial t}(\Phi_r(t,r_o,s_o,x,v),x,v)\right) \alpha_{\Psi_1(\Phi_r(t,r_o,s_o,x,v),x,v)} \\ \frac{\partial \Phi_1}{\partial t}(t,r_o,s_o,x,v) &= 0 \\ \frac{\partial \Phi_2}{\partial t}(t,r_o,s_o,x,v) &= 0 \end{cases}$$

Now it thus suffices to define $r(t,x,v) = \Phi_r(t,0,1,x,v)$ and $s(t,x,v) = \Phi_s(t,0,1,x,v)$. $\square$

Local characterization of projective equivalence

Thanks to the algebraic characterization of projective equivalence, we can see that in coordinates, the condition for two torsion-free superconnections ∇ and ∇' to be projectively equivalent can be written as $\Pi_{ij}^k = {\Pi'}_{ij}^k$, where

$$\Pi_{ij}^k = \Gamma_{ij}^k - \frac{1}{n-m+1} \cdot \left(\Gamma_{is}^s \cdot \delta_j^k (-1)^{\varepsilon_s} + \Gamma_{js}^s \cdot \delta_i^k \cdot (-1)^{\varepsilon_i \varepsilon_j + \varepsilon_s}\right) \ . \tag{3.28}$$

The Π_{ij}^k define the so-called *fundamental descriptive invariant* of the projective class of ∇.

Remark. In graded dimension $n|m$ with $n - m = -1$, formula (3.28) does not make sense. Actually, no such quantity as a fundamental descriptive invariant is known in this situation.

3.2.4 Projectively equivalent smooth families

We use the algebraic characterization of projective equivalence in order to generalize this equivalence to *smooth families* of torsion free connections on M, i.e., even smooth sections of $\mathcal{C}M$ whose local components Γ^i_{jk} in the local adapted coordinates are graded symmetric in the lower indices.

Definition. Two smooth families of torsion-free connections $\nabla : W \subset P \times M \to \mathcal{C}M$ and $\widehat{\nabla} : W \subset P' \times M \to \mathcal{C}M$ on M are called projectively equivalent if there exists a smooth family $\alpha : \{(p, p', x) : (p, x) \in W \text{ and } (p', x) \in W'\} \subset (P \times P') \times M \to {}^*TM$ of even 1-forms on M such that the family $S = \nabla - \widehat{\nabla}$ is given by

$$\iota(v, w)S(p, p', x) = \tfrac{1}{2} \cdot (v \cdot \iota(w)\alpha(p, p', x) + (-1)^{\varepsilon(v)\cdot\varepsilon(w)} \cdot w \cdot \iota(v)\alpha(p, p', x)) \tag{3.29}$$

for all homogeneous $v, w \in T_xM$.

3.3 The projective subalgebra of vector fields on $E_0^{n|m}$ preserves the projective class of the flat connection.

3.3.1 Preserving a projective class: vector fields

DEFINITION. We say that a smooth vector field X *preserves the projective structure* of ∇ when there exists a smooth 1-form α on M such that the tensor $L_X\nabla$ is given by

$$\iota(Y,Z)\mathrm{L}_X\nabla = \iota(Y,Z)(\alpha \vee \mathrm{id}) = \tfrac{1}{2}\Big(Y\cdot\iota(Z)\alpha + (-1)^{\varepsilon(Y).\varepsilon(Z)}Z\cdot\iota(Y)\alpha\Big)\ ,$$

where $\iota(Y,Z)\mathrm{L}_X\nabla$ is given by

$$\begin{aligned}\iota(Y,Z)\mathrm{L}_X\nabla = (-1)^{\varepsilon(X).(\varepsilon(Y)+\varepsilon(Z))}\cdot[X,\nabla_Y Z] - (-1)^{\varepsilon(X).(\varepsilon(Y)+\varepsilon(Z))}\cdot\nabla_{[X,Y]}Z\\ - (-1)^{\varepsilon(X).\varepsilon(Z)}\cdot\nabla_Y[X,Z]\ ,\end{aligned} \quad (3.30)$$

PROPOSITION 19. *A smooth vector field $X \in \Gamma(TE_0^{n|m})$ preserves the projective structure of the canonical flat connection ∇_0 if and only if $X = X^h$ for some $h \in \mathbf{B}\mathfrak{paut}(n+1|m,\mathcal{A})$.*

Proof. By definition, the Christoffel symbols of ∇_0 in the canonical coordinates of $E_0^{n|m}$ are zero. The condition for $X = X^i\cdot\partial_{x^i}$ to preserve the projective class of ∇_0 thus reads

$$(\partial_{x^j}\partial_{x^k}X^i)\cdot\partial_{x^i} = \tfrac{1}{2}\Big((-1)^{\varepsilon(\alpha).(\varepsilon_j+\varepsilon_k)+\varepsilon_j}\cdot\alpha_j\cdot\partial_{x^k} + (-1)^{\varepsilon(\alpha).(\varepsilon_j+\varepsilon_k)+\varepsilon_j.\varepsilon_k+\varepsilon_k}\cdot\alpha_k\cdot\partial_{x^j}\Big)\ , \quad (3.31)$$

where $\alpha = \alpha_i\cdot \mathrm{d}x^i$. Obviously, all X^h with $h\in\mathbf{B}\mathfrak{paut}(n+1|m,\mathcal{A})$ satisfy such an equation: for $h \in \mathbf{B}\mathfrak{g}_{(-1)}\cup\mathbf{B}\mathfrak{g}_{(0)}$, take $\alpha = 0$ while for $h=\xi\in\mathbf{B}\mathfrak{g}_{(1)}$, take $\alpha = 2\cdot(-1)^i\cdot\xi_i\cdot\mathrm{d}x^i$.

Conversely, equation (3.31) gives

$$\partial_{x^j}\partial_{x^k}X^i = \begin{cases}\tfrac{1}{2}\cdot(1+(-1)^{\varepsilon_i})\cdot\alpha_i, & \text{if } i=j=k\ ;\\ \tfrac{1}{2}\cdot(-1)^{(\varepsilon(\alpha)+\varepsilon_k).(\varepsilon_j+\varepsilon_k)}\cdot\alpha_k, & \text{if } i=j\neq k\ ;\\ \tfrac{1}{2}\cdot(-1)^{\varepsilon(\alpha)(\varepsilon_j+\varepsilon_k)+\varepsilon_j}\cdot\alpha_j\ , & \text{if } i=k\neq j\ ;\\ 0\ , & \text{if } i\notin\{j,k\}\ .\end{cases}$$

It can be shown from these equalities that all partial derivatives of the coefficient functions of α are zero. Therefore those coefficient functions are real constants and X^i reads as

$$X^i = \tfrac{1}{2}\cdot\alpha_s\cdot x^s x^i + a^i_{1,s}\cdot x^s + a^i_0$$

for some constants $a^k_{1,i}, a^k_0\in\mathbb{R}$. Finally, the conclusion is obtained from formulas (1.3). $\square$

A smooth family of vector fields

Let Z be the smooth family of all even fundamental vector fields associated with the action of the projective group on the projective space, i.e.,

$$Z : \mathfrak{paut}(n+1|m,\mathcal{A})_0 \times E_0^{n+m} \to TE_0^{n+m},\ (h,x) \mapsto X_x^h\ .$$

LEMMA 20. *The smooth family of even vector fields Z preserves the projective structure of the canonical flat connection ∇_0, i.e.,*

$$\nabla_0 + \mathrm{L}_Z \nabla_0 \sim \nabla_0\ .$$

Proof. From the proof of Proposition 19, we already know that for any $h \in \mathbf{B}\mathfrak{paut}(n+1|m,\mathcal{A})$, we have

$$\mathrm{L}_{X^h} \nabla_0 = \alpha_h \vee \mathrm{id}\ ,$$

where the smooth 1-form $\alpha_h = \alpha_{h,i} \cdot \mathrm{d}x^i\big|_x \in \Gamma({}^*TE_0^{n|m})$ is defined by

$$\alpha_{h,i} = \begin{cases} 2 \cdot (-1)^{\varepsilon_i} \cdot \xi_i \in \mathbb{R}\ , & \text{if } h = \xi \in \mathbf{B}\mathfrak{g}_{(1)}\ ; \\ 0\ , & \text{otherwise}\ . \end{cases}$$

We prolong these formula to $\mathfrak{paut}(n+1|m,\mathcal{A})_0$ by setting

$$\alpha_{h,i} = \begin{cases} 2 \cdot (-1)^{\varepsilon_i} \cdot \xi_i \in \mathcal{A}\ , & \text{if } h = \xi \in \mathfrak{g}_{(1)}\ ; \\ 0\ , & \text{otherwise}\ . \end{cases}$$

As a result, we obtain a smooth family of even 1-forms on $E_0^{n|m}$:

$$\alpha : \mathfrak{paut}(n+1|m,\mathcal{A})_0 \times E_0^{n|m} \to {}^*TE_0^{n|m}\ , (h,x) \mapsto \alpha_{h,i} \cdot \mathrm{d}x^i\big|_x\ .$$

Finally, it is straightforward, using formula (2.7), to check in local coordinates that we have

$$\mathrm{L}_Z \nabla_0(h,x) = \alpha(h,x) \vee \mathrm{id}\ .$$

□

3.3.2 Preserving a projective class: integration

Integrating with respect to the time parameter

First, given a smooth function $f : I \subset \mathcal{A}_0 \to \mathcal{A}$, we set for any $t_0 \in \mathbf{B}I$ and $t_1 \in I$ such that $[t_0, \mathbf{B}t_1] \subset I$,

$$\int_{t_0}^{t_1} f(t) \cdot \mathrm{d}t = \left(\int_{t_0}^{\cdot} f_0(t) \cdot \mathrm{d}t \right)^{\sim} (t_1) \ ,$$

where f_0 is the ordinary smooth function on $\mathbb{R}$ such that $f(t) = \widetilde{f_0}(t)$.

Then, we extend this definition to any smooth function f defined on an open subset W of $\mathcal{A}_0 \times M$ by setting, in local coordinates,

$$\int_{t_0}^{t_1} f(t, x, \xi) \cdot \mathrm{d}t = \sum_{I,J} \xi^I \cdot \frac{(x - \mathbf{B}x)^J}{J!} \cdot \left(\int_{t_0}^{\cdot} \frac{\partial f_I}{\partial x^J}(t, \mathbf{B}x) \cdot \mathrm{d}t \right)^{\sim} (t_1) \ ,$$

if f reads as $f = \sum_I \xi^I \cdot \widetilde{f_I}(t, x)$.

Finally, we extend the integration process to any smooth family of sections $\sigma : W \subset (\mathcal{A}_0 \times P) \times M \to E_\pi$ of a vector bundle $\pi : E_\pi \to M$: given a set of local trivializing sections $\{\mathbf{e}_j \in \Gamma_U(E_\pi)\}$, we set

$$\int_{t_0}^{t_1} \sigma(t, p, x) \cdot \mathrm{d}t = \sum_j \left(\int_{t_0}^{t_1} \sigma^j(t, p, x) \cdot \right) \cdot \mathbf{e}_j(x) \ ,$$

if $\sigma|_U = \sum_j \sigma^j \cdot \mathbf{e}_j$.

Remark. Integration as defined above is related to differentiation as recalled in 2.2.2:

$$\begin{aligned} \left(\partial_t \cdot \left(\int_{t_0}^{\cdot} \sigma(t, p, x) \cdot \mathrm{d}t \right) \right) (t_1) &= \sigma(t_1, p, x) \\ \left(\int_{t_0}^{t_1} (\partial_t \cdot \sigma)(t, p, x) \cdot \mathrm{d}t \right) &= \sigma(t_1, p, x) - \sigma(t_0, p, x) \ . \end{aligned}$$

These formulas are inherited from the classical relation between integration and differentiation because we defined things here from the classical notion through the local deomposition of smooth functions.

Integrating projective invariance

PROPOSITION 21. *If $X : P \times M \to TM$ is a smooth family of even vector fields that preserves the projective structure of $\nabla \in \Gamma(\mathcal{C}M)$, i.e., $\nabla + \mathrm{L}_X\nabla \sim \nabla$, then so does its flow, i.e., $\Phi_X^*\nabla \sim \nabla$.*

Proof. By definiton of the Lie derivative $\mathrm{L}_X\nabla$, the fact that X preserves the projective class of ∇ gives

$$\begin{aligned}
\partial_t \cdot (\Phi_X^*\nabla)(0,p,x) &= (\mathrm{L}_X\nabla)(p,x) \\
&= (\alpha \vee \mathrm{id})(p,x)\ ,
\end{aligned}$$

for some smooth family $\alpha : P \times M \to {}^*TM$ of even 1-forms. For any $t_1 \in \mathcal{A}_0$, using the fact that the flow map of X satisfies $\Phi_{X,(t_1+t,p)} = \Phi_{X,(t_1,p)} \circ \Phi_{X,(t,p)}$, we then find

$$\begin{aligned}
(\partial_t \cdot (\Phi_X^*\nabla))(t_1,p,x) &= \partial_t \cdot \Big((t,t_1,p,x) \mapsto (\Phi_X^*\nabla)_{(t_1+t,p)}(x)\Big)(0,t_1,p,x) \\
&= \partial_t \cdot \Big((t,t_1,p,x) \mapsto \Big(\mathcal{C}\Phi_{X,(-t_1,p)} \circ (\Phi_X^*\nabla)_{(t,p)} \circ \Phi_{X,(t_1,p)}(x)\Big)\Big)(0,t_1,p,x) \\
&= \Big(\vec{\mathcal{C}}\Phi_{X,(-t_1,p)} \circ (\partial_t \cdot (\Phi_X^*\nabla))_{(0,p)} \circ \Phi_{X,(t_1,p)}\Big)(x) \\
&= (\Phi_X^*(\mathrm{L}_X\nabla))(t_1,p,p,x) \\
&= (\Phi_X^*(\alpha \vee \mathrm{id}))(t_1,p,p,x)\ ,
\end{aligned}$$

where the third and fourth equalities are easily obtained from the definition of $\mathcal{C}\Phi$ and its underlying natural vector bundle functor $\vec{\mathcal{C}}$. Then, integration with respect to the even time parameter t yields

$$\begin{aligned}
(\Phi_X^*\nabla - \nabla)(t_1,p,x) &= \int_0^{t_1} (\partial_t \cdot (\Phi_X^*\nabla))(t,p,x) \cdot \mathrm{d}t \\
&= \int_0^{t_1} (\Phi_X^*(\alpha \vee \mathrm{id}))(t,p,p,x) \cdot \mathrm{d}t \\
&= \left(\int_0^{t_1} (\Phi_X^*\alpha)(t,p,p,x) \cdot \mathrm{d}t\right) \vee \mathrm{id}\ ,
\end{aligned}$$

showing that $\Phi_X^*\nabla$ is projectively equivalent to ∇. $\square$

COROLLARY 22. *The flow $\Phi_Z : \mathcal{A}_0 \times \mathfrak{paut}(n+1|m,\mathcal{A})_0 \times E_0^{n+m} \to E_0^{n+m}$ preserves the projective structure of the canonical flat connection ∇_0.*

CHAPTER 4

NATURAL PROJECTIVELY INVARIANT QUANTIZATION ON $\mathcal{A}$-MANIFOLDS

In this last chapter we first describe the problem of Natural Projectively Invariant Quantization (NPIQ) on supermanifolds in the language of $\mathcal{A}$-manifolds. Then, we establish a super analog of the classical relation between Projectively Equivariant Quantization (PEQ) and NPIQ: if a NPIQ exists, its restriction to the flat superspace endowed with the canonical flat connection gives a PEQ. The idea of the proof in the classical setting can be reused as soon as one adopts the "language of smooth families" developed in the other chapters. Actually, this language enables us to circumvent semantic problems arising from the fact that the flow of a super vector field is, in general, not smooth for a fixed value of the time parameter.

Having recovered the relation between NPIQ and PEQ, we describe the superization presented in [**?**] of M. BORDEMANN's method: with each torsion-free connection $[\nabla]$ one associates a unique linear connection, $\tilde{\nabla}$, on a line bundle $\tilde{M} \to M$; then one identifies symbols on M with suitable tensors on $\tilde{M}$; finally, one applies the so-called *standard ordering* on $\tilde{M}$ and project the result back to M, so that the whole procedure defines a projectively invariant quantization map.

This procedure is not valid when the superdimension is either 1 or -1. As a conclusion, we discuss the (open) problem of existence of a NPIQ in these peculiar cases.

REMARK. In this chapter, we denote by $\mathrm{d}x^1, \ldots, \mathrm{d}x^{n+m}$ the left dual basis of the canonical basis of local supervector fields $\partial_1, \ldots, \partial_{n+m}$ on M, i.e., we have here $\iota(\partial_{x^j})\mathrm{d}x^i = \delta^i_j$ for all i, j. In [**?**], $\{\mathrm{d}x^i\}$ stood for the *right* dual basis, which explains why some definition may seem different at first sight. Actually, developing things in coordinates shows that the formulas/computations here are exactly *the same* as in [**?**].

Contents

4.1 Natural Projectively Invariant Quantization generalizes Projectively Equivariant Quantization.

4.1.1 The bundle of densities

Let M be an $\mathcal{A}$-manifold of dimension $n|m$ and let $\{(U_a \subset M, \varphi_a : U_a \to E_0^{n|m})\}$ be the atlas of all charts for M. Let $\lambda \in \mathbb{R}$. We define $\mathcal{F}_\lambda(\varphi_{ba}) : \varphi_a(U_b \cap U_a) \to \mathrm{Aut}(\mathcal{A})$ by setting([1])

$$\mathcal{F}_\lambda(\varphi_{ba})(x)(a) = |\mathrm{Ber}(A_{ba}(x))|^{-\lambda} \cdot a ,$$

where $A_{ba}(x)$ is given in terms of a basis $\{e_i\}$ of $E^{n|m}$ and its left dual basis $\{{}^i e\}$ by

$$A_{ba}(x) = \sum_{k,l} {}^k e \cdot (\partial_{x^l} \varphi_{ba}^k(x)) \otimes e_l .$$

In other words, we have $\iota(h^k \cdot e_k)(A_{ba}(x)) = \sum_{k,l} h^k \cdot (\partial_{x^l} \varphi_{ba}^k(x)) \cdot e_l$ and the matrix representation of $A_{ba}(x)$ is thus the ordinary (not graded) transpose of that of $\mathrm{Jac}(\varphi_{ba})(x)$.

It follows from the chain rule and the properties of the Berezinian Ber (see [**?**, II.5]) that the functions $\Psi_{ba} = \mathcal{F}_\lambda(\varphi_{ba}) \circ \varphi_a : U_b \cap U_a \to \mathrm{Aut}(\mathcal{A})$ satisfy the cocycle conditions (B.1):

$$\begin{aligned}
\Psi_{aa}(x)(a) &= |\mathrm{Ber}(A_{aa}(\varphi_a(x)))|^{-\lambda} \cdot a = |\mathrm{Ber}(\mathrm{id}_{E^{n|m}})|^{-\lambda} \cdot a = a \\
\Psi_{cb}(x) \circ \Psi_{ba}(x) &= |\mathrm{Ber}(A_{cb}(\varphi_b(x)))|^{-\lambda} \cdot |\mathrm{Ber}(A_{ba}(\varphi_a(x)))|^{-\lambda} \cdot a \\
&= |\mathrm{Ber}(A_{ba}(\varphi_b(x))) \cdot \mathrm{Ber}(A_{cb}(\varphi_a(x)))|^{-\lambda} \cdot a \\
&= |\mathrm{Ber}(A_{b,a,c,b}(x))|^{-\lambda} \cdot a ,
\end{aligned}$$

where $A_{b,a,c,b}(x)$, the product of the matrix representations of $A_{cb}(\varphi_b(x))$ and $A_{ba}(\varphi_a(x))$ (as left $\mathcal{A}$-linear operators, in the middle coordinates), represents $A_{ca}(\varphi_a(x))$:

$$\begin{aligned}
(A_{b,a,c,b}(x))^i_j &= (A_{ba}(\varphi_a(x)))^i_k \cdot (A_{cb}(\varphi_b(x)))^k_j \\
&= (\partial_{x_a^i} \varphi_{ba}^k)(\varphi_a(x)) \cdot (\partial_{x_b^k} \varphi_{cb}^j)(\varphi_b(x)) \\
&= (\partial_{x_a^i} \varphi_{ca}^j)(\varphi_a(x)) .
\end{aligned}$$

The vector bundle $\mathcal{F}_\lambda M$ corresponding to these transition functions Ψ_{ba} is the *bundle of λ-densities* over M. Note that above a chart, any local section of $\mathcal{F}_\lambda M$ can be written as $\phi = f \cdot |Dx|^\lambda$, where f is a local smooth function while $|Dx|^\lambda$ stands for the local section whose local expression in the chart is the constant function 1.

[1] The function $|\cdot| : \mathrm{inv}\mathcal{A}_0 \to \mathrm{inv}\mathcal{A}_0$ is defined by $|a| = \mathrm{sign}(\mathbf{B}a) \cdot a$ while $\cdot^{-\lambda} : \{a \in \mathcal{A}_0 : \mathbf{B}a > 0\} \to \mathcal{A}_0$ is defined from the ordinary smooth function $\cdot^{-\lambda} : \widetilde{\mathbb{R} \setminus \{0\}} \to \mathbb{R} \setminus \{0\}$ by means of a Taylor expansion in the nilpotent part (see the definition of the functions $\widetilde{f_{i_1 \ldots i_r}}$ in Subsection A.2.2).

A functor

Definition. The bundle functor $\mathcal{F}_\lambda$ associates with an $\mathcal{A}$-manifold M its vector bundle of λ-densities while the image of a morphism $\Phi : W \subset P \times M \to N$ is the smooth collection $\mathcal{F}_\lambda\Phi : \{(p,\phi) : (p,\pi(\phi)) \in W\} \subset P \times \mathcal{F}_\lambda M \to \mathcal{F}_\lambda N$ of fiberwise even $\mathcal{A}$-linear maps defined locally as

$$\iota\left(a \cdot |Dx|^\lambda|_x\right)(\mathcal{F}_\lambda\Phi)_p = |\mathrm{Ber}(A_{\Phi_p}(x))|^{-\lambda} \cdot a \cdot |Dy|^\lambda|_{\Phi_p(x)} \ ,$$

where $A_{\Phi_p}(x)$ is given in terms of a basis $\{e_i\}$ of $E^{n|m}$ and its left dual basis $\{{}^ie\}$ by

$$\iota(h^k \cdot e_k)(A_{\Phi_p}(x)) = \sum_{k,l} h^k \cdot (\partial_{x^l}\Phi^k(p,x)) \cdot e_l \ .$$

Geometric Lie derivative

Proposition 23. *Let $X : P \times M \to TM$ be a smooth family of even vector fields on M. If $\phi \in \Gamma(\mathcal{F}_\lambda M)$ reads locally as $\phi = f \cdot |Dx|^\lambda$, we have*

$$(\mathrm{L}_X\phi)(p,x) = \left(X^j(p,x)(\partial_j f)(x) + \lambda \cdot (-1)^{\varepsilon_i} \cdot (\partial_{x^i}X^i)(p,x) \cdot f(x)\right) \cdot |Dx|^\lambda|_x \ . \qquad (4.1)$$

In particular, when $M = E_0^{n|m}$ and $p \in \mathbf{B}P$, we recover the Lie derivative of Chapter 1, i.e.,

$$(\mathrm{L}_X\phi)_p = \left(D_{X_p}(f) + \lambda \cdot \mathrm{div}(X_p) \cdot f\right) \cdot |Dx|^\lambda = \left(\mathrm{L}^\lambda_{X_p} f\right) \cdot |Dx|^\lambda \ .$$

Proof. It follows from the definition of Lie derivatives (Chapter 2) and the chain rule that

$$\begin{aligned}
(\mathrm{L}_X\Phi)(p,x) &= \left(\left(\partial_t \cdot \left((t,p,x) \mapsto |\mathrm{Ber}(A_{\Phi_{X,(-t,p)}}(\Phi_{X,(t,p)}(x)))|^{-\lambda} \cdot (f \circ \Phi_{X,(t,p)})\right)\right)(0,p,x)\right) \cdot |Dx|^\lambda|_x \\
&= \left(\left(\frac{\partial \Phi^i_X}{\partial t}(0,p,x)\right) \cdot \left(\frac{\partial f}{\partial x^i}(\Phi_X(0,p,x))\right)\right) \cdot |Dx|^\lambda|_x \\
&\quad - \left(\lambda \cdot \left(\partial_t \cdot \left((t,p,x) \mapsto \mathrm{Ber}(A_{\Phi_{X,(-t,p)}}(\Phi_{X,(t,p)}(x)))\right)\right)(0,p,x) \cdot f(x)\right) \cdot |Dx|^\lambda|_x \\
&= \left(X^i(p,x) \cdot (\partial_{x^i} f)(x) + \lambda \cdot \mathrm{str}\left(\partial_t A_{\Phi_{X,(t,p)}}(x)|_{t=0}\right) \cdot f(x)\right) \cdot |Dx|^\lambda|_x \ ,
\end{aligned}$$

where the graded trace, arising from $\mathrm{Jac(Ber)(id_E)} = \mathrm{str}$ (cf. [**?**, III.3.14]), is given by

$$\begin{aligned}
\mathrm{str}\left(\partial_t A_{\Phi_{X,(t,p)}}(x)|_{t=0}\right) &= \mathrm{str}\left(\sum_{k,l} {}^ke \cdot (\partial_t \cdot \partial_{x^l}\Phi^k_X(0,p,x)) \otimes e_l\right) \\
&= \sum_{k,l} (-1)^{\varepsilon_l \cdot (\varepsilon_k + \varepsilon_l + \varepsilon(X^k(p,x)))} \delta^k_l \cdot (\partial_{x^l} X^k(p,x)) \ ,
\end{aligned}$$

where $\varepsilon(X^k(p,x)) = \varepsilon_k$ since $X(p,x)$ is even. Hence the proposition. $\square$

4.1.2 The bundle of weighted graded symmetric tensors

Let M be an $\mathcal{A}$-manifold of dimension $n|m$ and let $\{(U_a \subset M, \varphi_a : U_a \to E_0^{n|m})\}$ be the atlas of all charts for M. Let $\delta \in \mathbb{R}$ and $r \in \mathbb{N}$. We define $\vee_\delta^r(\varphi_{ba}) : \varphi_a(U_b \cap U_a) \to \mathrm{Aut}(\vee^r E^{n|m})$, in terms of a basis $(e_1, ..., e_{m+n})$ of $E^{n|m}$, by setting

$$\begin{aligned}\iota(S^{k_1,\dots,k_r} \cdot e_{k_1} \vee \cdots \vee e_{k_r})((\vee_\delta^r(\varphi_{ba}))(x)) = \\ |\mathrm{Ber}(A_{ba}(x))|^{-\delta} \cdot S^{k_1,\dots,k_r} \cdot (\partial_{x^{k_1}} \varphi_{ba}^{l_1}(x)) \cdot e_{l_1} \vee \;\cdots\; \vee (\partial_{x^{k_r}} \varphi_{ba}^{l_r}(x)) \cdot e_{l_r} \ ,\end{aligned}$$

where $A_{ba}(x)$ is given by

$$A_{ba}(x) = \sum_{k,l} {}^k e \cdot (\partial_{x^l} \varphi_{ba}^k(x)) \otimes e_l \ .$$

In other words, we have

$$\iota(S^{k_1,\dots,k_r} \cdot e_{k_1} \vee \cdots \vee e_{k_r})((\vee^r(\varphi_{ba}))(x)) = \tilde{S}^{l_1,\dots,l_r} \cdot e_{l_1} \vee \cdots \vee e_{l_r} \ ,$$

where

$$\tilde{S}^{l_1,\dots,l_r} = |\mathrm{Ber}(A_{ba}(x))|^{-\delta} \cdot S^{k_1,\dots,k_r} \cdot (-1)^{\sum_{s=2}^r (\epsilon_{k_s} + \epsilon_{l_s}) \cdot (\sum_{t=1}^{s-1} \epsilon_t)} \cdot (\partial_{x^{k_1}} \varphi_{ba}^{l_1}(x)) \cdot \cdots \cdot (\partial_{x^{k_r}} \varphi_{ba}^{l_r}(x)) \ .$$

It follows from the chain rule that the functions $\vee_\delta^r(\varphi_{ba}) \circ \varphi_a : U_b \cap U_a \to \mathrm{Aut}(\vee^r E^{n|m})$ satisfy the cocycle conditions (B.1). The corresponding vector bundle $\vee_\delta^r M$ is called the *bundle of graded symmetric tensors of degree r and weight δ* over M.

Remark. For $\delta = 0$, the bundle of graded symmetric tensors of degree r over M is nothing but the r-th graded symmetric tensor power of the tangent bundle of M. In particular, $\vee_0^1 M = TM$. For a general δ, we have an isomorphism

$$\vee_\delta^r M \cong \mathcal{F}_\delta M \otimes \vee^r TM$$

and thus also an isomorphism of $\mathrm{C}^\infty(M)$-modules,

$$\mathcal{I}_\delta : \Gamma(\mathcal{F}_\delta M) \otimes_{\mathrm{C}^\infty(M)} \vee^r \Gamma(TM) \xrightarrow{\cong} \Gamma(\vee_\delta^r M) \ ,$$

induced by the canonical isomorphism

$$\Gamma(\mathcal{F}_\delta M) \otimes_{\mathrm{C}^\infty(M)} \vee^r \Gamma(TM) \cong \Gamma(\mathcal{F}_\delta M \otimes_M \vee^r TM) \ .$$

A functor

Definition. The bundle functor $\vee^r_\delta$ associates with an $\mathcal{A}$-manifold M its vector bundle of graded symmetric tensors of degree r and weight δ while the image of a morphism $\Phi : W \subset P \times M \to N$ is the smooth collection of fiberwise left $\mathcal{A}$-linear maps $\vee^r_\delta\Phi : \{(p,S) : (p,\pi(S)) \in W\} \subset P \times \vee^r_\delta M \to \vee^r_\delta N$ defined locally as

$$\begin{aligned}
&\iota\left(S^{i_1,\ldots,i_r} \cdot |Dx|^\delta \cdot \partial_{x^{i_1}}|_x \vee \cdots \vee \partial_{x^{i_r}}|_x\right)(\vee^r_\delta\Phi)_p \\
&= |\mathrm{Ber}(A_{\Phi_p}(x))|^{-\delta} \cdot S^{i_1,\ldots,i_r} \cdot |Dx|^\delta \cdot (-1)^{\sum_{s=2}^r (i_s+j_s)\cdot(\sum_{t=1}^{s-1} t)} \\
&\qquad \cdot (\partial_{x^{i_1}}\Phi^{j_1}(p,x)) \cdot \;\cdots\; \cdot (\partial_{x^{i_r}}\Phi^{j_r}(p,x)) \cdot \partial_{y^{j_1}}\big|_{\Phi(p,x)} \vee \cdots \vee \partial_{y^{j_r}}\big|_{\Phi(p,x)} \;. \qquad (4.2)
\end{aligned}$$

Geometric Lie derivative

Let $X : P \times M \to TM$ be a smooth family of even vector fields on M. If S reads locally as $S^{i_1,\ldots,i_r} \cdot |Dx|^\delta \cdot \partial_{x^{i_1}} \vee \cdots \vee \partial_{x^{i_r}} \in \Gamma(\vee^r_\delta M)$, we have

$$(\mathrm{L}_X S)(p,x) = \bar{S}^{i_1,\ldots,i_r}(p,x) \cdot |Dx|^\delta|_x \otimes \partial_{x^{i_1}}|_x \vee \cdots \vee \partial_{x^{i_r}}|_x$$

with

$$\begin{aligned}
\bar{S}^{i_1,\ldots,i_r}(p,x) \;=\;& X^l(p,x) \cdot (\partial_{x^l} S^{i_1,\ldots,i_r})(x) \\
& +\delta \cdot S^{i_1,\ldots,i_r}(x) \cdot (-1)^i \cdot (\partial_{x^i} X^i)(p,x) \\
& -\sum_{j=1}^r (-1)^{\sum(\epsilon_l+\epsilon_{i_j})(\epsilon_{i_1}+\ldots+\epsilon_{i_{j-1}})} \cdot S^{i_1,..i_{j-1},l,i_{j+1},.,i_r}(x) \cdot (\partial_{x^l} X^{i_j})(p,x)
\end{aligned}$$

In particular, if $M = E_0^{n|m}$ and $p \in \mathbf{B}P$, then if we see $(\mathrm{L}_X S)_p$ through the isomorphism

$$\Gamma(\vee^r_\delta E_0^{n|m}) \cong \mathcal{F}_\delta \otimes \vee^r \Gamma(TE_0^{n|m}) = \mathcal{S}^k_\delta \;,$$

then we have

$$(\mathrm{L}_X S)_p = \mathrm{L}^{\delta,r}_{X_p} S \;,$$

where $\mathrm{L}^\delta_{X_p} S$ stands for the Lie derivative of weighted tensor fields in the direction of smooth vector fields defined in Chapter 1 (formula (1.6)).

4.1.3 The bundle of differential operators

Let M be an $\mathcal{A}$-manifold of dimension $n|m$ and let $\{(U_a \subset M, \varphi_a : U_a \to E_0^{n|m})\}$ be the atlas of all charts for M. Let $\lambda, \mu \in \mathbb{R}$ and $r \in \mathbb{N}$. We set $\mathcal{D}^r_{\lambda,\mu}(E^{n|m}) = \oplus_{s=0}^r \vee^s E^{n|m}$ and we define $\mathcal{D}^r_{\lambda,\mu}(\varphi_{ba}) : \varphi_a(U_a \cap U_b) \to \mathcal{D}^r_{\lambda,\mu}(E)$ by analogy with the transformation law of the coefficient functions of differential operators from λ-densities to μ-densities under a change of chart from (U_a, φ_a) to (U_b, φ_b): in terms of a basis $(e_1, ..., e_{m+n})$ of E, we thus set

$$\iota\left(\sum_{s=0}^r D^{i_1,\ldots,i_s} \cdot e_{i_1} \vee \cdots \vee e_{i_s}\right)(\mathcal{D}^r_{\lambda,\mu}(\varphi_{ba}))(x) = \sum_{s=0}^r \bar{D}^{i_1,\ldots,i_s}(\varphi_{ba}(x)) \cdot e_{i_1} \vee \cdots \vee e_{i_s} \ ,$$

where $\bar{D}^{i_1,\ldots,i_r}$ are the coefficient functions in the chart (U_b, φ_b) of the differential operator $\bar{D}$ acting on local smooth functions on $U_a \cap U_b$, given by

$$\bar{D}(f) = |\mathrm{Ber}(A_{ba})|^{-\mu} \circ D(|\mathrm{Ber}(A_{ba})|^{\lambda} \cdot f) \ ,$$

with D standing for the differential operator $\bar{D}$ acting on local smooth functions on $U_a \cap U_b$ with (constant) coefficients functions $D^{i_1,\ldots,i_r}$ in the chart (U_a, φ_a).

Remark. Since only real constants are smooth, our definition holds only for real coefficients $D^{i_1,\ldots,i_r}$. However, we can extend the definition to coefficients in $\mathcal{A}$ by left $\mathcal{A}$-linearity.

By construction, the functions $(\mathcal{D}^r_{\lambda,\mu}(\varphi_{ba})) \circ \varphi_a : U_b \cap U_a \to \mathrm{Aut}(\mathcal{D}^r(E))$ satisfy the cocycle conditions (B.1). The corresponding vector bundle $\mathcal{D}^r_{\lambda,\mu}M$ is called the *bundle of linear differential operators of order* r from λ-densities to μ-densities.

A functor

Definition. The bundle functor $\mathcal{D}^r$ associates with an $\mathcal{A}$-manifold M its bundle of linear differential operators of order r from λ-densities to μ-densities while the image of a morphism $\Phi : W \subset P \times M \to N$ is the smooth collection $\mathcal{D}^r_{\lambda,\mu}\Phi : \{(p, D) : (p, \pi(D)) \in W\} \subset P \times \vee^r M \to \vee^r N$ defined locally by

$$\iota\left(\sum_{s=0}^r D^{i_1,\ldots,i_r} \cdot \partial_{x^{i_1}}|_x \vee \cdots \vee \partial_{x^{i_r}}|_x\right)((\mathcal{D}^r_{\lambda,\mu}\Phi)_p) = \sum_{s=0}^r \bar{D}^{i_1,\ldots,i_r}(\Phi_p(x)) \cdot \partial_{y^{i_1}}|_{\Phi_p(x)} \vee \cdots \vee \partial_{y^{i_r}}|_{\Phi_p(x)} \ ,$$

where $\bar{D}^{i_1,\ldots,i_r}$ are the coefficient functions of the differential operator $\bar{D}$ on local smooth functions, given by

$$\bar{D}(f) = |\mathrm{Ber}(A_{\Phi_p})|^{-\mu} \circ D(|\mathrm{Ber}(A_{\Phi_p})|^{\lambda} \cdot f) \ ,$$

where $D = \sum_{s=0}^r D^{i_1,\ldots,i_r} \cdot \partial_{x^{i_1}} \circ \cdots \circ \partial_{x^{i_r}}$.

Geometric Lie derivative

Let $X : P \times M \to TM$ be a smooth family of even vector fields on M. There is a correspondence

$$\mathcal{I}_{\lambda,\mu} : \Gamma(\mathcal{D}^r_{\lambda,\mu}M) \xrightarrow{\cong} \mathcal{D}^r_{\lambda,\mu}(\Gamma(\mathcal{F}_\lambda M), \Gamma(\mathcal{F}_\lambda M)) \ ,$$

given locally by

$$\mathcal{I}_{\lambda,\mu}\left(\sum_{s=0}^{r} D^{i_1,\ldots,i_r} \cdot \partial_{x^{i_1}} \vee \cdots \vee \partial_{x^{i_r}}\right) = \sum_{s=0}^{r} D^{i_1,\ldots,i_r} \cdot |Dx|^\delta \otimes \partial_{x^{i_1}} \circ \cdots \circ \partial_{x^{i_r}} \ ,$$

where $\partial_{x^{i_1}} \vee \cdots \vee \partial_{x^{i_r}}$ is the local section of $\mathcal{D}^r_{\lambda,\mu}M$ with local expression $e_{i_1} \vee \cdots \vee e_{i_r}$ in the coordinates $(x^1, \ldots, x^{n+m})$. Through that correspondence, we have

$$L_X D = \mathrm{L}_X \circ D - D \circ \mathrm{L}_X$$

In particular, when $M = E_0^{n|m}$ and $p \in \mathbf{B}P$, we recover through the correspondence $\Gamma(\mathcal{D}^r_{\lambda,\mu}E_0^{n|m}) \cong \mathcal{D}_{\lambda,\mu}$ the (algebraic) Lie derivative of differential operators defined in Chapter 1 (formula 1.8), i.e.,

$$(\mathrm{L}_X D)_p = \mathrm{L}_{X_p} \circ D - D \circ \mathrm{L}_{X_p} \ .$$

Symbol map

Let $\lambda, \mu \in \mathbb{R}$. We define the *symbol map*

$$\sigma : \bigcup_{r\in\mathbb{N}} \widetilde{\Gamma}(\mathcal{D}^r_{\lambda,\mu}M) \to \bigoplus_{r\in\mathbb{N}} \widetilde{\Gamma}(\vee^r_\delta M)$$

by setting, in local coordinates,

$$\sigma(D)(p,x) = D^{i_1,\ldots,i_r}(p,x) \cdot \partial_{x^{i_1}}|_x \vee \cdots \vee \partial_{x^{i_r}}|_x$$

if $D(p,x) = D^{i_1,\ldots,i_r}(p,x) \cdot \partial_{x^{i_1}}|_x \vee \cdots \vee \partial_{x^{i_r}}|_x +$ lower degree terms.

Remark. The map σ is well-defined because the transformation law of the higher order terms of differential operators under a change of chart from (U_a, φ_a) to (U_b, φ_b) coincide with $\vee^r_\delta(\varphi_{ba})$.

Remark. For $M = E_0^{n|m}$, the restriction to $\Gamma(\mathcal{D}^r_{\lambda,\mu}M)$ of this symbol map coincide, through the correspondences $\mathcal{I}_\delta$ and $\mathcal{I}_{\lambda,\mu}$, with the principal symbol operator $\sigma_r : \mathcal{D}^k_{\lambda,\mu} \to \mathcal{S}^k_\delta$ defined in Chapter 1 (formula (1.9)).

4.1.4 Natural Projectively Invariant Quantization

Quantizations

An *extended quantization* on M is an even regular bijection

$$Q : \oplus_{k\in\mathbb{N}}\widetilde{\Gamma}(\vee^k_\delta M) \to \bigcup_{k\in\mathbb{N}} \widetilde{\Gamma}(\mathcal{D}^k_{\lambda,\mu}M)$$

such that:

(Lin.) The map Q is left $\mathcal{A}$-linear and even:

$$\begin{cases} Q(S+S')(p,p',x) &= Q(S)(p,x)+Q(S')(p',x) \;, \\ Q(\mathcal{A}\cdot S)(a,p,x) &= a\cdot(Q(S)(p,x)) \;, \\ \epsilon(Q(S)_p)=\epsilon(S_p) \;. \end{cases}$$

In particular, Q is $\mathbb{R}$-linear.

(Quant.) The map Q is symbol-preserving:

$$\sigma(Q_M(S)) = S \;.$$

Definition. A *quantization on* M is a map $\oplus_{k\in\mathbb{N}}\Gamma(\vee^k_\delta M) \to \bigcup_{k\in\mathbb{N}}\Gamma(\mathcal{D}^k_{\lambda,\mu}M)$ that is the restriction to smooth sections of an extended quantization.

NPIQ

An *extended natural quantization* on $\mathcal{A}\text{-}\widetilde{\mathrm{Man}}_{n|m}$ is a collection $\mathcal{Q}=\{Q^k : k\in\mathbb{N}\}$ of natural operators $Q^k : (\mathcal{C}\times\vee^k_\delta) \to \mathcal{D}^k_{\lambda,\mu}$, i.e., a collection of maps

$$Q^k_M : \widetilde{\Gamma}(\mathcal{C}M)\times\widetilde{\Gamma}(\vee^k_\delta M) \to \widetilde{\Gamma}(\mathcal{D}^k_{\lambda,\mu}M) \;,$$

such that for any $\nabla\in\widetilde{\Gamma}(\mathcal{C}M)$, the maps $Q^k_M(\nabla,\cdot)$ ($k\in\mathbb{N}$) define a quantization on M. Finally, an extended natural quantization Q_M is called *projectively invariant* if one has

$$Q_M(\nabla,\cdot) = Q_M(\nabla',\cdot)$$

whenever ∇ and ∇' are projectively equivalent.

Definition. A *natural projectively invariant quantization (NPIQ) on* $\mathcal{A}\text{-}\mathrm{Man}_{n|m}$ is a collection of maps $Q^k_M : \Gamma(\mathcal{C}M)\times\Gamma(\vee^k_\delta M)\to\Gamma(\mathcal{D}^k_{\lambda,\mu}M)$ that can be extended to smooth families in order to form an extended natural projectively invariant quantization on $\mathcal{A}\text{-}\widetilde{\mathrm{Man}}_{n|m}$.

4.1.5 From NPIQ to PEQ

Let Z be the smooth family of all even fundamental vector fields associated with the action of the projective group on the projective space, i.e.,

$$Z : \mathfrak{paut}(n+1|m,\mathcal{A})_0 \times E_0^{n+m} \to TE_0^{n+m},\ (h,x) \mapsto X_x^h\ .$$

LEMMA 24. *Let $\tilde{Q}$ be an extended quantization on $M = E_0^{n|m}$. If $\tilde{Q}$ is equivariant with respect to the Lie derivative in the direction of Z, then its restriction Q to smooth sections is projectively equivariant in the sense of Chapter 1.*

Proof. We need to show that

$$\mathrm{L}^{\mathrm{alg}}_{X^h}(Q(S)) = Q(\mathrm{L}^{\mathrm{alg}}_{X^h}S)\ .$$

for all $h \in \mathbf{B}\mathfrak{paut}(n+1|m,\mathcal{A})$. However, by $\mathbb{R}$-linearity, it is enough to show this equality for a basis $\{e_i\}$ of the $\mathcal{A}$-vector space $\mathfrak{paut}(n+1|m,\mathcal{A})$.

For even elements e_i, the result is immediate because we have already noticed that the geometric and algebraic Lie derivatives of differential operators and symbols (i.e., weighted graded symmetric tensor fields) coincide when $M = E_0^{n|m}$ and the parameter is in the body. Indeed, since $e_i \in \mathbf{B}\mathfrak{paut}(n+1|m,\mathcal{A})_0$, we have

$$\begin{aligned} \mathrm{L}^{\mathrm{alg}}_{X^{e_i}}(Q(S))(x) &= \mathrm{L}_Z(Q(S))(e_i,x) \\ &= \tilde{Q}(\mathrm{L}_Z S)(e_i,x) \\ &= Q(\mathrm{L}^{\mathrm{alg}}_{X^{e_i}}S)(x)\ , \end{aligned}$$

where the last equality is obtained using the regularity property of $\tilde{Q}$.

Unfortunately, odd elements e_i do not belong to $\mathfrak{paut}(n+1|m,\mathcal{A})_0$, the parameter space of Z. Nevertheless, $\mathfrak{paut}(n+1|m,\mathcal{A})_0$ contains the linear combinations of the odd elements e_i with odd coefficients and those coefficients are nothing but the odd coordinates in $\mathfrak{paut}(n+1|m,\mathcal{A})_0$. In order to show equivariance of Q with respect to e_i, the idea is differentiate the Z-equivariance of Q in the direction of the odd coordinate h^i.

To this aim, we first notice that for differential operators (resp. symbols), we have

$$(\mathrm{L}_Z D)(h,x) = \sum_i h^i \cdot \mathrm{L}^{\mathrm{alg}}_{X^{e_i}} D \quad \left(\text{resp. } (\mathrm{L}_Z S)(h,x) = \sum_i h^i \cdot \mathrm{L}^{\mathrm{alg}}_{X^{e_i}} S\right)\ ,$$

if $h = \sum_i h^i \cdot e_i$. This is easily seen from the local expression of the Lie derivatives and of the vector fields X^h (see Chapter 1).

Then, using the hypothesis, we get

$$\begin{aligned}
\sum_i h^i \cdot \left(\left(\mathrm{L}^{\mathrm{alg}}_{X^{e_i}}(Q(S))\right)(x)\right) &= \left(\mathrm{L}_Z(Q(S))\right)(h,x) \\
&= \tilde{Q}(\mathrm{L}_Z S)(h,x) \\
&= \tilde{Q}\left(\sum_{i=0}^{d} \mathcal{A}\cdot \mathrm{L}^{\mathrm{alg}}_{X^{e_i}} S\right)(h^1,\ldots,h^d,x) \\
&= \sum_i h^i \cdot \left(Q(\mathrm{L}^{\mathrm{alg}}_{X^{e_i}} S)(x)\right) \ ,
\end{aligned}$$

where $d = \dim(\mathfrak{paut}(n+1|m,\mathcal{A}))$. Note that the third equality follows from the regularity property of $\tilde{Q}$ while the fourth equality can be obtained thanks to the left $\mathcal{A}$-linearity of $\tilde{Q}$.

Finally, since both sides of the equation are smooth families of sections of the bundle of differential operators, we can differentiate with respect to the coordinate h^i and get the equivariance of Q with respect to the the odd vector fields X^{e_i}. □

Proposition 25. *If $\tilde{\mathcal{Q}}$ is an extended NPIQ on $\mathcal{A}$-$\mathrm{Man}_{n|m}$, then*

$$Q_0 = \tilde{\mathcal{Q}}_{E_0^{n|m}}(\nabla_0,\cdot)\Big|_{\oplus_{k\in\mathbb{N}}\Gamma(\vee^k_\delta E_0^{n|m})}$$

is equivariant with respect to the Lie derivative in the direction of Z.

Proof. For any $(h,x)\in \mathfrak{paut}(n+1|m,\mathcal{A})_0 \times E_0^{n|m}$ and any $S\in\oplus_{k\in\mathbb{N}}\Gamma(\vee^k_\delta E_0^{n|m})$, we have

$$\begin{aligned}
\mathrm{L}_Z(Q_0(S))(h,x) &= \partial_t\cdot\left(\Phi_Z^*\left(\tilde{\mathcal{Q}}_{E_0^{n|m}}(\nabla_0,S)\right)\right)(t=0,h,x)) \\
&= \partial_t\cdot\left(\tilde{\mathcal{Q}}_{E_0^{n|m}}\left((\Phi_Z^*\nabla_0),(\Phi_Z^*S)\right)\right)(t=0,h,x) \\
&= \partial_t\cdot\left(\tilde{\mathcal{Q}}_{E_0^{n|m}}\left(\nabla_0,(\Phi_Z^*S)\right)\right)(t=0,h,x) \\
&= \tilde{\mathcal{Q}}_{E_0^{n|m}}(\nabla_0,\partial_t\cdot(\Phi_Z^*S))(t=0,h,x) \\
&= Q_0(\mathrm{L}_Z S)(h,x)\ .
\end{aligned}$$

The first equality is nothing but the definition of the Lie derivative. The second equality follows from the naturality property of $\tilde{\mathcal{Q}}$. The third equality follows from corollary 22 and the projective invariance of $\tilde{\mathcal{Q}}$. The fourth equality is a consequence of Peetre theorem for local regular even $\mathcal{A}$-linear operators actions on smooth families of sections of vector bundles (see Section 2.4). Finally, the last equality follows from the regularity property of $\tilde{\mathcal{Q}}$ and the definition of the Lie derivative. □

Corollary 26. *If $\mathcal{Q}$ is a NPIQ on $\mathcal{A}$-$\mathrm{Man}_{n|m}$, then $Q=\mathcal{Q}_{E_0^{n|m}}(\nabla_0,\cdot)$ is a PEQ on $E_0^{n|m}$.*

4.2 M. BORDEMANN's construction of a NPIQ can be adapted on $\mathcal{A}$-manifolds.

4.2.1 Thomas bundle and Thomas manifold

Let M be an $\mathcal{A}$-manifold of dimension $n|m$ and let $\{(U_a \subset M, \varphi_a : U_a \to E_0)\}$ be the atlas of all charts for M. We introduce $\mathcal{T}(\varphi_{ba}) : U_a \cap U_b \to \mathrm{Aff}(\mathcal{A})$ by setting

$$\mathcal{T}(\varphi_{ba})(x)(a) = a + \log |\mathrm{Ber}(A_{ba}(x))| \,, \tag{4.3}$$

where $A_{ba}(x)$ is given in terms of a basis $\{e_i\}$ of $E^{n|m}$ and its left dual basis $\{{}^i e\}$ by $\iota(h^k \cdot e_k)(A_{ba}(x)) = \sum_{k,l} h^k \cdot (\partial_{x^l} \varphi^k_{ba}(x)) \cdot e_l$.

REMARK. The function $\log : \{a \in \mathcal{A}_0 : \mathbf{B}a > 0\} \to \mathcal{A}_0$ the unique smooth function (see [**?**, III.5.25]) such that for any $a \in \mathcal{A}_0$ with $\mathbf{B}a > 0$, we have

$$\mathbf{B}(\log a) = \log(\mathbf{B}a) \,.$$

It follows from the chain rule that the functions $\mathcal{T}(\varphi_{ba}) \circ \varphi_a : U_b \cap U_a \to \mathrm{Aff}(\mathcal{A})$ satisfy the cocycle conditions (B.1), thus defining the *Thomas bundle* $\mathcal{T}M$ *of* M. Note that above a chart, any local section of $\mathcal{T}M$ can be written as $\phi = f + \log(|Dx|)$, where $\log(|Dx|)$ stands for the local section whose local expression in the chart is the constant function 0.

REMARK. The computations showing that the cocycle conditions are satisfied are the same as for the bundles of densities (see Subsection 4.1.1). Moreover, up to the log operation, the transition functions of the (super) Thomas bundle correspond to the transformation law of sections of the Berezinian sheaf (see [**?**] for a formal definition).

DEFINITION. The bundle functor $\mathcal{T}$ associates with an $\mathcal{A}$-manifold M its Thomas bundle while the image of a morphism $\Phi : W \subset P \times M \to N$ is the smooth collection $\mathcal{T}\Phi : \{(p, z) : (p, \pi(z)) \in W\} \subset P \times \mathcal{T}M \to \mathcal{T}N$ defined in fibered coordinates as

$$(\mathcal{T}\Phi)_p \left(a + \log(|Dx|)|_x\right) = a + \log |\mathrm{Ber}(A_{\Phi_p}(x))| + \log(|Dx|)|_{\Phi_p(x)} \,.$$

DEFINITION. The *Thomas manifold* $\tilde{M}$ of M is the even subspace of $\mathcal{T}M$ made of those points whose image through any local trivialization in the $\mathrm{Aff}(\mathcal{A})$-atlas constructed above lie in $\mathcal{A}_0$. In other words, $\tilde{M}$ is an $\mathcal{A}$ manifold with local coordinates $(x^0, x^1, \ldots, x^{n+m})$, where the coordinates $x^1, \ldots, x^{n+m}$ transform as the coordinates of M while the extra even coordinate x^0 transforms according to formula (4.3).

Densities on M are equivariant functions on $\tilde{M}$.

By analogy with the classical situation, we set $\mathcal{E} = \partial_{x^0}$ to represent the partial derivative with respect to the extra even coordinate of $\tilde{M}$ and we call it the *Euler vector field* of $\tilde{M}$. The fact that $\mathcal{E}$ is well-defined is easily seen from (4.3) and the transformation law of the components of a vector field X under a change of coordinates.

Densities on M identify with some superfunctions on $\tilde{M}$. More precisely, we can associate with a λ-density expressed locally as $\Phi = f \cdot |Dx|^\lambda$ the superfunction $\tilde{\phi}$ given by

$$\tilde{\phi}(x^0, x^1, \ldots, x^{n+m}) = f(x^1, \ldots, x^{n+m}) \cdot \exp(\lambda \cdot x^0). \tag{4.4}$$

It follows directly from the transformation law of densities that $\tilde{\phi}$ is well-defined. Moreover, it is λ-equivariant in the sense that

$$\mathrm{L}_{\mathcal{E}} \tilde{\phi} = \lambda \cdot \tilde{\phi}.$$

Conversely, from a λ-equivariant superfunction f on $\tilde{M}$, one defines a λ-density ϕ_f on M by setting

$$\phi_f(x^1, ..., x^n) = f(x^0, x^1, ..., x^n) \cdot \exp(-\lambda \cdot x^0) \cdot |Dx|^\lambda \tag{4.5}$$

for an arbitrary x^0. Because of the equivariance property of f, the derivative of ϕ_f with respect to x^0 is zero and the density is well-defined. This way, we establish a one-to-one correspondence between λ-densities on M and λ-equivariant superfunctions on $\tilde{M}$.

4.2.2 Projectively invariant lift of torsion-free connections

Let ∇ be a torsion-free connection on M. We are going to define a lifted torsion-free connection $\tilde{\nabla}$ on $\tilde{M}$ in terms of horizontal lifts of supervector fields on M.

Horizontal lift of vector fields

Definition. In the coordinates of $\tilde{M}$, the *horizontal lift* to $\tilde{M}$ of a super vector field $X = X^i \cdot \partial_{x^i}$ on M is defined by

$$X^{h(\nabla)} = -(-1)^{\varepsilon_s} \cdot X^i \cdot \Gamma_{is}^s \cdot \partial_{x^0} + X^i \cdot \partial_{x^i} \,. \tag{4.6}$$

The fact that this super vector field $X^{h(\nabla)}$ is well-defined can be checked in local coordinates.

Remark. The fact that the local components in formula (4.6) do indeed transform according to the transformation law of super vector fields on $\tilde{M}$ is a consequence of the following facts. On the one hand, knowing [?, III.3.14] that Jac(Ber)(id) = str, we can show that([2])

$$X^i \cdot \partial_{x^i} \log(|\mathrm{Ber}(A_{ba}(x))|) \quad = \quad X^i \cdot \mathrm{str}({}^j e \cdot (\partial_{x^i} A_{ba}(x))^i_k \cdot (A^{-1}_{ba}(x))^k_j \otimes e_i) \;,$$

where $\partial_{x^i} A_{ba}(x)$ is given in terms of a basis $\{e_i\}$ of $E^{n|m}$ by

$$\sum_{k,l} {}^k e \cdot (-1)^{\varepsilon_i . \varepsilon_k} \cdot (\partial_{x^i} \cdot \partial_{x^l} \varphi^k_{ba}(\varphi_a(x)) \otimes e_l \;.$$

On the other hand, writing $\bar{X}^i$ (resp. $\bar{\Gamma}^i_{jk}$) the local components of X (resp. the Christoffel symbols of ∇) in another coordinate system, the transformation law of super vector fields on M and of Christoffel symbols (formula (3.11)) gives

$$-(-1)^{\varepsilon_s} \cdot \bar{X}^i \cdot \bar{\Gamma}^s_{is} = -(-1)^{\varepsilon_r} \cdot X^t \cdot \Gamma^r_{tr} - (-1)^{\varepsilon_s} \cdot X^i \cdot \partial_{x^i} \varphi^v_{ba} \cdot \partial_{x^v} \partial_{x^s} \varphi^k_{ba} \cdot \partial_{x^k} \varphi^s_{ba} \;.$$

From these formulas, the computations consists in applying the chain rule extensively.

Graded traces of the curvature tensor

Remember that from the curvature tensor R of ∇, i.e.,

$$\mathrm{R}(X,Y)Z = \nabla_X \nabla_Y Z - (-1)^{\varepsilon(X)\varepsilon(Y)} \cdot \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

the super-Ricci tensor Ric and the tensor strR are defined as graded traces, namely

$$\mathrm{Ric}(Y,Z) = \mathrm{str}(X \mapsto \mathrm{R}(X,Y)Z),$$
$$\mathrm{strR}(X,Y) = \mathrm{str}(Z \mapsto \mathrm{R}(X,Y)Z) \;.$$

Equivalently, in coordinates, Ric and strR are given by

$$\mathrm{Ric}(Y,Z) = (-1)^{\varepsilon_i(\varepsilon_i + \varepsilon(Y) + \varepsilon(Z))} \cdot \iota(\mathrm{R}(\partial_{x^i}, Y)Z)\mathrm{d}x^i$$
$$\mathrm{strR}(X,Y) = (-1)^{\varepsilon_i} \cdot \iota(\mathrm{R}(X,Y)\partial_{x^i})\mathrm{d}x^i \;.$$

Remark. Note that $\mathrm{R}(X,Y)Z$ is *left* $\mathrm{C}^\infty(M)$-linear in X and *right* $\mathrm{C}^\infty(M)$-linear in Z. Therefore Ric is obtained by means of a left graded trace while strR is obtained by means of a right graded trace (see [?, II.5]).

[2] Note that in [?], it is shown that $\partial_{x^i}(\mathrm{Ber}(A)) = (\mathrm{Ber}A) \cdot \mathrm{str}((\partial_{x^i} A) \cdot A^{-1})$.

Projectively invariant lift of torsion-free connections

We denote by r the following multiple of a supersymmetric part of the Ricci tensor of ∇: for homogeneous $X, Y \in \Gamma(TM)$, we set

$$\mathrm{r}(X,Y) = \frac{1}{2(n-m-1)}(\mathrm{Ric}(X,Y) + (-1)^{\varepsilon(X)\varepsilon(Y)} \cdot \mathrm{Ric}(Y,X)) \ .$$

Now let ∇ be a torsion-free connection on M. With notations of (4.4), we set

$$\tilde{\nabla}_{X^{h(\nabla)}} Y^{h(\nabla)} = (\nabla_X Y)^{h(\nabla)} - \frac{1}{2} \cdot \widetilde{\mathrm{strR}(X,Y)} \cdot \mathcal{E} + (n-m+1) \cdot \widetilde{\mathrm{r}(X,Y)} \cdot \mathcal{E}$$

$$\tilde{\nabla}_{X^{h(\nabla)}} \mathcal{E} = \tilde{\nabla}_{\mathcal{E}} X^{h(\nabla)} = \frac{-1}{n-m+1} \cdot X^{h(\nabla)}, \quad \tilde{\nabla}_{\mathcal{E}} \mathcal{E} = \frac{-1}{n-m+1} \cdot \mathcal{E} \ .$$

Proposition 27. *The quantities Π_{ij}^k introduced in (3.28) can be used to express the Christoffel symbols of the lifted connection $\tilde{\nabla}$. These Christoffel symbols are given by*

$$\tilde{\Gamma}_{ij}^k = \Pi_{ij}^k \ , \quad \tilde{\Gamma}_{0\mathfrak{a}}^{\mathfrak{c}} = \tilde{\Gamma}_{\mathfrak{a}0}^{\mathfrak{c}} = \frac{-\delta_{\mathfrak{a}}^{\mathfrak{c}}}{n-m+1},$$

$$\tilde{\Gamma}_{ij}^0 = \frac{n-m+1}{n-m-1} \cdot \left(\partial_{x^q} \Pi_{ij}^q - \Pi_{qi}^p \cdot \Pi_{pj}^q\right) \cdot (-1)^{\varepsilon_q(\varepsilon_q + \varepsilon_i + \varepsilon_j)},$$

where i, j, k ranges from 1 to $n+m$ while $\mathfrak{a}, \mathfrak{c}$ ranges from 0 to $n+m$.([3])

Proof. The result is obtained after long but straightforward local computations. ☐

Corollary 28. *The map $\nabla \mapsto \tilde{\nabla}$ is projectively invariant.*

Definition. The lifted connection $\tilde{\nabla}$ on $\tilde{M}$ is called the *projectively invariant lift* of ∇.

Remark. We need to assume that the superdimension $n-m$ is neither 1 nor -1 for the above formulas to make sense. The case $n-m=1$ is somehow the super prolongation of the fact that M. Bordemann's construction fails for a 1-dimensional smooth manifold. The case $n-m=-1$ has no classical counterpart since negative dimensions do not appear in the context of ordinary smooth manifolds. Note that when $n-m=-1$, the quantities Π_{ij}^k themselves cannot be defined.

Remark. The lifted connection $\tilde{\nabla}$ is associated in a natural way with the connection ∇. Moreover, $\tilde{\nabla}$ is such that $\mathrm{L}_{\mathcal{E}} \tilde{\nabla} = 0$, where $\mathrm{L}_{\mathcal{E}} \tilde{\nabla}(X,Y) = [\mathcal{E}, \tilde{\nabla}_X Y] - \tilde{\nabla}_{[\mathcal{E},X]} Y - \tilde{\nabla}_X [\mathcal{E}, Y]$. This invariance is due to the invariance of $\mathcal{E}$, of the horizontal lifts and of the functions $\widetilde{\mathrm{strR}(X,Y)}$ and $\widetilde{\mathrm{r}(X,Y)}$.

[3]These formulas first appeared in [?].

4.2.3 Construction of the NPIQ

Our goal in this section is to lift in a natural and projectively invariant way a symbol S on M to a tensor $\tilde{S}$ on $\tilde{M}$. To this aim, we define in a first step a horizontal lift of S via the horizontal lift of supervector fields (4.6). In a second step, we define a map which transforms equivariant tensors on $\tilde{M}$ into symbols on M. We prove that the restriction of this map to the divergence-free tensors (with respect to $\tilde{\nabla}$) is a bijection. The natural and projectively invariant lift is then the inverse map of this "descent" application.

Horizontal lift of symbols

Since a symbol S of degree k on M is locally a sum of terms of the form $\phi \otimes \partial_{i_1} \vee \cdots \vee \partial_{i_k}$, it suffices to define the horizontal lift on symbols of this form and to extend it additively.

Definition. In coordinates, the horizontal lift of a symbol $S = \phi \otimes \partial_{i_1} \vee \cdots \vee \partial_{i_k}$ of weight δ is a symbol of weight 0 on $\tilde{M}$:

$$S^{h(\nabla)} = \tilde{\phi} \otimes \partial_{i_1}^{h(\nabla)} \vee \cdots \vee \partial_{i_k}^{h(\nabla)} .$$

The horizontal lift of a symbol S is δ-*equivariant*, i.e. $\mathrm{L}_{\mathcal{E}} S^{h(\nabla)} = \delta \cdot S^{h(\nabla)}$. In the sequel, we denote by $\Gamma(\vee^k \tilde{M})^\delta$ the space of δ-equivariant tensor fields of degree k on $\tilde{M}$.

Remark. The horizontal lift of a δ-density on M (i.e., a tensor of degree 0) to a superfunction on $\tilde{M}$ coincides with the correspondence given in (4.4).

Descent map

Using the fact that tensor fields of degree k on $\tilde{M}$ can be locally decomposed in the basis $\partial_1^{h(\nabla)}, \ldots, \partial_{n+m}^{h(\nabla)}, \mathcal{E}$, it is enough to define the descent map on a tensor of the form

$$S = \sum_{l=0}^{k} \sum_{i_1,\ldots,i_{k-l}} \varphi^{i_1,\ldots,i_{k-l},0,\ldots,0} \otimes \partial_{i_1}^{h(\nabla)} \vee \cdots \vee \partial_{i_{k-l}}^{h(\nabla)} \vee \mathcal{E}^l. \tag{4.7}$$

For any $S \in \Gamma(\vee^k \tilde{M})^\delta$ expressed as in (4.7), we set

$$\Psi(S) = \sum_{i_1,\ldots,i_k} \varphi_0^{i_1,\ldots,i_k} \otimes \partial_{i_1} \vee \cdots \vee \partial_{i_k},$$

where $\varphi_0^{i_1,\ldots,i_k}(x^1, \ldots, x^{n+m}) = \varphi^{i_1,\ldots,i_k}(x^0, x^1, \ldots, x^{n+m}) \cdot \exp(x^0)$ for an arbitrary x^0 (cf. (4.5)). It is easy to check that $\Psi(S)$ is a well-defined symbol of weight δ.

Moreover, the map Ψ is surjective: if A_k is a symbol of degree k and weight δ on M, then any tensor field of the form

$$A_k^{h(\nabla)} + A_{k-1}^{h(\nabla)} \vee \mathcal{E} + \ldots + A_0^{h(\nabla)} \vee \mathcal{E}^k \ , \tag{4.8}$$

where each A_{k-j} $(j = 1, \ldots, k)$ is a symbol of degree $k - j$ and weight δ, is such that $\Psi(S) = A_k$.

Covariant derivative of symbols

A symbol reads locally as a sum of terms of the form $\phi \cdot \partial_{i_1} \vee \cdots \vee \partial_{i_k}$, where ϕ is a local density. We already have a covariant derivative of vector fields. By means of the Lie derivative of equivariant functions on $\tilde{M}$ in the direction of horizontal lift of vector fields, we can define a covariant derivative of densities on M.

Definition. The *covariant derivative of a δ-density* $\phi \in \Gamma(\mathcal{F}_\delta M)$ in the direction of a vector field $X \in \Gamma(TM)$ is the δ-density associated with the δ-equivariant function $\mathrm{L}_{X^{h(\nabla)}}\widetilde{\phi}$ in the sense of Subsection 4.2.1.([4])

In coordinates, using formula (4.1), we obtain

$$\nabla_X \phi = \big(D_X(f) - (-1)^{\varepsilon_s} \cdot \delta \cdot X^i \cdot \Gamma_{is}^s \cdot f\big) \cdot |Dx|^\delta \ ,$$

if $\phi = f \cdot |Dx|^\delta$.

Definition. Given a 1-form $\alpha = \alpha_i \cdot \mathrm{d}x^i$ and a symbol $S = \phi \otimes \partial_{i_1} \vee \cdots \vee \partial_{i_k}$, we set

$$\iota(\alpha)(S) = \sum_{j=1}^k (-1)^{\varepsilon(\alpha).(\varepsilon(\phi)+\varepsilon(i_1)+\ldots+\varepsilon(i_{j-1}))} \cdot \phi \otimes \partial_{i_1} \vee \cdots \overset{(j)}{(-1)^{\varepsilon_{i_j}} \cdot \alpha_{i_j}} \cdots \vee \partial_{i_k} \ ,$$

where $(-1)^{\varepsilon_{i_j}} \cdot \alpha_{i_j}$ replaces ∂_{i_j}. Moreover, we extend this definition to covariant tensor of degree l by setting $\iota(\alpha^1 \vee \cdots \vee \alpha^l) = \iota(\alpha^1) \circ \ldots \circ \iota(\alpha^l)(S)$.([5])

Definition. The covariant derivative with respect to ∇ of a symbol $S = \phi \otimes \partial_{i_1} \vee \cdots \vee \partial_{i_k}$ in the direction of a supervector field X is defined by

$$\begin{aligned} \nabla_X(S) &= \nabla_X \phi \otimes \partial_{i_1} \vee \cdots \vee \partial_{i_k} \\ &\quad + \sum_{j=1}^k (-1)^{\varepsilon(X).(\varepsilon(\phi)+\varepsilon(i_1)+\ldots+\varepsilon(i_{j-1}))} \cdot \phi \otimes \partial_{i_1} \vee \cdots \overset{(j)}{\nabla_X \partial_{i_j}} \cdots \vee \partial_{i_k}. \end{aligned}$$

[4]The function $\mathrm{L}_{X^{h(\nabla)}}\widetilde{\phi}$ is δ-equivariant because $\mathrm{L}_{\mathcal{E}}$ commutes with all $\mathrm{L}_{X^{h(\nabla)}}$.

[5]With respect to [?], there is an additional $(-1)^{\varepsilon_{i_j}}$ because here $\mathrm{d}x^i$ stands for the *left* dual basis of ∂_{x^i} while in [?] we denoted by $\mathrm{d}x^i$ the *right* dual basis.

Divergence of symbols

Definition. The operator of *divergence* with respect to ∇ is the map

$$\mathrm{Div}_\nabla : \bigoplus_{k\in\mathbb{N}_0} \Gamma(\vee^k_\delta M) \to \bigoplus_{k\in\mathbb{N}} \Gamma(\vee^k_\delta M) : S \mapsto \sum_{j=1}^{n+m} \iota(\mathrm{d}x^j)(\nabla_{\partial_{x^j}} S).$$

For $M = E_0^{n|m}$, $\delta = 0$, $k = 1$, we have $\Gamma(\vee^k_\delta M) = \mathrm{Vect}(E_0^{p|q})$. Then, if $\nabla = \nabla_0$, we have

$$\begin{aligned}\mathrm{div}_{\nabla_0}(X) &= \sum_j \iota(\mathrm{d}x^j)((\partial_{x^j} X^i)\cdot \partial_{x^i}) \\ &= \sum_j (-1)^{\varepsilon_j\cdot(\varepsilon_j+\varepsilon(X)+\varepsilon_i)}(\partial_{x^j} X^i)\cdot(-1)^{\varepsilon_j}\cdot \delta_i^j \\ &= \sum_j (-1)^{\varepsilon_j\cdot(\varepsilon(X)+\varepsilon_j)}(\partial_{x^j} X^j)\ ,\end{aligned}$$

if $X = X^i \cdot \partial_{x^i}$. In other words, the divergence operator on symbols defined above is generalization of the divergence operator on vector fields defined in Chapter 1 (formula 1.5).

Projectively invariant lift of symbols

If δ is not critical, the restriction of Ψ to the divergence-free tensors with respect to $\tilde\nabla$ is a bijection. Indeed, the condition of zero divergence allows to fix the symbols A_{k-j} in (4.8) because of the following proposition (whose proof is exactly the same as in [**?**]).

Proposition 29. *If $j \in \mathbb{N}$, $l \in \mathbb{N}$ and if $A \in \Gamma(\vee^j_\delta M)$, then we have*

$$\begin{aligned}\mathrm{Div}_{\tilde\nabla}(A^{h(\nabla)} \vee \mathcal{E}^l) &= (\mathrm{Div}_\nabla A)^{h(\nabla)} \vee \mathcal{E}^l + 2(n-m+1)(\iota(\mathrm{r})A)^{h(\nabla)} \vee \mathcal{E}^{l+1} \\ &\quad -l\cdot\gamma_{2j+l}\cdot A^{h(\nabla)}\vee\mathcal{E}^{l-1}\ ,\end{aligned}$$

where the coefficients γ_{2j+l} are those defined in (1.2.3).

More precisely, the condition of zero divergence gives the following equations (for $0 < l < k$):

$$\begin{cases} A_{k-1} = \frac{1}{\gamma_{2k-1}}\cdot \mathrm{Div}_\nabla A_k, \\ A_{k-(l+1)} = \frac{1}{(l+1)(\gamma_{2k-(l+1)})}\cdot\big(\mathrm{Div}_\nabla A_{k-l} + 2(n-m+1)\cdot\iota(\mathrm{r})A_{k-(l-1)}\big)\,. \end{cases}$$

Finally, the projectively invariant lift of a symbol S on M, denoted by $\tilde S$, is obtained by applying to S the inverse of this bijection. Note that projective invariance follows from the fact that the zero divergence condition depends only on $\tilde\nabla$.

Construction of the NPIQ

Definition. If T is a graded symmetric covariant tensor of degree l with values in λ-densities, $\nabla_{\mathrm{s}}T$ is the supersymmetric covariant tensor

$$(\nabla_s T)(X_1,\dots,X_{l+1}) = \sum_{\sigma\in S_{l+1}} (-1)^{\epsilon_{l+1}+\varepsilon(T)\varepsilon(X_{\sigma(1)})}\cdot(\nabla_{X_{\sigma(1)}}(T(X_{\sigma(2)},\dots,X_{\sigma(l+1)}))$$
$$-\sum_{j=2}^{l+1}(-1)^{\varepsilon(X_{\sigma(1)})(\varepsilon(X_{\sigma(2)})+\ldots+\varepsilon(X_{\sigma(j-1)}))}\cdot T(X_{\sigma(2)},\dots,\nabla_{X_{\sigma(1)}}X_{\sigma(j)},\dots,X_{\sigma(l+1)})),$$

where $X_1,\dots,X_{l+1}$ are super vector fields and where ϵ_{l+1} is the sign of the permutation σ' induced by σ on the ordered subset of all odd elements among $X_1,\dots,X_{l+1}$.

Definition. If $\phi\otimes X_1\vee\cdots\vee X_k$ is a graded symmetric contravariant tensor of degree k and if $\psi\cdot\alpha_1\vee\cdots\vee\alpha_k$ is a graded symmetric covariant tensor of degree k, then

$$\langle\phi\otimes X_1\vee\cdots\vee X_k,\psi\otimes\alpha_1\vee\cdots\vee\alpha_k\rangle =$$
$$\phi\cdot\psi\otimes(-1)^{\varepsilon(\psi)(\varepsilon(X_1)+\ldots+\varepsilon(X_k))}\cdot\iota(X_1)\circ\cdots\circ\iota(X_k)(\alpha_1\vee\cdots\vee\alpha_k)\ ,$$

where the interior product ι is defined in the same way as in Definition 4.2.3. One extends this operation by bilinearity to arbitrary symmetric tensors of degree k.

The explicit formula

Theorem 30. *If $n-m\neq\pm1$ and δ is not a critical value, then the collection of maps $Q^k_M:\Gamma(\mathcal{C}M)\times\Gamma(\vee^k_\delta M)\to\Gamma(\mathcal{D}^k_{\lambda,\mu}M)$ given by*

$$Q^k_M(\nabla,S)(\phi)=\Psi\left(\langle\tilde{S},\tilde{\nabla}^k_s\tilde{\phi}\rangle\right)\ , \tag{4.9}$$

defines a projectively invariant natural quantization for supermanifolds of dimension $(n|m)$.

Proof. The proof goes as in [?]. Note that the right-hand of formula (4.9) is μ-equivariant because of the invariance of $\tilde{\nabla}$, the δ-equivariance of $\tilde{S}$ and the λ-equivariance of $\tilde{f}$. The fact that the maps Q^k_M can be extended to smooth families in order to form an extended NPIQ comes from the fact that the M. Bordemann's construction can be performed the same way for smooth families of connections, symbols and densities. In practice, one just have to add parameters in the local formulas, taking care to fix the parameters only *after* having applied the partial derivatives (note that all our local formulas are nothing but polynomials in the partial derivatives of the local components of the objects and that no partial derivatives in the direction of the parameter space will appear when passing to smooth families). □

REMARK. When $n-m \neq \pm 1$, $M = \mathbb{R}^{n|m}$ and $\nabla = \nabla_0$, formula (4.9) recovers the unique $\mathfrak{pgl}(n+1|m)$-equivariant quantization found in [?]. It is interesting to notice the problem there was solved without any hypothesis on the superdimension.

4.2.4 The case $n-m=1$

The hypothesis $n-m \neq 1$ is the analogue in the super context of the fact that M. BORDEMANN's method [?] does not apply for 1-dimensional smooth manifolds.

The classical setting

Actually, the problem of natural and projectively invariant quantization on 1-dimensional smooth manifolds turns out to be very peculiar. In this case, it is easily shown that the difference between any two torsion-free linear connections can be expressed as $\alpha \vee \mathrm{id}$ for some 1-form α. Consequently, all torsion-free linear connections are projectively equivalent, and the quest for a natural projectively invariant quantization amounts to the quest for a natural bijection from symbols to differential operators. As it is well-known (it is for instance a consequence of [?, Theorem 3]), such a natural bijection does not exist. Notice that for symbols of order two, the theory of natural operators [?] imposes for a natural projectively invariant quantization to be of the form

$$Q(\nabla, S)(f) = \langle S, \nabla^2 f \rangle + a \cdot \langle \mathrm{Div}_\nabla S, \nabla f \rangle + b \cdot \langle \mathrm{Div}^2_\nabla S, f \rangle + c \cdot \langle \iota(\mathrm{Ric}) S, f \rangle \tag{4.10}$$

where $a, b, c \in \mathbb{R}$. The condition of projective invariance yields a system of equations for a, b, c which admits no solution in dimension $n = 1$ (cf. [?]).

The super setting

If we make the assumption that a natural projectively invariant quantization must write under the form (4.10), with all objects being replaced by their super analogues, then the system of equations provided by the condition of projective invariance has no solutions when $n-m=1$. Therefore, unless there are more natural operators for supermanifolds than the superizations of the classical ones, a natural projectively invariant quantization does not exist in this case.

4.2.5 The case $n - m = -1$

In [?], P. Mathonet and F. Radoux were able to build a $\mathfrak{pgl}(n+1|m)$-equivariant quantization without restriction on the superdimension. However, the case $n - m = -1$ required an ad-hoc construction because of the peculiarities of the Lie superalgebra $\mathfrak{pgl}(n+1|n+1)$.

In our case, the problem lies in the very definition of the quantities Π^i_{jk} used in the construction of the connection $\tilde{\nabla}$ on $\tilde{M}$ associated with a projective class of connections on M. The manifold $\tilde{M}$ is thus unhelpful here.

When $n-m=-1$, the Christoffel symbols of a connections have a special property. Indeed, the local quantities $\sum_s (-1)^{\varepsilon_s} \Gamma^s_{is}$ are projectively invariant because the factor $n-m+1$ appears from the graded trace when passing from a connection ∇ to a projectively equivalent one $\nabla + (\alpha \vee \mathrm{id})$. It follows that the horizontal lift of symbols (of any order) is projectively invariant (remember that in general, we have to introduce a condition of zero divergence to get projective invariance). Unfortunately, although we are able to lift symbols and densities to $\tilde{M}$ in a projectively invariant manner, the lack of lifted connection on $\tilde{M}$ prevents us from applying the standard ordering on $\tilde{M}$ as in M. Bordemann's method.

Also because of the projective invariance of the local quantities $\sum_s (-1)^{\varepsilon_s} \Gamma^s_{is}$, the divergence of symbols of order one is projectively invariant. Therefore, the formula

$$Q^1_M(\nabla, S)(f) = \langle S, \nabla f \rangle + t \cdot \langle \mathrm{Div}_\nabla S, f \rangle \tag{4.11}$$

defines a 1-parameter family of natural projectively invariant quantizations for symbols of order one. This result agrees with the phenomenon observed in [?]. Also, for symbols of order two, the formula

$$Q^2_M(\nabla, S)(f) = \langle S, \nabla^2 f \rangle + \langle \mathrm{Div}_\nabla S, \nabla f \rangle$$

turns out to be projectively invariant.

We conjecture that similar formulas can be obtained for higher order symbols and that a natural projectively invariant quantization exists when $n - m = -1$.

The graded dimension $0|1$

As for the case, $n - m = 1$, there is a degenerate situation with $n - m = -1$: $\mathcal{A}$-manifolds of graded dimension $0|1$. In this situation, a smooth function reads locally as

$$f(\xi) = a + b \cdot \xi \ ,$$

where $a, b \in \mathbb{R}$. In particular, transition functions between local charts are just multiplication by (nonzero) real numbers.

About the problem of quantization, note that there are no differential operators (resp. symbols) of order greater than one: locally, a differential operator reads as

$$D(f)(\xi) = D_0(\xi) \cdot f(\xi) + D_1(\xi) \cdot (\partial_\xi f)(\xi) \ ,$$

where D_0 and D_1 are local smooth functions. In this (very reduced) case, formula (4.11) thus give a solution for all possible orders.

Remark. Since Christoffel symbols of torsion-free connections have to be graded symmetric in the lower indices, there is a unique (canonical, flat) torsion-free connection in dimension $0|1$, given locally by

$$\nabla_X Y = (X^1 \cdot \partial_\xi Y^1) \cdot \partial_\xi \ ,$$

if $X = X^1 \cdot \partial_\xi$ and $Y = Y^1 \cdot \partial_\xi$.

Appendix A

A Quick Introduction to $\mathcal{A}$-Manifolds

In the language introduced by G. Tuynman [?], the definition of supermanifolds follows the lines of the usual definition of smooth manifolds: one starts with local models together with a notion of smoothness for maps between them; then, one considers sets which are locally homeomorphic to the local models and one transports the notion of smoothness by means of these local homeomorphisms (called the local charts).

In this appendix, we first recall the main ingredient of the formalism: free graded modules over a $\mathbb{Z}_2$-graded algebra $\mathcal{A}$. Then we describe local models and smooth maps between them. Finally, we give the definition of $\mathcal{A}$-manifolds and smooth maps between them.

Remark. This appendix is essentially a quick overview of [?, Chapter II: Linear algebra of free graded $\mathcal{A}$-modules and Chapter III: Smooth functions and $\mathcal{A}$-manifolds]. The presentation here aims to make the thesis reasonnably self-contained, accessible to someone who did not (yet) read Tuynman's book. The reader interested in a more thorough study (with all proofs) is invited to read the original source [?].

Contents

A.1 An $\mathcal{A}$-vector space E is a free graded $\mathcal{A}$-module with an equivalence class of bases.

A.1.1 The algebra $\mathcal{A}$

By $\mathcal{A}$, we will always mean a $\mathbb{Z}_2$-graded commutative infinite-dimensional real algebra with unit $1_{\mathcal{A}}$. Moreover,

(i) the canonical map $\mathbb{R} \to \mathcal{A}, r \mapsto r.1_{\mathcal{A}}$ defines an embedding of $\mathbb{R}$ as a real subalgebra of the even part $\mathcal{A}_0$;

(ii) we have $\mathcal{A} = \mathbb{R}.1_{\mathcal{A}} \oplus \mathcal{N}$, where the set $\mathcal{N}$ of *nilpotent elements* is defined by

$$\mathcal{N} : \{a \in \mathcal{A} \mid \exists k \in \mathbb{N} : a^k = 0\} \, .$$

The subset $\mathcal{N}$ is an ideal in $\mathcal{A}$. It can be decomposed as the direct sum $\mathcal{N} = (\mathcal{N} \cap \mathcal{A}_0) \oplus \mathcal{A}_1$.

Definition. The canonical projection

$$\mathbf{B} : \mathcal{A} \to \mathcal{A}/\mathcal{N} \cong \mathbb{R}$$

is called the *body map* of $\mathcal{A}$.

The body map $\mathbf{B} : \mathcal{A} \to \mathbb{R}$ is $\mathbb{R}$-linear and the embedding of $\mathbb{R}$ as a subalgebra of $\mathcal{A}$ is a canonical section of $\mathbf{B}$, i.e., $\mathbf{B}(r.1_{\mathcal{A}}) = r$ for all $r \in \mathbb{R}$. By abuse of notation, one often writes $a = r + n$ when $\mathbf{B}(a) = r$ and $n = a - r \cdot 1_{\mathcal{A}} \in \mathcal{N}$.

Example 31. If X is an infinite-dimensional real vector space, the exterior algebra $\mathcal{A} = \bigwedge X$ has the properties required above. The $\mathbb{Z}_2$-grading is given by

$$\mathcal{A} = \bigwedge X = \left(\bigoplus_{k \in 2\mathbb{N}} \bigwedge^k X \right) \oplus \left(\bigoplus_{k \in 2\mathbb{N}+1} \bigwedge^k X \right) \, .$$

Moreover, we have also the decomposition

$$\mathcal{A} = \left(\bigwedge^0 X \right) \oplus \left(\bigoplus_{k \geqslant 1} \bigwedge^k X \right) \, ,$$

where

$$\bigwedge^0 X \cong \mathbb{R} \quad \text{and} \quad \mathcal{N} = \bigoplus_{k \geqslant 1} \bigwedge^k X \, .$$

A.1.2 Free graded $\mathcal{A}$-modules

Since $\mathcal{A}$ is not commutative ($\mathcal{A}$ is *graded* commutative), the notions of left and right modules do not coincide. By an *$\mathcal{A}$-module E* (without the adjectives left or right) we will always mean a graded $\mathcal{A}$-bimodule for which the left and right actions are related by the formula

$$a \cdot x = (-1)^{\epsilon(a).\epsilon(x)} \cdot x \cdot a \ .$$

Remember that for a graded $\mathcal{A}$-module E, the parity of an operator $a\cdot : E \to E$ is the parity of $a \in \mathcal{A}$, i.e., we have $\varepsilon(a \cdot x) = \varepsilon(a) + \varepsilon(x)$.

Definition. A *free graded $\mathcal{A}$-module of graded dimension* $p|q$ ($p, q \in \mathbb{N}$) is an $\mathcal{A}$-module

$$E = E_0 \oplus E_1$$

for which there exists a basis $\{e_1, \dots, e_p, e_{p+1}, \dots, e_{p+q}\}$ (over $\mathcal{A}$) such that

$$e_1, \dots, e_p \in E_0 \ , \quad e_{p+1}, \dots, e_{p+q} \in E_1 \ .$$

We say that $\{e_i\}$ is an *ordered homogeneous basis* of the $\mathcal{A}$-module E.

Remark. If E is a free graded $\mathcal{A}$-module of graded dimension $p|q$, then any ordered homogeneous basis of E has p even elements followed by q odd elements.

If $\{e_i\}$ is a basis of E, any element $x \in E$ can be written in a unique way as $x = \sum_i x^i \cdot e_i$. If $\{e_i\}$ is an ordered homogeneous basis, then the subspace E_0 of all even elements consists of those points in E for which we have $\varepsilon(x^i) = \varepsilon(e_i)$ for all i. In particular, E_0 should not be confused with the $\mathcal{A}$-linear subspace generated by $e_1, \dots, e_p$.([1])

Left and right $\mathcal{A}$-linear maps

Definition. A map $\phi : E \to F$ between $\mathcal{A}$-modules is *left (resp. right) $\mathcal{A}$-linear* if for any $a \in \mathcal{A}$ and any $x \in E$, we have

$$\phi(a \cdot x) = a \cdot \phi(x) \quad (\text{resp. } \phi(x \cdot a) = \phi(x) \cdot a)$$

Remark. If $\phi : E \to F$ is an even left (resp. right) $\mathcal{A}$-linear map between two free graded $\mathcal{A}$-modules, then $\mathcal{A}$ is also right (resp. left) $\mathcal{A}$-linear. For instance, if ϕ is left $\mathcal{A}$-linear and even, we have $\phi(x \cdot a) = (-1)^{\epsilon(a).\epsilon(x)} \cdot \phi(a \cdot x) = (-1)^{\epsilon(a).\epsilon(x)} \cdot a \cdot \phi(x) = \phi(x) \cdot a$.

[1] The subspace E_0 is not an $\mathcal{A}$-submodule of E because $\mathcal{A}_1 \cdot E_0 \subset E_1$.

The body functor

For a free graded $\mathcal{A}$-module E, the set of *nilpotent vectors* is defined as

$$\mathcal{N}_E = \{x \in E \mid \exists a \in \mathcal{A}\backslash\{0\} : a \cdot x = 0\} .$$

It is easy to show that $\mathcal{N}_E$ consists of those elements in E that have nilpotent coefficients with respect to an arbitrary basis of E (since $\mathcal{N}$ is an ideal in $\mathcal{A}$, the property will automatically be true for all bases of E).

Definition. The canonical projection $\mathbf{B} : E \to \mathbf{B}E = E/\mathcal{N}_E$ is called the *body map* of E.

This body map is $\mathbb{R}$-linear and for any $a \in \mathcal{A}$ and any $x \in E$, we have $\mathbf{B}(a \cdot x) = \mathbf{B}(a).\mathbf{B}(x)$. Moreover, the body map of the trivial module $E = \mathcal{A}$ coincide with the body map of the algebra $\mathcal{A}$ and this body map is thus a morphism of real superalgebras.

The image $\mathbf{B}E$ is a $\mathbb{Z}_2$-graded real vector space of graded dimension $p|q$. The $\mathbb{Z}_2$-grading is given by

$$\mathbf{B}E = \mathbf{B}E_0 \oplus \mathbf{B}E_1 .$$

Moreover, if $\{e_i\}$ is an ordered homogeneous basis of E (over $\mathcal{A}$), then $\{\mathbf{B}(e_i)\}$ is an ordered homogeneous basis of $\mathbf{B}E$ (over $\mathbb{R}$).

Proposition 32 ([**?**]). *Given a left (resp. right) $\mathcal{A}$-linear map $\phi : E \to F$ between two free graded $\mathcal{A}$-modules, there is a unique $\mathbb{R}$-linear map $\mathbf{B}\phi : \mathbf{B}E \to \mathbf{B}F$ making commutative the following diagram:*

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[d, "\mathbf{B}"'] & F \arrow[d, "\mathbf{B}"] \\ \mathbf{B}E \arrow[r, "\mathbf{B}\phi"'] & \mathbf{B}F \end{tikzcd}$$

Moreover, if ϕ is homogeneous of parity α, then so is $\mathbf{B}\phi$. Finally, if $\chi : F \to G$ is another $\mathcal{A}$-linear map, then

$$\mathbf{B}(\chi \circ \phi) = \mathbf{B}\chi \circ \mathbf{B}\phi .$$

In view of Proposition 32, we thus have a (parity-preserving) functor $\mathbf{B}$ from the category of free graded $\mathcal{A}$-modules and left (resp. right) $\mathcal{A}$-linear maps to the category of real super vector spaces with $\mathbb{R}$-linear maps.

A.1.3 $\mathcal{A}$-vector spaces

If $\{e_i\}$ is a basis of the free graded $\mathcal{A}$-module E, then $\mathbf{B}E$ can be identified to the set of points that have real coefficients by means of the $\mathbb{R}$-linear bijection

$$\mathrm{Span}_{\mathbb{R}}(\{e_i\}) \to \mathbf{B}E,\ \sum_i r^i \cdot e_i \mapsto \sum_i r^i \cdot \mathbf{B}(e_i)\ . \tag{A.1}$$

However, the set of points that have real coefficients is not stable under all changes of basis.

Definition.

- Two bases $\{e_i\}$ and $\{f_j\}$ of a free graded $\mathcal{A}$-module of graded dimension $p|q$ are said to be *equivalent* if they are related to each other by a matrix with real coefficients , i.e., if there exist numbers $a^i_j \in \mathbb{R}$ such that $f^j = \sum_i a^i_j \cdot e_i$ for all j.
- An $\mathcal{A}$-*vector space* is a free graded $\mathcal{A}$-module together with an equivalence class of bases containing an ordered homogeneous basis.([2])

The real vector subspace $\mathrm{Span}_{\mathbb{R}}(\{e_i\})$ is independent of the choice of the basis $\{e_i\}$ in the equivalence class, and so is the isomorphism (A.1). The inverse map of this isomorphism thus provides a canonical embedding of $\mathbf{B}E$ in E and, by abuse of notation, one often writes

$$x = \mathbf{B}(x) + n\ ,$$

where $n = \sum_i (x^i - \mathbf{B}(x^i)) \cdot e_i \in \mathcal{N}_E$.

Smooth $\mathcal{A}$-linear maps

Definition. An $\mathcal{A}$-linear map (left or right) $\phi : E \to F$ is *smooth* if it sends the points with real coefficients in (a basis of) E on points with real coefficients in (a basis of) F, i.e.,

$$\phi(\mathbf{B}E) \subset \mathbf{B}F\ ,$$

where $\mathbf{B}E$ (resp. $\mathbf{B}F$) is seen as a subspace of E (resp. F) through the canonical embedding.

A left (resp. a right) $\mathcal{A}$-linear map $E \to F$ is completely determined by its values on a basis of E. Therefore, we have a one-to-one correspondance between smooth left (resp. right) $\mathcal{A}$-linear maps $E \to F$ and $\mathbb{R}$-linear maps $\mathbf{B}E \to \mathbf{B}F$.

[2] In the sequel, by a basis of E, we will always mean an ordered homogeneous basis in the equivalence class of bases attached with E.

An equivalence of categories

Proposition 33 ([?]). *The body functor* $\mathbf{B}$ *defines an equivalence of categories between the category of* $\mathcal{A}$*-vector spaces with smooth left (resp. right)* $\mathcal{A}$*-linear maps and the category of* $\mathbb{Z}_2$*-graded real vector spaces with* $\mathbb{R}$*-linear maps.*

Proof. We already know that $\mathbf{B}$ is fully faithful. Moreover, $\mathbf{B}$ is essentially surjective: it is easy to see that for any real super vector space V, the set $\mathbf{G}V = \mathcal{A} \otimes_{\mathbb{R}} V$ has a canonical structure of an $\mathcal{A}$-vector space whose body is isomorphic to V. □

As a consequence of Proposition 33, one can say that there is no real difference between $\mathcal{A}$-vector spaces with their smooth linear maps and real super vector spaces with their linear maps. However, it is important to remember that in order to get a one-to-one correspondence at the level of morphisms, one must choose between left and right $\mathcal{A}$-linear maps: for every odd linear map between $\mathbb{R}$-vector spaces, there exist two smooth odd maps between $\mathcal{A}$-vector spaces: a right and a left $\mathcal{A}$-linear one.

A.2 On the even part of an $\mathcal{A}$-vector space, one can define smooth functions and their derivatives.

A.2.1 The De Witt topology

Definition. The De Witt topology on an $\mathcal{A}$-vector space E is the coarsest topology for which the body map $\mathbf{B} : E \to \mathbf{B}E$ is continuous: a subset $U \subset E$ is open in E if and only if $U = \mathbf{B}^{-1}(V)$ for some open subset V in (the finite dimensional real vector space) $\mathbf{B}E$. All subsets of E, and in particular E_0, are then equipped with the relative topology.

It follows from the definition that open subsets $U \subset E_0$ are saturated with nilpotent elements in the sense that $U + (\mathcal{N}_E \cap E_0) = U$. Moreover, through the canonical embedding of $\mathbf{B}E$ as a subspace of E, the body $\mathbf{B}U$ of an open subset $U \subset E_0$ consists of those points in U with real coordinates (in all bases of E).

Remark. The De Witt topology on E is not Hausdorff. Indeed, two distinct points $x, y \in E$ such that $\mathbf{B}(x) = \mathbf{B}(y)$ cannot be separated by disjoint open subsets.

A.2.2 Smooth functions and their derivatives

For $\mathcal{A}$-valued functions, defining smoothness in terms of limits is problematic because the topology on $\mathcal{A}$ is not Hausdorff (limits need not be unique). G. Tuynman [?] uses an alternative definition which, in the classical context, is equivalent to the standard one. Then, he gives the following description, which we take as definition.

Definition. Let U be an open subset of E_0. A map $f : U \subset E_0 \to \mathcal{A}$ is *smooth* if and only if given a basis of E, there exist ordinary smooth functions $f_{i_1,\dots,i_r} \in \mathrm{C}^\infty(\mathbf{B}U \subset \mathbb{R}^p, \mathbb{R})$ such that f reads (in the left coordinates) as

$$f(x^1,\dots,x^p,\xi^1,\dots,\xi^q) = \sum_{r=0}^{q} \sum_{1\leqslant i_1<\dots<i_r\leqslant q} \xi^{i_1}\cdot \;\cdots\; \cdot\xi^{i_r}\cdot \widetilde{f_{i_1\dots i_r}}(x^1,\dots,x^p)\,, \qquad \text{(A.2)}$$

where

$$\widetilde{f_{i_1\dots i_r}}(x^1,\dots,x^p) = \sum_{\alpha\in\mathbb{N}^p} \frac{(x-\mathbf{B}(x))^\alpha}{\alpha!}\cdot(\partial^\alpha f_{i_1\dots i_r})(\mathbf{B}x^1,\dots,\mathbf{B}x^p)\,.$$

This definition of smoothness can be extended to functions valued in an $\mathcal{A}$-vector space F as follows: a map $f : U \subset E_0 \to F$ is smooth if and only if given a basis $\{e'_j\}$ of F, we have $f(x,\xi) = \sum_j f_j(x,\xi)\cdot e'_j$, where all functions $f_j : U \subset E_0 \to \mathcal{A}$ are smooth.

The space of smooth functions

The space $\mathrm{C}^\infty(U, \mathcal{A})$ of smooth functions on an open subset $U \subset E_0$ is made of continuous functions. Moreover, it has the following important properties (cf. [**?**, III;1.24]):

- Being a smooth function is a local property, stable under composition.
- The set $\mathrm{C}^\infty(U, \mathcal{A})$ is a graded commutative $\mathbb{R}$-algebra with unit under pointwise addition and multiplication of functions. The $\mathbb{Z}_2$-grading is given by

$$\mathrm{C}^\infty(U, \mathcal{A})_\alpha = \{f \in \mathrm{C}^\infty(U, \mathcal{A}) : \mathrm{im}(f) \subset \mathcal{A}_\alpha\} \quad (\alpha = 0, 1) \; .$$

- Given a basis of E, the collection of real functions $f_{i_1,\ldots,i_r}$ appearing in (A.2) is uniquely determined by the function f and we have an identification (as real superalgebras)

$$\mathrm{C}^\infty(U, \mathcal{A}) \simeq \mathrm{C}^\infty(\mathbb{R}^p, \mathbb{R}) \otimes \bigwedge \mathbb{R}^q \; .$$

Moreover, if F is an $\mathcal{A}$-vector space, then the space $\mathrm{C}^\infty(U, F)$ of F-valued smooth functions on $U \subset E_0$ is a free graded $\mathrm{C}^\infty(U, \mathcal{A})$-module with the same graded dimension as F, the $\mathbb{Z}_2$-grading of $\mathrm{C}^\infty(U, F)$ being given by

$$\mathrm{C}^\infty(U, F)_\alpha = \{f \in \mathrm{C}^\infty(U, F) : \mathrm{im}(f) \subset F_\alpha\} \quad (\alpha = 0, 1) \; .$$

Fixing of variables

Being a smooth function is stable under fixing of variables to real values, but given a smooth

$$f : P \times U \to \mathcal{A}$$

and an element $p \in P$, the induced function

$$f_p = f(p, \cdot) : U \to \mathcal{A}$$

is, in general, not smooth. Indeed, the decomposition (A.2) for f does not, in general, induce a similar decomposition for $f(p, \cdot)$ because the coordinates of p still appear while p is no longer considered as a variable.

Example 34. The map $\mathrm{id} : \mathcal{A}_0 \to \mathcal{A},\ x \mapsto x$ is smooth (we have $\mathrm{id}(x) = \mathbf{B}x + n_x = \widetilde{\mathrm{id}_{\mathbb{R}}}(x)$) but a constant map $c_a : \mathcal{A}_0 \to \mathcal{A},\ x \mapsto a = \mathbf{B}a + n_a$ is smooth if and only if $a = \mathbf{B}a \in \mathbb{R}$. More generally, a constant map $U \to F$ is smooth if and only if the constant value is a point with real coordinates, i.e., an element of $\mathbf{B}F \subset F$.

Partial derivatives

From the decomposition (A.2) of smooth functions, we can transport the partial derivatives from ordinary smooth functions to $\mathcal{A}$-valued smooth functions.

Definition. If a function $f \in \mathrm{C}^\infty(U,\mathcal{A})$ reads as (A.2), its partial derivatives are the smooth functions

$$\partial_{x^k} f(x^1,\ldots,x^p,\xi^1,\ldots,\xi^q) = \sum_{r=0}^{q} \sum_{1\leqslant i_1<\cdots<i_r\leqslant q} \xi^{i_1}\cdot \;\cdots\; \cdot \xi^{i_r}\cdot \partial_k f_{i_1\ldots i_r}(\widetilde{\ }x^1,\ldots,x^p)\ ,$$

and

$$\partial_{\xi^l} f(x^1,\ldots,x^p,\xi^1,\ldots,\xi^q) = \sum_{r=0}^{q} \sum_{1\leqslant i_1<\cdots<i_r\leqslant q} \sum_{j=1}^{r} (-1)^{j-1}\cdot \xi^{i_1}\cdot\;\cdots\;\xi^{i_{j-1}}\cdot \delta_l^{i_j}\cdot \xi^{i_{j+1}}\;\cdots\;\cdot \xi^{i_r}$$
$$\cdot\, \widetilde{f_{i_1\ldots i_r}}(x^1,\ldots,x^p)\ . \quad \text{(A.3)}$$

The maps ∂_{x^k} (resp. ∂_{ξ^l}) are even (resp. odd) $\mathbb{R}$-linear derivations of the algebra $\mathrm{C}^\infty(U,\mathcal{A})$. Moreover, these derivations commute in the graded sense, i.e., we have

$$[\partial_{x^k},\partial_{x^l}] = [\partial_{x^k},\partial_{\xi^l}] = [\partial_{\xi^k},\partial_{\xi^l}] = 0\ ,$$

where the brackets stand for the graded commutator of derivations.

Remark. The definition ∂_{x^k} and ∂_{ξ^l} can be extended to F-valued smooth functions by setting $(\partial_{x^k} f)^j = \partial_{x^k}(f^j)$ and $(\partial_{\xi^l} f)^j = \partial_{\xi^l}(f^j)$.

The chain rule

When we do not need to distinguish between even and odd coordinates on E_0, we often write $(y^1,\ldots,y^{p+q})$ instead of $(x^1,\ldots,x^p,\xi^1,\ldots,\xi^q)$. Accordingly, the partial derivatives are thus written as ∂_{y^i} with $i = 1,\ldots,p+q$.

This being said, we can now state an important property of the partial derivatives: the *chain rule*. This rule says that if U' is an open subset, if $f \in \mathrm{C}^\infty(U,F)$ is such that $f(U)\subset U'$ and if $g\in \mathrm{C}^\infty(U',G)$, then for any $x\in U$, we have

$$\partial_{y^i}(g\circ f)(x) = (\partial_{y^i} f^j)(x)\cdot(\partial_{z^j} g)(f(x))$$

where the z^j are (left) coordinate functions (with respect to a fixed basis) on F.

A.3 An $\mathcal{A}$-manifold M is a set covered by local charts valued in the even part E_0 of an $\mathcal{A}$-vector space.

From a local model E_0, one defines $\mathcal{A}$-manifolds in terms of charts and smooth transition functions as in the classical case.

DEFINITION. Let M be a topological space.

- A *chart* of M is a pair (U, φ), where $U \subset M$ and $\varphi : U \to O$ is a homeomorphism between U and an open subset $O \subset E_0$ in the even part of an $\mathcal{A}$-vector space E.
- An *atlas* of M is a collection of charts $\{(U_a, \varphi_a) : U_a \to O_a \subset E_0\}$ such that $\bigcup_a U_a = M$ and $\varphi_b \circ \varphi_a^{-1} \in \mathrm{C}^\infty(\varphi_a(U_a \cap U_b), \varphi_b(U_a \cap U_b))_0$ whenever $U_a \cap U_b \neq \emptyset$.

If M is a topological space endowed with an atlas $\{(U_a, \varphi_a) : U_a \to O_a \subset E_0\}$, we say that M is *modeled* on the $\mathcal{A}$-vector space E and by a chart of M, we will always mean an E_0-valued chart compatible with all charts in the atlas of M.

If M is modeled on E, the *body of* M is made of those points in M with real coordinates in a chart of M, i.e.,

$$\mathbf{B}M = \{x \in M : \varphi_a(x) \in \mathbf{B}O_a \text{ for some chart } (U_a, \varphi_a : U_a \to O_a) \text{ of } M\} .$$

Moreover, we define the *body map* of M, $\mathbf{B} : M \to \mathbf{B}M$, by setting

$$\mathbf{B}|_{U_a} = \varphi_a^{-1} \circ \mathbf{B} \circ \varphi_a ,$$

where $\mathbf{B}$ in the right-hand side is the body map of E.

REMARK. The body map of M is well-defined and $\mathbf{B}M$ is actually made of those points that have real coordinates in all charts of M. This is due to the fact that transition functions between charts are smooth maps between $\mathcal{A}$-vector spaces and thus they commute with the body maps of those $\mathcal{A}$-vector spaces.

DEFINITION. An $\mathcal{A}$-*manifold* is a topological space M endowed with an atlas such that $\mathbf{B}M$ is an ordinary smooth manifold (in particular, $\mathbf{B}M \subset M$ must be a second countable Hausdorff topological space with the relative topology). By definition, the *graded dimension* of an $\mathcal{A}$-manifold M is the graded dimension of its model $\mathcal{A}$-vector space.

If M is an $\mathcal{A}$-manifold of dimension $p|q$, the smooth manifold $\mathbf{B}M$ is of dimension p. The odd dimension disappears because the charts of M lie in the even part of the model $\mathcal{A}$-vector space and thus the odd coordinates become zero when taking the body map.

Smooth maps

Definition. A map $f : M \to N$ between two $\mathcal{A}$-manifolds is smooth (we write $f \in \mathrm{C}^\infty(M, N)$) if for any chart (U, φ) of M and any chart (V, ψ) of N, the local expression of f in these charts, i.e., $\psi \circ f \circ \varphi^{-1}$, is smooth on $\varphi(U \cap f^{-1}(V))$.

If $f \in \mathrm{C}^\infty(M, N)$, then f is continuous and the map $\mathbf{B}f : \mathbf{B}M \to \mathbf{B}N$ defined by $\mathbf{B}f = f|_{\mathbf{B}M}$ is smooth (between ordinary smooth manifolds). This map $\mathbf{B}f$ is called the *body* of f.

Remark. When $N = F$ is an $\mathcal{A}$-vector space([3]), the space $\mathrm{C}^\infty(M, F)$ is a free graded module over the real superalgebra $\mathrm{C}^\infty(M, \mathcal{A})$.

[3] Any $\mathcal{A}$-vector space E can be seen as the even part of a larger $\mathcal{A}$-vector space $E^\#$ (see [**?**, III.A.26]). In particular, any $\mathcal{A}$-vector space is an $\mathcal{A}$-manifold.

Appendix B

A Quick Introduction to Fiber Bundles over $\mathcal{A}$-Manifolds

In this appendix, we first recall what is a fiber bundle (this requires the notion of smooth action of an $\mathcal{A}$-Lie group on an $\mathcal{A}$-manifold) and how it can be encoded by a collection of maps associated with an atlas of M. Then we look at an important class of fiber bundles, namely vector bundles. Finally, we introduce the notion of affine bundle.

Remark. This appendix is a quick and incomplete overview of [**?**, Chapter IV: Bundles].([1]) The presentation here aims to make the thesis reasonnably self-contained, accessible to someone who did not (yet) read Tuynman's book. The reader interested in a more thorough study (with all proofs) is invited to read the original source [**?**].

[1] Although it does not appear in [**?**], we present here the definition of affine bundles because it is closely related to the definition of vector bundles.

Contents

B.1 A fiber bundle is an $\mathcal{A}$-manifold fibered by means of a locally trivial smooth surjection.

B.1.1 Smooth actions of $\mathcal{A}$-Lie groups

Definition.

- An *$\mathcal{A}$-Lie group* is an $\mathcal{A}$-manifold G which is also a group, with the property that the group operations are smooth.
- A *smooth left action* of an $\mathcal{A}$-Lie group G on an $\mathcal{A}$-manifold M is a smooth map

  $$\Phi : G \times M \to M$$

  that is an action of the group G on the set M.
- A smooth left action $\Phi : G \times M \to M$ is *pseudo-effective* if for any $\mathcal{A}$-manifold N and any smooth map $\psi : N \to G$, we have

  $$\big(\forall n \in N : \Phi_{\psi(n)} = \mathrm{id}_M\big) \;\Rightarrow\; (\forall n \in N : \psi(n) = \mathrm{e}_G)\ .$$

  where e_G is the identity element of G.

For a fixed element $g \in G$, the map

$$\Phi_g = \Phi(g, \cdot) : M \to M$$

is a homeomorphism. However, Φ_g is, in general, not smooth. This is because being smooth is, in general, not stable under fixing variables to nonreal values. This being said, if $g \in \mathbf{B}G$ (i.e. g is a point with real coordinates), then Φ_g is smooth.

Example 35. Let F be an $\mathcal{A}$-vector space. The group $G = \mathrm{Aut}(F)$ of even $\mathcal{A}$-linear invertible maps $F \to F$ is an $\mathcal{A}$-Lie group. This $\mathcal{A}$-Lie group acts smoothly on F (seen as an $\mathcal{A}$-manifold) by means of the evaluation of automorphisms, i.e., the action is

$$\Phi : \mathrm{Aut}(F) \times F \to F\ ,\ x \mapsto \Phi_T(x) = T(x)\ .$$

This action is pseudo-effective: $\phi_{\Psi(n)} = \mathrm{id}_F$ precisely means that $\psi(n) = \mathrm{id}_F = e_G$.

B.1.2 Fiber bundles over $\mathcal{A}$-manifolds

Definition. Let B, M and F be $\mathcal{A}$-manifolds. Let G be an $\mathcal{A}$-Lie group with a smooth and pseudo-effective left action on F. Finally, let $\pi : B \to M$ be a smooth surjection.

- A *local trivializing F-chart* of π is a triple (U, φ, ψ), where (U, φ) is a chart of M and

  $$\psi : \pi^{-1}(U) \to U \times F$$

  is a diffeomorphism such that $\pi_U \circ \psi = \pi$, where π_U denotes the canonical projection on U. In other words, there is a commutative diagram

  $$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\pi_U"] \\ & U & \end{tikzcd}$$

- A *G-atlas of trivializing F-charts* of π is a collection of *G-compatible* local trivializing F-charts $\{(U_a, \varphi_a, \psi_a)\}$ of π. This means that $\{(U_a, \varphi_a)\}$ is an atlas of M and that

  $$\psi_b \circ \psi_a^{-1}(x, f) = (x, \psi_{ba}(x)(f))$$

  for some *transition functions* $\psi_{ba} \in \mathrm{C}^\infty(U_a \cap U_b)$ whenever $U_a \cap U_b \neq \emptyset$. .

- A *fiber bundle* over M with typical fiber F and structure group G is a smooth surjection

  $$\pi : B \to M$$

  together with a G-atlas of trivializing F-charts.

- A *fiber bundle map* between $\pi : B \to M$ and $\pi' : B' \to M'$ is a smooth surjection map $\Phi : B \to B'$ above a smooth map $\phi : M \to M'$, i.e., there is a commutative diagram

  $$\begin{tikzcd} B \arrow[r, "\Phi"] \arrow[d, "\pi"'] & B' \arrow[d, "\pi'"] \\ M \arrow[r, "\phi"] & M' \end{tikzcd}$$

Each fiber $\pi_x = \pi^{-1}(\{x\})$ is homeomorphic to F. Indeed, any point of M lies in a local trivializing F-chart and if (U, φ, ψ) is a local trivializing F-chart of π, then for any $x \in U$, the map $\psi^{-1}(x, \cdot) : F_\pi \to \pi_x$ is a homeomorphism.

Remark. If $x \in \mathbf{B}M$, this homeomorphism is a diffeomorphism.

The cocyle conditions

If $\{(U_a, \varphi_a, \psi_a)\}$ is a G-atlas of trivializing F-charts of $\pi : B \to M$, the pseudo-effectiveness of the action of G ensures that the maps $\Psi_{ba} : U_b \cap U_a \to G$ satisfy the *cocycle conditions*:

$$\begin{cases} \psi_{aa}(x) = \mathrm{e}_G & \text{for all } x \in U_a \ , \\ \psi_{cb}(x) \cdot_G \psi_{ba}(x) = \psi_{ca}(x) & \text{for all } x \in U_a \cap U_b \cap U_c \ , \end{cases} \tag{B.1}$$

where $\cdot_G$ stands for the group operation in G.

Conversely, it is shown in [**?**, Construction 1.24] that, given the atlas $\{(U_a, \varphi_a)\}$ of all charts of M together with a collection of smooth maps $\Psi_{ba} : U_b \cap U_a \to G$ satisfying the *cocycle condition* (B.1), one can build an $\mathcal{A}$-manifold B together with a projection $\pi : B \to M$ and an G-atlas $\{(U_a, \varphi_a, \psi_a)\}$ of trivializing F-charts for which the maps Ψ_{ba} are the transition functions between charts.

Remark. In practice, one can thus define a fiber bundle by giving, for an atlas $\{(U_a, \varphi_a)\}$ of M, a collection of maps Ψ_{ba} satisfying the cocycle conditions (B.1).

B.2 A vector bundle is a bundle whose structure group consists of automorphisms of an $\mathcal{A}$-vector space.

B.2.1 $\mathcal{A}$-vector space as $\mathcal{A}$-manifolds

Definition. Let F be an $\mathcal{A}$-vector space of graded dimension $p|q$.

- By definition, ΠF is an $\mathcal{A}$-vector space such that ΠF coincide with F as a left $\mathcal{A}$-module, but with the parity reversed, i.e., we have

  $$(\Pi F)_0 = F_1 \quad \text{and} \quad (\Pi F)_1 = F_0 \ .$$

  The map $\hat{\mathrm{id}}_F : F \to \Pi F,\ x \mapsto x$ is an odd left $\mathcal{A}$-linear bijection by means of which ΠF inherits an equivalence class of basis: if $\{f_i\}$ is a basis of F, then $\{\bar{f}_i = \hat{\mathrm{id}}(f_i)\}$ is a basis of ΠF.

- The $\mathcal{A}$-vector space $F^{\#}$ is the direct sum of $\mathcal{A}$-vector spaces

  $$F^{\#} = F \oplus \Pi F \ .$$

Any $\mathcal{A}$-vector space F is an $\mathcal{A}$-manifold. Indeed, F can be identified to the even part of the $\mathcal{A}$-vector space $F^{\#}$. More precisely, for any basis $\{e_i\}$ of E, the map

$$F \to (F^{\#})_0 \ , \ \sum_i x^i \cdot f_i \mapsto \sum_i (x^i)_{\varepsilon(f_i)} \cdot f_i + \sum_i (x^i)_{\varepsilon(f_i)+1} \cdot \bar{f}_i$$

is a (global) chart of F.

B.2.2 Vector bundles over $\mathcal{A}$-manifolds

Definition. A *vector bundle* over an $\mathcal{A}$-manifold M is a fiber bundle

$$\pi : E_\pi \to M$$

whose typical fiber F_π is an $\mathcal{A}$-vector space and whose structure group is $\mathrm{Aut}(F_\pi)$.

For any $x \in M$, the fiber π_x inherits (by means of the local trivializations) from the typical fiber F_π a canonical structure of free graded $\mathcal{A}$-module (see [**?**, Chapter IV, Discussion 3.2]). In particular, each fiber π_x has a canonical origin 0_x.

Homogeneous parts of a vector bundle

Definition. If $\pi : E_\pi \to M$ is a vector bundle, its even part $E_\pi^{(0)}$ (resp. its odd part $E_\pi^{(1)}$) is the subspace of E_π made of those points that lie in the even (resp. the odd) part of their fiber, i.e., for $\alpha = 0, 1$, we have

$$E_\pi^{(\alpha)} = \{e \in E_\pi : e \in \left(\pi_{\pi(e)}\right)_\alpha\} .$$

If we restrict the projection π to these subspaces, we obtain two fiber bundles $\pi^{(0)} : E_\pi^{(0)} \to M$ and $\pi^{(1)} : E_\pi^{(1)} \to M$ with typical fibers $(F_\pi)_0$ and $(F_\pi)_1$, respectively. Those new fiber bundles are not vector bundles because $(F_\pi)_0$ and $(F_\pi)_1$ are not $\mathcal{A}$-modules ($(F_\pi)_0$ and $(F_\pi)_1$ are only $\mathcal{A}_0$-modules). However, their fibered product over M is a vector bundle reconstructing the vector bundle $\pi : E_\pi \to M$, i.e., we have $E_\pi^{(0)} \times_M E_\pi^{(1)} \cong E_\pi$.

If the base M is an $\mathcal{A}$-manifold of graded dimension $n|m$ and if and the F_π is an $\mathcal{A}$-vector space of graded dimension $p|q$, then the total space E_π is an $\mathcal{A}$-manifold of graded dimension is not $(n+p)|(m+q)$ but rather $(n+p+q)|(m+q+p)$. This is because the dimension of F_π as an $\mathcal{A}$-manifold is the graded dimension of the $\mathcal{A}$-vector space $F_\pi^{\#}$ such that $F_\pi \cong (F_\pi^{\#})_0$, namely $(p+q)|(q+p)$.([2]) For the same reason, the $\mathcal{A}$-manifold $E_\pi^{(0)}$ is of graded dimension $(n+p|m+q)$ while $E_\pi^{(1)}$ is of graded dimension $(n+q|m+p)$.

Remark. At the level of sections, we have $\Gamma(E_\pi^{(\alpha)}) = \Gamma(E_\pi)_\alpha$.

Morphisms between vector bundles

Definition ([**?**, Chapter IV, Definition 3.5]). Let $\pi : E_\pi \to M$ and $\pi' : E_{\pi'} \to N$ be vector bundles with typical fibers F_π and F'_π respectively. A fiber bundle map $\Phi : E_\pi \to E_{\pi'}$ inducing a map $\phi : M \to N$ is called a *(left linear) vector bundle morphism* if the restriction $\Phi_x = \Phi|_{\pi_x}$ to any fiber is left $\mathcal{A}$-linear, i.e.,

$$\Phi_x \in \mathrm{Hom}_L(\pi_x, \pi'_{\phi(x)})$$

The map Φ is said to be *of parity* α if all linear maps Φ_x are of parity α. Similar definitions hold for right linear vector bundle morphisms. A *vector bundle isomorphism* is an even vector bundle morphism which is at the same time an isomorphism of fiber bundles. The vector bundle $\pi : E_\pi \to M$ is said to be *(globally) trivializable* if it is isomorphic as vector bundle to the *trivial vector bundle* $\mathrm{pr}_1 : M \times F_\pi \to M$.

[2]Remember [**?**, III.1.26] that $F_\pi^{\#} = F_\pi \oplus (\Pi F_\pi)$, where Π stands for the parity reversal operation.

B.3 An affine bundle is a bundle whose structure group consists of affine isomorphisms of an $\mathcal{A}$-vector space.

B.3.1 Affine bundles over $\mathcal{A}$-manifolds

DEFINITION. An *affine bundle* over M is a fiber bundle $\pi : Z_\pi \to M$ whose typical fiber F_π is an $\mathcal{A}$-vector space and whose structure group is $\mathrm{Aff}(F_\pi)$, the group of $\mathcal{A}$-affine maps $F_\pi \to F_\pi$ whose linear part is an even invertible $\mathcal{A}$-linear map and whose translation part is even.

EXAMPLE 36. Since $\mathrm{Aut}(F_\pi) \subset \mathrm{Aff}(F_\pi)$, vector bundles are affine bundles.

Any affine bundle $\pi : Z_\pi \to M$ has an underlying vector bundle. Indeed, if $\{(U_a, \varphi_a, \psi_a)\}$ is an $\mathrm{Aff}(F_\pi)$-atlas of trivializing F_π-charts or π, the collection $\{\vec{\psi}_{ba}\}$ of all $\mathcal{A}$-linear parts of the transition functions ψ_{ba} determine a vector bundle $\vec{\pi} : E_{\vec{\pi}} \to M$.

Moreover, each fiber π_x inherits (by means of the local trivializations) from F_π (seen as an $\mathcal{A}$-affine space modeled on itself) a canonical structure of affine space modeled on the free graded $\mathcal{A}$-module $\vec{\pi}_x$, i.e., if $\Psi : \pi^{-1}(U) \to U \times F_\pi$ is a fiberwise affine trivialization of π, we have

$$z + e = \Psi^{-1}(x, \pi_{F_\pi} \circ \Psi(z) + \pi_{F_\pi} \circ \vec{\Psi}(e))$$

for all $z \in \pi_x$ and $e \in \vec{\pi}_x$. However, the fibers π_x do not come with a canonical origin because affine transition functions do not preserve the origin of F_π.

The even part of an affine bundle

DEFINITION. If $\pi : Z_\pi \to M$ is an affine bundle, its even part $Z_\pi^{(0)}$ is the subspace of Z_π made of those points whose image through any local trivialization in the $\mathrm{Aff}(F_\pi)$-atlas of π lie in the even part of the typical fiber F_π, i.e., we have

$$Z_\pi^{(0))} = \{z \in Z_\pi : z \in \left(\pi_{\pi(z)}\right)_0\} \,.$$

If we restrict the projection π to this subspace, we obtain a fiber bundle $\pi^{(0)} : Z_\pi^{(0)} \to M$. This fiber bundle inherits translations by elements in the even part of the underlying vector bundle of π.

REMARK. A similar definition for an odd part $Z_\pi^{(1))}$ would, in general, not make sense here because transition functions are valued in $\mathrm{Aff}(F_\pi)$ and this space does not preserve the odd part of the typical fiber.

Morphisms between affine bundles

DEFINITION. Let $\pi : Z_\pi \to M$ and $\pi' : Z_{\pi'} \to N$ be affine bundles with typical fibers F_π and F'_π respectively. A fiber bundle map $\Phi : Z_\pi \to Z_\pi$ inducing a map $\phi : M \to N$ is called a *left affine bundle morphism* if the restriction $\Phi_x = \Phi|_{\pi_x}$ to any fiber is an $\mathcal{A}$-affine map whose linear part is left $\mathcal{A}$-linear

$$\Phi_x \in \mathrm{aff}_L(\pi_x, \pi'_{\phi(x)}) \ .$$

The map Φ is said to be *even* if all affine maps Φ_x have even $\mathcal{A}$-linear part and even translation part. An *affine bundle isomorphism* is an even affine bundle morphism which is at the same time an isomorphism of fiber bundles. The affine bundle $\pi : F_\pi \to M$ is said to be (globally) *trivializable* if it is isomorphic as affine bundle to the trivial affine bundle $\mathrm{pr}_1 : M \times F_\pi \to M$.