On differentially dissipative dynamical systems

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Abstract: Dissipativity is an essential concept of systems theory. The paper provides an extension of dissipativity, named differential dissipativity, by lifting storage functions and supply rates to the tangent bundle. Differential dissipativity is connected to incremental stability in the same way as dissipativity is connected to stability. It leads to a natural formulation of differential passivity when restricting to quadratic supply rates. The paper also shows that the interconnection of differentially passive systems is differentially passive, and provides preliminary examples of differentially passive electrical systems.

Keywords: Dissipativity, incremental stability, contraction analysis

1. INTRODUCTION

Dissipativity, Willems (1972a); Willems (1972b), plays a central role in the analysis of open systems to reduce the analysis of complex systems to the study of the interconnection of simpler components. Dissipativity is a fundamental tool in nonlinear control design Sepulchre et al. (1997); van der Schaft (1999), widely adopted in industrial applications. Typical examples are provided by applications on electro-mechanical devices modeled within the port-hamiltonian framework, Ortega et al. (2001). Passivity-based designs conveniently connect the physical modeling of mechanical and electrical interconnections and the stability properties required by applications.

In a nonlinear setting, applications like regulation, observer designs, and synchronization call for incremental notions of stability, Angeli (2000); Angeli (2009). Several results in the literature propose extensions of passivity to guarantee connections to incremental properties. For example, in the theory of equilibrium independent passivity, Hines et al. (2011); Jayawardhana et al. (2007), the dissipation inequality refers to pairs of system trajectories, one of which is a fixed point. The incremental passivity of Desoer and Vidyasagar (1975) and Stan and Sepulchre (2007) characterizes a passivity property of solutions pairs, through the use of incremental storage functions reminiscent of the notion of incremental Lyapunov functions of Angeli (2000), and supply rates of the form $Q := \Delta y^T \Delta u$, for $\Delta y := y_1 - y_2$ and $\Delta u := u_1 - u_2$, where u_i and y_i refers to input/output signals.

Incremental passivity is equivalent to passivity for linear systems. It has been used in nonlinear control for regulation, Pavlov and Marconi (2008), and synchronization purposes, Stan and Sepulchre (2007). Yet, it requires the construction of a storage function in the extended space of paired solutions, a difficult task in general, and the a priori formulation of the supply rate based on the difference between signals, which does not take into account the possible nonlinearities of the state and external spaces. A motivation for the present work partly come A motivation for the present paper partly comes from the role of incremental properties in ant windup design of induction motors Sepulchre et al. (2011) and the difficulty to establish those properties in models that integrate magnetic saturation, see Example 5 in the present paper.

A different approach to the characterization of incremental properties is provided by contraction, a differential concept The theory developed in Lohmiller and Slotine (1998) recognizes that the infinitesimal approximation of a system carries information about the behavior of its solutions set. It provides a variational approach to incremental stability, based on the linearization of the system, without explicitly constructing the distance measuring the convergence of solutions towards each other.

Following this basic idea, the present paper proposes a dissipativity theory based on the infinitesimal variations of dynamical systems along their solutions. We call it *differential dissipativity* because it is classical dissipativity lifted to the tangent bundle of the system manifold. In analogy with the classical relation between storage functions and Lyapunov functions, the proposed notion of *differential storage function* for differential dissipativity is paired to the notion of Finsler-Lyapunov function recently proposed in Forni and Sepulchre (2012), which plays a role in connecting differential dissipativity and incremental stability. The preprint van der Schaft (2013) is an insightful complementary effort in that direction, connecting the framework to the early concept of prolonged system in nonlinear control Crouch and van der Schaft (1987).

The are many potential advantages in developing a differential version of dissipativity theory. First of all, differential dissipativity is equivalent to dissipativity for linear sys-

tems. In the nonlinear setting, the fact that the infinitesimal approximation of a nonlinear system is a linear time-varying system opens the way to a characterization of *differential passivity* - differential dissipativity with quadratic supply rates - that falls in the linear setting of Willems (1972b). Moreover, differential dissipativity provides an input-output characterization of the dynamical system in the infinitesimal neighborhood of each trajectory, which leads to state-dependent differential supply rates. This is of relevance to tailor the dissipativity property to nonlinear state and external variables spaces.

The content of the paper is developed in analogy with classical results on dissipativity. The instrumental notion of displacement dynamical system is provided in Section 2. Differential dissipativity and differential passivity are formulated in Sections 3 and 4, Examples of differentially passive electromechanical systems are proposed in Section 5. Conclusion follows. Proofs are in appendix. This paper is an extended version of Forni and Sepulchre (2013).

Notation. The exposition of the differential dissipativity approach is developed on manifolds following the notation of Absil et al. (2008) and Do-Carmo (1992).

Given a manifold $\mathcal{M}$, and a point x of $\mathcal{M}$, $T_x\mathcal{M}$ denotes the *tangent space* of $\mathcal{M}$ at x. $T\mathcal{M} := \bigcup_{x\in\mathcal{M}}\{x\}\times T_x\mathcal{M}$ is the *tangent bundle*. Given two manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ and a mapping $F : \mathcal{M}_1 \to \mathcal{M}_2$. F is of class C^k, $k \in \mathbb{N}$, if the function $\hat{F} = \varphi_2 \circ F \circ \varphi_1^{-1} : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ is of class C^k, where $\varphi_1 : \mathcal{U}_x \subset \mathcal{M}_1 \to \mathbb{R}^{d_1}$ and $\varphi_2 : \mathcal{U}_{F(x)} \subset \mathcal{M}_2 \to \mathbb{R}^{d_2}$ are smooth charts. The *differential of F at x* is denoted by $DF(x)[\cdot] : T_x\mathcal{M}_1 \to T_{F(x)}\mathcal{M}_2$. A *curve* γ on a given manifold $\mathcal{M}$ is a mapping $\gamma : I \subset \mathbb{R} \to \mathcal{M}$. For simplicity we sometime use $\dot{\gamma}(t)$ or $\frac{d\gamma(t)}{dt}$ to denote $D\gamma(t)[1]$. Specifically, this notation is adopted when the variable t in γ refers to time.

I_n is the identity matrix of dimension n. Given a vector v, v^T denotes the transpose vector of v. Given a matrix M we say that $M \geq 0$ or $M \leq 0$ if $v^T M v \geq 0$ or $v^T M v \leq 0$, for each v, respectively. Given the vectors $\{v_1, \ldots, v_n\}$, $\text{Span}(\{v_1, \ldots, v_n\}) := \{v \mid \exists \lambda_1, \ldots \lambda_n \in \mathbb{R} \text{ s.t. } v = \sum_{i=1}^n \lambda_i v_i\}$. A locally Lipschitz function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to *class $\mathcal{K}$* if it is strictly increasing and $\alpha(0) = 0$; it belongs to *class $\mathcal{K}_\infty$* if, moreover, $\lim_{r\to+\infty}\alpha(r) = +\infty$.

A *distance* (or *metric*) $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_{\geq 0}$ on a manifold $\mathcal{M}$ is a positive function that satisfies $d(x,y) = 0$ if and only if $x = y$, for each $x,y \in \mathcal{M}$ and $d(x,z) \leq d(x,y) + d(y,z)$ for each $x,y,z \in \mathcal{M}$. If $d(x,y) = 0$ but $x \neq y$ we say that d is a *pseudo-metric*. A set $\mathcal{S} \subset \mathcal{M}$ is bounded if $\sup_{x,y\in\mathcal{S}} d(x,y) < \infty$ for any given distance d on $\mathcal{M}$. A curve $\gamma : I \to \mathcal{M}$ is *bounded* when its image is bounded. Given a manifold $\mathcal{M}$, a set of *isolated points* $\Omega \subset \mathcal{M}$ satisfies: for any distance function d on $\mathcal{M}$ and any given pair x_1, x_2 in Ω, there exists an $\varepsilon > 0$ such that $d(x_1, x_2) \geq \varepsilon$. Given $f : \mathcal{Z} \to \mathcal{Y}$ and $g : \mathcal{X} \to \mathcal{Z}$, the *composition* $f \circ g$ assigns to each $p \in \mathcal{X}$ the value $f \circ g(p) = f(g(p)) \in \mathcal{Y}$. Given a function $f : \mathbb{R}^n \to \mathbb{R}^m$, the matrix of partial derivatives is denoted as $\partial_x f(x)$ (Jacobian). $\partial_{xx}f(x)$ denotes the Hessian of $f(x)$.

2. DISPLACEMENT DYNAMICAL SYSTEMS

Taking inspiration from the dissipativity paper of Willems (1972a) and from the (state-space) behavioral framework in Willems (1991), given smooth manifolds $\mathcal{M}$ and $\mathcal{W}$, a time-invariant dynamical system Σ is represented by algebraic-differential equations of the form

$$F(x, \dot{x}, w) = 0 , \qquad (1)$$

where $F : T\mathcal{M} \times \mathcal{W} \to \mathbb{R}^p$, $p \in \mathbb{N}$, $x \in \mathcal{M}$ is the state, and w collects the external variables. The behavior of Σ is given by the set of absolutely continuous curves $(x,w)(\cdot) : \mathbb{R} \to \mathcal{M} \times \mathcal{W}$ that satisfy $F(x(t), Dx(t)[1], w(t)) = 0$ for (almost) all $t \in \mathbb{R}$. Given $w = (u,y)$, u - input, y - output, and $(x,u,y)(\cdot) \in \Sigma$, we say that $x(\cdot)$ is a *solution* to (1) from the initial condition $x(0) \in \mathcal{M}$ under the action of the input $u(\cdot)$.

In what follows we assume that $(x,w)(\cdot) \in \Sigma$ are C^2 functions. When the external variables are organized into input and output variables, i.e. $w = (u,y)$, we also assume existence, unicity, and forward completeness of solutions for each initial condition x_0 and input $u(\cdot)$. Note that under mild regularity assumptions on F, if $u(\cdot) \in C^2$, every $(x,u,y)(\cdot) \in \Sigma$ is a C^2 curve, as clarified in Chapter IV, Section 4, of Boothby (2003).

Under these assumptions, the *displacement dynamical system* $\delta\Sigma$ induced by Σ is represented by

$$F(x, \dot{x}, w) = 0 \qquad (2a)$$
$$DF(x, \dot{x}, w)[\delta x, \dot{\delta x}, \delta w] = 0 , \qquad (2b)$$

and it is given by the set of C^1 curves $(x, \delta x, w, \delta w)(\cdot) : \mathbb{R} \to T\mathcal{M} \times T\mathcal{W}$ that satisfy (2) for each $t \in \mathbb{R}$.

Following the interpretation proposed in Lohmiller and Slotine (1998), given a point $(x,w) \in \mathcal{M} \times \mathcal{W}$, a tangent vector $(\delta x, \delta w) \in T_x\mathcal{M} \times T_w\mathcal{W}$ represents an infinitesimal variation - or displacement - on (x,w). In this sense $\delta\Sigma$ characterizes the infinitesimal difference between every two neighborhood solutions, that is, the infinitesimal variations $\delta x(\cdot)$ on the solutions $x(\cdot)$ to (1). A graphical representation of a displacement is proposed in Figure 1. The intuitive notion of infinitesimal variation is made precise in Remark 1.
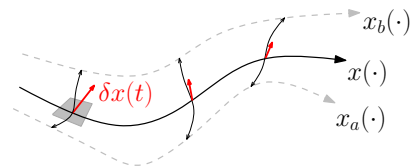

Fig. 1. The tangent vector $\delta x(t)$ represents an infinitesimal variation on $x(t)$. Given an input curve $u(\cdot)$ and its infinitesimal variation $\delta u(\cdot)$, the time-evolution of $\delta x(\cdot)$ along a given solution $x(\cdot)$ to (1) must satisfy (2). A precise characterization is given in Remark 1.

Remark 1. For each $s \in [0,1]$, consider a (parameterized) curve $(x,w)(\cdot, s) : \mathbb{R} \to \mathcal{M} \times \mathcal{W} \in \Sigma$. We assume that $(x,w)(\cdot, \cdot) \in C^2$. An infinitesimal variation on $(x,w)(\cdot, s)$ is given by $(\delta x, \delta w)(\cdot, s) := (Dx(\cdot, s)[0,1], Dw(\cdot, s)[0,1])$. As a matter of fact, $(x, \delta x, w, \delta w)(\cdot, s) \in \delta\Sigma$ for each $s \in [0,1]$. In fact, by chain rule [1],

$$\begin{aligned}
0 &= DF(x(t,s), \dot{x}(t,s), w(t,s))[0,1] \\
&= DF(\ldots)[\delta x(t,s), D\dot{x}(t,s)[0,1], \delta w(t,s)] \qquad (3) \\
&= DF(\ldots)[\delta x(t,s), \dot{\delta x}(t,s), \delta w(t,s)]
\end{aligned}$$

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[1] The differential in the right-hand side of the first identity refers to the mapping from $\mathbb{R} \times [0,1]$ to $\mathbb{R}^p$. The one in the right-hand side of the second identity refers to the mapping from $T\mathcal{M} \times \mathcal{W}$ to $\mathbb{R}^p$.

where the third identity follows from the fact that $x(\cdot, \cdot)$ is a C^2 function, by assumption (in local coordinates $\partial_s \partial_t x(t, s) = \partial_t \partial_s x(t, s)$. ⌟

When the manifold $\mathcal{M}$ is equipped with a Finsler metric $|\delta x|_x$ (see, for example, Tamássy (2008); Bao et al. (2000)), the time-evolution of $|\delta x(t)|_{x(t)}$ along the solutions $(x(\cdot), \delta x(\cdot))$ to (2) measures the contraction of the dynamical system Σ, that is, the tendency of solutions to converge towards each other. The connection between the displacement dynamical system $\delta\Sigma$ and incremental stability properties have been exploited in the seminal paper of Lohmiller and Slotine (1998), and in many other works, e.g. Lewis (1949); Aghannan and Rouchon (2003); Pavlov et al. (2004); Wang and Slotine (2005); Fromion and Scorletti (2005); Pham and Slotine (2007); Russo et al. (2010). A unifying framework for contraction based on the extension of Lyapunov theory to the tangent bundle has been recently proposed in Forni and Sepulchre (2012).

3. DIFFERENTIALLY DISSIPATIVE SYSTEMS

We develop the theory of differential dissipativity mimicking classical dissipativity, Willems (1972a); Sepulchre et al. (1997); van der Schaft (1999). In analogy to the intuitive interpretation of a storage function as the energy of the system, it is convenient to view the *differential storage function* $S : T\mathcal{M} \rightarrow \mathbb{R}_{\geq 0}$ as the infinitesimal energy associated to the infinitesimal variation $\delta x(\cdot)$ on a given solution $x(\cdot)$. This energy can be either increased or decreased through the supply provided by external sources, as prescribed by a *differential supply rate Q*.

Definition 1. Consider a manifold $\mathcal{M}$ and a set of isolated points $\Omega \subset \mathcal{M}$. For each $x \in \mathcal{M}$, consider a subdivision of $T_x\mathcal{M}$ into a *vertical distribution* $\mathcal{V}_x \subset T_x\mathcal{M}$

$$\mathcal{V}_x = \mathrm{Span}(\{v_1(x), \ldots, v_r(x)\}), \quad 0 \leq r < d, \quad (4)$$

and a *horizontal distribution* $\mathcal{H}_x \subseteq T_x\mathcal{M}$ complementary to $\mathcal{V}_x$, i.e. $\mathcal{V}_x \oplus \mathcal{H}_x = T_x\mathcal{M}$, given by

$$\mathcal{H}_x = \mathrm{Span}(\{h_1(x), \ldots, h_q(x)\}), \quad 0 < q \leq d - r \quad (5)$$

where v_i, $i \in \{1, \ldots, r\}$, and h_i, $i \in \{1, \ldots, q\}$, are C^1 vector fields.

A function $S : T\mathcal{M} \rightarrow \mathbb{R}_{\geq 0}$ is a *differential storage function* for the dynamical system Σ in (1) if there exist $c_1, c_2 \in \mathbb{R}_{\geq 0}$, $p \in \mathbb{R}_{\geq 1}$, and $K : T\mathcal{M} \rightarrow \mathbb{R}_{\geq 0}$ such that

$$c_1 K(x, \delta x)^p \leq S(x, \delta x) \leq c_2 K(x, \delta x)^p \quad (6)$$

for all $(x, \delta x) \in T\mathcal{M}$, where S and K satisfies the following conditions:

(i) S and K are C^1 functions for each $x \in \mathcal{M}$ and $\delta x \in \mathcal{H}_x \setminus \{0\}$;
(ii) S and K satisfy $S(x, \delta x) = S(x, \delta x_h)$ and $K(x, \delta x) = K(x, \delta x_h)$ for each $(x, \delta x) \in T\mathcal{M}$ such that $(x, \delta x) = (x, \delta x_h) + (x, \delta x_v)$, $\delta x_h \in \mathcal{H}_x$, and $\delta x_v \in \mathcal{V}_x$.
(iii) $K(x, \delta x) > 0$ for each $x \in \mathcal{M} \setminus \Omega$ and $\delta x \in \mathcal{H}_x \setminus \{0\}$.
(iv) $K(x, \lambda \delta x) = \lambda S(x, \delta x)$ for each $\lambda > 0$, $x \in \mathcal{M}$, and $\delta x \in \mathcal{H}_x$;
(v) $K(x, \delta x_1 + \delta x_2) < K(x, \delta x_1) + K(x, \delta x_2)$ for each $x \in \mathcal{M} \setminus \Omega$ and $\delta x_1, \delta x_2 \in \mathcal{H}_x \setminus \{0\}$ such that $\delta x_1 \neq \lambda \delta x_2$ and $\lambda \in \mathbb{R}$ (strict convexity). ⌟

Definition 2. A function $Q : \mathcal{M} \times T\mathcal{W} \rightarrow \mathbb{R}$ is a *differential supply rate* for the dynamical system Σ in (1) if

$$\int_0^t |Q(x(\tau), w(\tau), \delta w(\tau)| d\tau < \infty \quad (7)$$

for each $t \geq 0$ and each $(x, \delta x, w, \delta w)(\cdot) \in \delta\Sigma$. ⌟

The function S provides a non-negative value $S(x, \delta x)$ to each $\delta x \in T_x\mathcal{M}$. When $\mathcal{V}_x = \emptyset$, a suggestive notation for $K(x, \delta x)$ is $|\delta x|_x$ - a non-symmetric norm on each tangent space $T_x\mathcal{M}$ - which immediately connects the differential storage to the idea of an energy of the displacement δx, since $c_1 |\delta x|_x^p \leq S(x, \delta x) \leq c_2 |\delta x|_x^p$. From Definition 1 it is possible to identify differential storage functions S and horizontal Finsler-Lyapunov functions V, introduced in Section VIII of Forni and Sepulchre (2012). Therefore the existence of a differential storage S endows $\mathcal{M}$ with the structure of a pseudo-metric space, which plays a central role in connecting differential dissipativity to incremental stability. Restricting a differential storage to horizontal distributions is convenient in many situations where contraction takes place only in certain directions. For example, let $\mathcal{M}$ be the state space and suppose that the output $y \in \mathcal{Y}$ is given by $y = h(x)$ where $h : \mathcal{M} \rightarrow \mathcal{Y}$ is a differentiable function. Then, in coordinates, $\delta y^T \delta y$ is a possible candidate storage function with horizontal distribution $\mathcal{H}_x$ given by the span of the columns of the matrix $\partial_x h(x)^T \partial_x h(x)$. With this storage, the state-space $\mathcal{M}$ becomes a pseudo-metric space, while the output space $\mathcal{Y}$ becomes a metric space. Further details are collected in Remark 2.

Remark 2. Suppose that for each $x \in \mathcal{M}$, $\mathcal{H}_x = T_x\mathcal{M}$, and take $\Omega = \emptyset$. Then, K is a Finsler structure on $\mathcal{M}$ (see, for example, Tamássy (2008); Bao et al. (2000)). Then, we can define the length of a curve as $L(\gamma) := \int_I K(\gamma(s), D\gamma(s)[1]) ds$. The induced distance d between any two points $x_0, x_1 \in \mathcal{M}$ is given by $d(x_0, x_1) := \inf_{\Gamma(x_0, x_1)} L(\gamma)$, where $\Gamma(x_0, x_1)$ is the set of piecewise C^1 curves in $[0, 1] \rightarrow \mathcal{M}$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. For the case $\mathcal{H}_x \neq T_x\mathcal{M}$, we have the identity $L(\gamma) = \int_I K(\gamma(s), \Pi_h(D\gamma(s)[1])) ds$, where the function $\Pi_{\mathcal{H}}(\cdot)$ projects every tangent vector $v_x \in T_x\mathcal{M}$ into $\Pi_{\mathcal{H}}(v_x) \in \mathcal{H}_x$. In this case, $L(\gamma)$ measures only the horizontal contribution of γ, and the induced $d(x_0, x_1) := \inf_{\Gamma(x_0, x_1)} L(\gamma)$, is only a pseudo-distance on $\mathcal{M}$, since $d(x_0, x_1) = 0$ for some $x_0 \neq x_1$. An extended discussion and examples are provided in Sections IV and VIII of Forni and Sepulchre (2012). ⌟

We can finally provide the definition of differential dissipativity. We emphasize that differential dissipativity is just dissipativity lifted to the tangent bundle.

Definition 3. The dynamical system Σ in (1) is *differentially dissipative* with respect to the differential supply rate Q if there exists a differential storage function S such that

$$S(x(t), \delta x(t)) - S(x(0), \delta x(0)) \leq \int_0^t Q(x(\tau), w(\tau), \delta w(\tau)) d\tau$$

$$(8)$$

for all $t \geq 0$ and all $(x, \delta x, w, \delta w)(\cdot) \in \delta\Sigma$ in (2). When Q is independent on x, that is, $Q : T\mathcal{W} \rightarrow \mathbb{R}$, we say that Σ is *uniformly differentially dissipative*. ⌟

Exploiting the assumption $S \in C^1$, (8) is equivalent to

$$\frac{d}{dt} S(x(t), \delta x(t)) \leq Q(x(t), w(t), \delta w(t)). \quad (9)$$

We conclude the section by illustrating a first connection between differential dissipativity and incremental stability.

Theorem 1. Suppose that the dynamical system Σ represented by (1) is differentially dissipative with differential storage S and differential supply rate Q. Suppose also that for $w = (u, y)$, u - input, y - output, it holds that $Q(x, u, y, 0, \delta y) = 0$ for each $x \in \mathcal{M}$, and each $(u, y, 0, \delta y) \in T\mathcal{W}$. Then, there exists a class $\mathcal{K}$ function α such that

$$d(x_1(t), x_2(t)) \leq \alpha(d(x_1(0), x_2(0))) \qquad (10)$$

for each $t \geq 0$ and each $(x_1, u_1, y_1)(\cdot), (x_2, u_2, y_2)(\cdot) \in \Sigma$, such that $u_1(\cdot) = u_2(\cdot)$, where d is the pseudo-distance induced by $S^{\frac{1}{p}}$, with p degree of homogeneity of S (see Definition 1). ⌟

Note that if $\mathcal{H}_x = T_x\mathcal{M}$, then d is a distance on $\mathcal{M}$, thus Theorem 1 guarantees that Σ is incrementally stable for any feedforward input signal $u(\cdot)$.

4. DIFFERENTIAL PASSIVITY

Following the approach of Willems (1972b), we formulate differential passivity as the restriction of differential dissipativity to quadratic supply rates. To this end, we consider the external variable manifold $\mathcal{W}$ as the product of an input vector space $\mathcal{U}$ and an output vector space $\mathcal{Y}$ such that $\mathcal{U} = \mathcal{Y}$. A consequence of working with a vector space $\mathcal{W}$ is that $T_w\mathcal{W} = \mathcal{W}$ for each $w \in \mathcal{W}$. In what follows, we will use $u \in \mathcal{U}$ to denote the input and $y \in \mathcal{Y}$ to denote the output.

For each $x \in \mathcal{M}$, let $\mathbb{W}_x$ be a $(0, 2)$-tensor field on $\mathcal{W}$ that provides an inner product on each tangent space $T_w\mathcal{W} = \mathcal{W}$, denoted by $\langle \cdot, \cdot \rangle_{\mathbb{W}_x}$. For simplicity of the exposition, we write $\langle \delta y, \delta u \rangle_{\mathbb{W}_x}$ to denote $\langle (\delta y, 0), (0, \delta u) \rangle_{\mathbb{W}_x}$, or $\langle \delta y, \delta y \rangle_{\mathbb{W}_x}$ to denote $\langle (0, \delta y), (0, \delta y) \rangle_{\mathbb{W}_x}$.

Definition 4. For each $x \in \mathcal{M}$, let $\mathbb{W}_x$ be a $(0, 2)$-tensor field on $\mathcal{W}$. A dynamical system Σ is *differentially passive* if it is differentially dissipative with respect to a differential supply rate of the form

$$Q(x, u, \delta u, y, \delta y) := \langle \delta y, \delta u \rangle_{\mathbb{W}_x}. \qquad (11)$$

Σ is *uniformly* differentially passive whenever Q is independent on x. Finally, we say that Σ is *strictly* differentially passive if there exists a function α of class $\mathcal{K}$ such that (9) is restricted to $\dot{S} \leq -\alpha(S(x, \delta x)) + Q$. ⌟

As in passivity, the next theorems show that the feedback interconnection of differentially passive systems is differentially passive.

Theorem 2. Let Σ_1 and Σ_2 be (strictly) uniformly differentially passive dynamical systems. Suppose that $\mathcal{W}_1 = \mathcal{W}_2$ and that their supply rates are based on the same $(0, 2)$-tensor $\mathbb{W}$. Then, the dynamical system Σ arising from the feedback interconnection

$$u_1 = -y_2 + v_1 \ , \ u_2 = y_1 + v_2, \qquad (12)$$

is (strictly) uniformly differentially passive from $v = (v_1, v_2) \in \mathcal{U}_1 \times \mathcal{U}_2$ to $y = (y_1, y_2) \in \mathcal{Y}_1 \times \mathcal{Y}_2$. ⌟

Theorem 3. Let Σ_1 and Σ_2 be (strictly) differentially passive dynamical systems. Suppose that $\mathcal{W}_1 = \mathcal{W}_2$ and that their supply rates are based on the $(0, 2)$-tensors $\mathbb{W}_{x_1}$ for $x_1 \in \mathcal{M}_1$ and $\mathbb{W}_{x_2}$ for $x_2 \in \mathcal{M}_2$, respectively.

Then, the dynamical system Σ arising from the feedback interconnection

$$\begin{aligned} u_1 &= -k_2(x_2) + v_1 & k_2 : \mathcal{M}_2 \to \mathcal{M}_1 \in C^2 \\ u_2 &= k_1(x_1) + v_2 & k_1 : \mathcal{M}_1 \to \mathcal{M}_2 \in C^2 \end{aligned} \qquad (13)$$

is differentially passive from $v = (v_1, v_2)$ to $y = (y_1, y_2)$, provided that

$$\langle \delta y_1, Dk_2(x_2)[\delta x_2] \rangle_{\mathbb{W}_{x_1}} = \langle \delta y_2, Dk_1(x_1)[\delta x_1] \rangle_{\mathbb{W}_{x_2}} \quad (14)$$

for each $x_1 \in \mathcal{M}_1$ and each $x_2 \in \mathcal{M}_2$. ⌟

The state-feedback interconnection in (13) is in contrast with the classical passivity approach that looks at systems as input/output operators. However, differently from classical passivity and from uniform differential passivity, differential passivity is an input/output characterization of the system that depends on the *trajectories*, geometrically expressed by a different tensor $\mathbb{W}_x$ for each $x \in \mathcal{M}$. This lack of uniformity with respect to the solutions of the system requires extra-effort at interconnection, as shown by (14). In this sense, the key role of the state-feedback (13) is to *equalize* the two tensors $\mathbb{W}_{x_1}$ and $\mathbb{W}_{x_2}$, to achieve the desired interconnected behavior. Despite the state dependence, Theorem 3 can be conveniently used for design.

Example 1. Consider the dynamical system Σ of equations

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \qquad x \in \mathbb{R}^n, \ y, u \in \mathbb{R}^q \ ; \qquad (15)$$

whose induced displacement dynamical system $\delta\Sigma$ is represented by (15) and

$$\begin{cases} \dot{\delta x} = \partial_x f(x)\delta x + [\partial_x g(x)u]\delta x + g(x)\delta u \\ \delta y = \partial_x h(x)\delta x. \end{cases} \qquad (16)$$

Let $W(x)$ a symmetric matrix for each $x \in \mathcal{M}$. Σ is differentially passive with differential supply rate $\delta y^T W(x)\delta u$ if there exist a matrix $M(x) = \partial_{xx}m(x)$, where $m : \mathbb{R}^n \to \mathbb{R}$, and an invertible matrix Π such that

$$\begin{aligned} M(x)^T \partial_x[M(x)f(x)] &\leq 0 \\ M(x)g(x) &= \Pi \ , \\ \partial_x h(x)^T W(x) &= M(x)^T \Pi \end{aligned} \qquad (17)$$

In fact, define $S(x, \delta x) := \frac{1}{2}\delta x^T M(x)^T M(x)\delta x$. Then,

$$\begin{aligned} \dot{S} &= \delta x^T M(x)\partial_x(M(x)f(x))\delta x + \\ &\quad + \delta x^T M(x)^T \partial_x(M(x)g(x)u)\delta x + \\ &\quad + \delta x^T M(x)^T M(x)g(x)\delta u \\ &\leq \delta x^T M(x)^T \partial_x(\Pi u)\delta x + \delta x^T M(x)^T \Pi \delta u \\ &= \delta x^T h(x)^T W(x)\delta u \\ &= \delta y^T W(x)\delta u \ . \end{aligned} \qquad (18)$$
⌟

Example 2. Consider the dynamical system Σ given by

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) + i(x)u \end{aligned} \qquad x \in \mathbb{R}^n, \ y, u \in \mathbb{R}^q \ ; \qquad (19)$$

whose displacement dynamics is given by

$$\begin{aligned} \dot{\delta x} &= \partial_x f(x)\delta x + [\partial_x g(x)u]\delta x + g(x)\delta u \\ \delta y &= \partial_x h(x)\delta x + [\partial_x i(x)u]\delta x + i(x)\delta u. \end{aligned} \qquad (20)$$

Let $W(x)$ a symmetric matrix for each $x \in \mathcal{M}$. Σ is differentially passive with differential supply rate $\delta y^T W(x)\delta u$ if there exists a matrix $M(x) = \partial_{xx}m(x)$, where $m : \mathbb{R}^n \to \mathbb{R}$, such that

$$\begin{aligned} M(x)^T \partial_x [M(x)f(x)] &\leq 0 \\ [\partial_x h(x)]^T W(x) &= M(x)^T M(x) g(x) \\ [\partial_x i(x)u]^T W(x) &= M(x)^T \partial_x [M(x)g(x)u] \\ i(x)^T W(x) &\geq 0. \end{aligned} \quad (21)$$

for each $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^d$. In fact, using the differential storage $S(x,\delta x) := \frac{1}{2}\delta x^T M(x)^T M(x)\delta x$, we get

$$\begin{aligned} \dot{S} &\leq \delta x^T \underbrace{M(x)[\partial_x M(x)g(x)u]}_{[\partial_x i(x)u]^T W(x)} \delta x + \\ &\quad + \delta x^T \underbrace{M(x)^T M(x)g(x)}_{[\partial_x h(x)]^T W(x)} \delta u \\ &= \delta y^T W(x)\delta u - \delta u^T i(x)^T W(x)\delta u \\ &\leq \delta y^T W(x)\delta u. \end{aligned} \quad (22)$$
⌟

As a final example of the section, we reconsider Example 1 to illustrate Theorem 3.

Example 3. Consider two systems Σ_1 and Σ_2 satisfying (17) respectively with matrices $M_1(x_1) = \partial_{xx}m_1(x), W_1(x_1)$ and $M_2(x_2) = \partial_{xx}m_2(x), W_2(x_2)$, and constant matrices Π_1 and Π_2. The closed-loop system given by the feedback interconnection (13) is differentially passive provided that

$$\begin{cases} \partial_{x_2}k_2(x_2) = \Pi_2^T M_2(x_2) \\ \partial_{x_1}k_1(x_1) = \Pi_1^T M_1(x_1) \end{cases} \quad (23)$$

This is an immediate consequence of Theorem 3, since

$$\begin{aligned} \delta y_1^T W_1(x_1)\partial_{x_2}k_2(x_2)\delta x_2 &= \delta y_1^T W_1(x_1)\Pi_2^T M_2(x_2)\delta x_2 \\ &= \delta y_1^T W_1(x_1)W_2(x_2)\delta y_2 \\ &= \delta x_1^T M_1(x_1)^T \Pi_1 W_2(x_2)\delta y_2 \\ &= \delta x_1^T [\partial_{x_1}k_1(x_1)]^T W_2(x_2)\delta y_2 \end{aligned} \quad (24)$$

as required by (14). A graphical interpretation of (23) is provided in Figure 2.
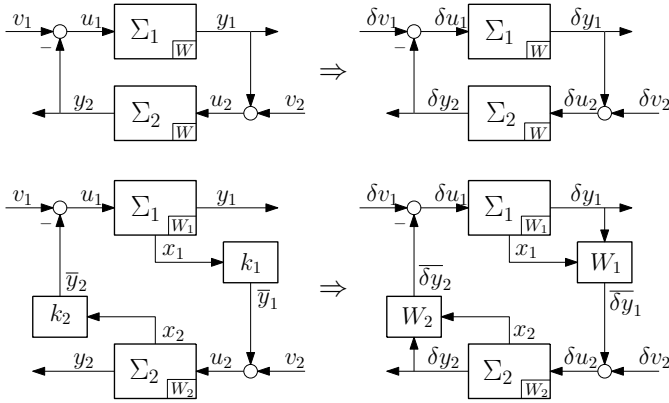⌟

Fig. 2. Interconnection of uniform differential passive systems (top). Interconnection of differential passive systems (bottom). $k_1(x_1)$ and $k_2(x)$ define an interconnection on (15) that induces on the displacement dynamics (16) the cancellation $-\delta y_1^T W_1(x_1)W_2(x_2)\delta y_2 + \delta y_2^T W_2(x_2)W_1(x_1)\delta y_1 = 0$. They also define new output functions $\overline{\delta y_i}$ that guarantee uniform differential passivity.

We conclude the section by extending Theorem 1. The next theorem shows that a differentially passive dynamical system with "excess" of output differential passivity behaves like a filter: its steady-state output depends only on the signal at the input.

Theorem 4. Let Σ be a differentially passive dynamical system with

- differential storage S such that $\mathcal{V}_x = \emptyset$ for each x;
- differential supply rate $Q := \langle \delta y, \delta u\rangle_{\mathbb{W}_x} - \langle \delta y, \delta y\rangle_{\mathbb{W}_x}$ such that $\langle \delta y, \delta y\rangle_{\mathbb{W}_x} > 0$ for each $\delta y \in \mathcal{Y}\setminus\{0\}$ and each $x \in \mathcal{M}$ (excess of output passivity).

Let $\overline{u}(\cdot) : \mathbb{R}_{\geq 0} \to \mathcal{U}$ be a C^2 input signal and suppose that every curve $\xi(\cdot) := (x, \overline{u}, y)(\cdot) \in \Sigma$ remains bounded.

Then, for any pair $(x_1, \overline{u}, y_1)(\cdot), (x_2, \overline{u}, y_2)(\cdot) \in \Sigma$,
$$\lim_{t\to\infty} |y_1(t) - y_2(t)| = 0 . \quad (25)$$
⌟

The hypothesis of the theorem guarantees incremental stability of Σ - a consequence of Theorem 1. If Σ is strictly differentially passive, then Theorem 4 can be strengthened towards incremental asymptotic stability. Finally, the case of $\mathcal{V}_x \neq \emptyset$ is not taken into account here but it presents similarities with the analysis of Section 2.3.2 in Sepulchre et al. (1997), about passivity with semidefinite storage functions and stability.

5. EXAMPLES OF DIFFERENTIALLY PASSIVE ELECTRICAL CIRCUITS

In the first example below we show the differential passivity of a simple nonlinear RC circuit. Differential passivity is also used in the second example below to develop an feed-forward control strategy for an induction motor with flux saturations.

Example 4. [Nonlinear RC circuit]
Consider the simple circuit reproduced in Figure 3. The nonlinearity of the circuit is due to the nonlinear relation $v_c = \mu(q_c)$ between the charge q_c and the voltage v_c of the capacitor. We suppose that $\mu(q_c)$ is differentiable and strictly increasing.
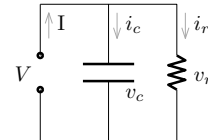

Fig. 3. V, I - external voltage and current. v_c, i_c - capacitor voltage and current. v_r, i_r - resistor voltage and current.

The algebraic-differential description of the circuit is given by the constitutive relations of each component and by Kirchhoff laws,

$$\begin{cases} \dot{q}_c = i_c \\ v_c = \mu(q_c) \\ v_r = Ri_r, R > 0 \end{cases} ; \quad \begin{cases} I = i_c + i_r \\ V = v_c \\ v_c = v_r \end{cases} . \quad (26)$$

Following (2), the displacement dynamical systems is thus represented by (26) and by the set of equations

$$\begin{cases} \dot{\delta q_c} = \delta i_c \\ \delta v_c = \partial_{q_c}\mu(q_c)\delta q_c \\ \delta v_r = R\delta i_r \end{cases} ; \quad \begin{cases} \delta I = \delta i_c + \delta i_r \\ \delta V = \delta v_c \\ \delta v_c = \delta v_r \end{cases} . \quad (27)$$

The circuit is differentially passive from V to I with differential storage $S(q_c, \delta q_c) := \frac{1}{2}\delta q_c^2$. In fact, define $W(q_c) := [\partial_{q_c}\mu(q_c)]^{-1}$, then

$$\begin{aligned}
\dot{S} &= \delta q_c \partial_{q_c} \mu(q_c) \delta q_c \\
&= W(q_c) \delta v_c \delta i_c \\
&= W(q_c) \delta V (\delta I - \delta i_r) \qquad (28) \\
&= W(q_c) \delta V \delta I - W(q_c) \delta v_r \delta i_r \\
&\leq W(q_c) \delta V \delta I,
\end{aligned}$$

where the last identity follows from the fact that $W(q_c)$ is greater than 0 for each value of q_c, and $\delta v_r \delta i_r = R \delta i_r^2 \geq 0$.

Example 5. [Induction motor with flux saturation]
We revisit the model proposed in Sullivan et al. (1996). The model is developed in a rotating frame at speed ω_s. The rotor speed is denoted by ω_r. Rotor and stator magnetic flux vectors are denoted respectively by φ_r and φ_s. Rotor and stator currents are given by i_r and i_s. The analysis below takes into account only the electrical part of the motor. The mechanical equations are thus not detailed. Indeed, for $\varphi_r, \varphi_s, i_r, i_s \in \mathbb{C}$, the differential relations are given by

$$\dot{\omega}_r = h(\omega_r, \varphi_r, \varphi_s, \tau_{load}) \qquad (29a)$$

$$\dot{\varphi}_r = -j\omega_g \varphi_r - R_r i_r \qquad (29b)$$

$$\dot{\varphi}_s = -j\omega_s \varphi_s - R_s i_s + u_s \qquad (29c)$$

where $\omega_g = \omega_s - \omega_r$, and R_r and R_s are rotor and stator resistances. τ_{load} is the (disturbance) load, and u_s is a control input. The motor model is completed by the algebraic relations between currents and fluxes, given by

$$\begin{cases} i_r = F_r(\varphi_r) + (\frac{1}{L_r} + \frac{1}{L_l})\varphi_r - \frac{1}{L_l}\varphi_s \\ i_s = F_s(\varphi_s) + (\frac{1}{L_s} + \frac{1}{L_l})\varphi_s - \frac{1}{L_l}\varphi_r. \end{cases} \qquad (30)$$

L_r, L_s, and L_l are the usual inductances adopted in classical linear flux-current models, while the nonlinear C^2 functions F_r and F_s characterize the flux saturation. For instance, F_r satisfies a relation of the form $F_r(\varphi_r) = f(|\varphi_r|)\varphi_r$ where f is a monotonically increasing sector function, that is, $f(s) \geq 0$ and $f(s)' \geq 0$, for each $s \geq 0$. These assumptions guarantee that

$$\partial_{\varphi_r} F(\varphi_r) = f'(|\varphi_r|)\frac{\varphi_r \varphi_r^T}{|\varphi_r|} + f(|\varphi_r|)I \geq 0. \qquad (31)$$

Indeed, the current i_r may grow faster than the flux φ_r (for $F_r \neq 0$), which characterizes a limited increase of the flux despite large increments of the currents. Similar assumptions hold for F_s. Note that the alignment of current and flux vectors is preserved.

In what follows we will use Σ to denote the dynamical system represented by (29) and (30). Using $\varphi := (\varphi_r, \varphi_s)$ and $i := (i_r, i_s)$, Σ is given by the set of C^2 curves $\xi(\cdot) := (\varphi, i, \omega_r, \omega_s, u_s)(\cdot)$ that satisfy (29) and (30) for each $t \geq 0$.

The analysis proposed below is based on the introduction of a new dynamical system, the *virtual dynamical system* (see, for example, Wang and Slotine (2005)), represented by (29b), (29c) and (30), where the relation between the rotor speed ω_r and the flux φ is disregarded. To distinguish between the induction motor and the associated virtual system, we use over-lined variables: $\overline{\varphi} := (\overline{\varphi}_r, \overline{\varphi}_s)$ and $\overline{i} := (\overline{i}_r, \overline{i}_s)$. Indeed, for each $\xi(\cdot) = (\varphi, i, \omega_r, \omega_s, u_s)(\cdot) \in \Sigma$, $\overline{\Sigma}_{\xi(\cdot)}$ is the virtual dynamical system given by the set of

curves $(\overline{\varphi}, \overline{i}, \omega_r, \omega_s, u_s)(\cdot)$ that satisfy (29b), (29c) and (30) (expressed in the over-lined variables).

The crucial relation between Σ and the virtual system $\overline{\Sigma}_{\xi(\cdot)}$ is that if $\xi(\cdot) \in \Sigma$, then $\xi(\cdot) \in \overline{\Sigma}_{\xi(\cdot)}$. Exploiting this relation, it is possible to infer properties of Σ from the properties of the virtual dynamical system $\overline{\Sigma}_{\xi(\cdot)}$.

For the virtual system $\overline{\Sigma}_{\xi(\cdot)}$, $\omega_r(\cdot)$ and $\omega_s(\cdot)$ are exogenous signal acting uniformly on each solution $\overline{\varphi}(\cdot)$. Therefore for both $\omega_s(\cdot)$ and $\omega_g(\cdot)$ one can consider $\delta\omega_g = \delta\omega_s = 0$ (see Remark 1). The virtual displacement dynamical system is thus given by (29b), (29c) and (30) (expressed in the over-lined variables) and by

$$\begin{cases} \delta\dot{\overline{\varphi}}_r = -j\omega_g\delta\overline{\varphi}_r - R_r\delta\overline{i}_r \\ \delta\dot{\overline{\varphi}}_s = -j\omega_s\delta\overline{\varphi}_s - R_s\delta\overline{i}_s + \delta u_s \end{cases} \qquad (32)$$

$$\begin{cases} \delta\overline{i}_r = \partial F_r(\overline{\varphi}_r)\delta\overline{\varphi}_r + (\frac{1}{L_r} + \frac{1}{L_l})\delta\overline{\varphi}_r - \frac{1}{L_l}\delta\overline{\varphi}_s \\ \delta\overline{i}_s = \partial F_s(\overline{\varphi}_s)\delta\overline{\varphi}_s + (\frac{1}{L_s} + \frac{1}{L_l})\delta\overline{\varphi}_s - \frac{1}{L_l}\delta\overline{\varphi}_r. \end{cases} \qquad (33)$$

(32) and (33) characterize respectively a differentially passive dynamical system and a differentially passive static nonlinearity. For (32), consider the differential storage $V = \frac{\delta\overline{\varphi}_r^2}{2R_r} + \frac{\delta\overline{\varphi}_s^2}{2R_s}$. Then,

$$\dot{V} = -\delta\overline{\varphi}_r\delta\overline{i}_r - \delta\overline{\varphi}_s\delta\overline{i}_s + \frac{1}{R_s}\delta\overline{\varphi}_s\delta u_s \qquad (34)$$

which establish uniform differential passivity from $(-\delta\overline{i}, \delta u_s)$ to $(\delta\overline{\varphi}, \delta\overline{\varphi}_s)$ of the dynamical system represented by (29b), (29c) (expressed in the over-lined variables).

On the other hand, for (33) we get $0 \leq \delta\overline{i}^T \delta\overline{\varphi} =$

$$= \delta\overline{\varphi}^T \left(\underbrace{\begin{bmatrix} \partial F_r(\overline{\varphi}_r) + \frac{1}{L_r} & 0 \\ 0 & \partial F_r(\overline{\varphi}_s) + \frac{1}{L_s} \end{bmatrix}}_{>0} + \underbrace{\begin{bmatrix} \frac{1}{L_l} & -\frac{1}{L_l} \\ -\frac{1}{L_l} & \frac{1}{L_l} \end{bmatrix}}_{\geq 0} \right) \delta\overline{\varphi}.$$

$$(35)$$

From (34) and (35), the combination of (32) and (33) guarantees that $\overline{\Sigma}_{\xi(\cdot)}$ is strictly uniformly differentially passive from u_s to φ_s, for each $\xi(\cdot) \in \Sigma$. In fact,

$$\dot{V} \leq -\delta\overline{\varphi}^T \underbrace{M(\overline{\varphi})}_{>0} \delta\overline{\varphi} + \frac{1}{R_s}\delta\overline{\varphi}_s\delta u_s, \qquad (36)$$

where $M(\varphi)$ is the quantity between brackets in (35). Because $M(\varphi) > 0$, for $\delta u_s = 0$ (feedforward signal), Theorem 4 guarantees that

$$\lim_{t\to\infty} |\overline{\varphi}_1(t) - \overline{\varphi}_2(t)| = 0 \qquad (37)$$

for all $(\overline{\varphi}_1, \overline{i}_1, \omega_r, \omega_s, u_s)(\cdot)$, $(\overline{\varphi}_2, \overline{i}_2, \omega_r, \omega_s, u_s)(\cdot)$ in $\overline{\Sigma}_{\xi(\cdot)}$. Note that the boundedness of these curves is guaranteed for bounded signals $u_s(\cdot)$ by the combination of the effect of the dissipative terms in (32) and the alignment between currents and fluxes in (33).

The incremental property (37) of the virtual system $\overline{\Sigma}_{\xi(\cdot)}$ can be used to provide an feedforward control design for Σ. For illustration purposes, in what follows we consider the goal of asymptotically regulate φ_r towards a prescribed flux configuration φ_r^*.

From (37), achieving the goal for the virtual system $\overline{\Sigma}_{\xi(\cdot)}$ is straightforward: if $((\varphi_r^*, \varphi_s^*), i^*, \omega_r, \omega_s, u_s)(\cdot) \in \overline{\Sigma}_{\xi(\cdot)}$ then each curve $(\overline{\varphi}, \overline{i}, \omega_r, \omega_s, u_s)(\cdot) \in \overline{\Sigma}_{\xi(\cdot)}$ satisfies $\lim_{t\to\infty} |\overline{\varphi}(t) - (\varphi_r^*, \varphi_s^*)(t)| = 0$. Indeed, from (29b), (29c), and (30), the feedforward input $u_s(\cdot)$ given by

$$\varphi_s^* := -\frac{L_l}{R_r}\dot{\varphi}_r^* - L_l[j\omega_g + (\frac{1}{L_r} + \frac{1}{L_l})]\varphi_r^* - F_r(\varphi_r^*)$$
$$u_s := [j\omega_s + R_s(\frac{1}{L_s} + \frac{1}{L_l})]\varphi_s^* + R_s F_s(\varphi_s^*) - \frac{1}{L_l}r + \dot{\varphi}_s^*$$

$$(38)$$

guarantees that $((\varphi_r^*, \varphi_s^*), i^*, \omega_r, \omega_s, u_s)(\cdot) \in \overline{\Sigma}_{\xi(\cdot)}$.

The reader will notice that for any given selection of $\xi(\cdot) := (\varphi, i, \omega_r, \omega_s, u_s)(\cdot) \in \Sigma$, with $u_s(\cdot)$ given in (38), the curve $((\varphi_r^*, \varphi_s^*), i^*, \omega_r, \omega_s, u_s)(\cdot)$ belongs to $\overline{\Sigma}_{\xi(\cdot)}$. This is a consequence of the fact that $u_s(\cdot)$ is formulated by taking into account explicitly $\omega_s(\cdot)$ and $\omega_g(\cdot)$. Thus, exploiting the fact that if $\xi(\cdot) \in \Sigma$, then $\xi(\cdot) \in \overline{\Sigma}_{\xi(\cdot)}$, we can conclude that

$$\lim_{t\to\infty} |\varphi(t) - (\varphi_r^*, \varphi_s^*)(t)| = 0 \qquad (39)$$

for all $(\varphi, i, \omega_r, \omega_s, u_s)(\cdot) \in \Sigma$ with $u_s(\cdot)$ in (38). A similar (but dynamic) design of $\overline{u}$ can be provided for the regulation of φ_s to φ_s^*. ⌟

6. CONCLUSIONS

The concept of differential dissipativity is introduced as a natural extension of differential stability for open systems. The differential storage $S(x, \delta x)$ is inspired from the Finsler-Lyapunov function of Forni and Sepulchre (2012) and has the interpretation of (infinitesimal) energy of a displacement δx along a solution curve through x. Extending the role of dissipativity theory for analysis and design of interconnections in the tangent bundle offers a novel way to study incremental stability (or contraction) properties of nonlinear systems.

Appendix A. PROOFS

Proof of Theorem 1 [Sketch]. In accordance with Remark 1, we can consider curves in $\delta\Sigma$ for $\delta u(\cdot) = 0$. In fact, for any given pair of curves $(x_1, \overline{u}, y_1)(\cdot), (x_2, \overline{u}, y_2)(\cdot) \in \Sigma$, the associated parameterization satisfies $u(\cdot, s) = \overline{u}(\cdot)$, that is, $Du(t, s)[0, 1] = 0$ for each t and s.

As a consequence, by differential dissipativity, we have $\dot{S} \leq 0$. Because the differential storage S is also a non-increasing horizontal Finsler-Lyapunov function, (10) is a consequence of Theorem 3 in Forni and Sepulchre (2012) Moreover, the case of differential storages S with $\mathcal{H}_x = T_x\mathcal{M}$, is a consequence of Theorem 1 in Forni and Sepulchre (2012). □

Proof of Theorem 2 Define the differential storage $S := S_1 + S_2$ [2]. The functions α_1 and α_2 below must be set to zero for the weaker property of uniform differential passivity.

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[2] When clear from the context, we drop the arguments of the functions to simplify the notation.

$$\begin{aligned}\dot{S} &\leq -\alpha_1(S_1) - \alpha_2(S_2) + \langle \delta y_1, -\delta y_2 + \delta v_1\rangle_{\mathbb{W}} \\ &\quad + \langle \delta y_2, \delta y_1 + \delta v_2\rangle_{\mathbb{W}} \\ &= -\alpha_1(S_1) - \alpha_2(S_2) - \langle \delta y_1, \delta y_2\rangle_{\mathbb{W}} + \langle \delta y_2, \delta y_1\rangle_{\mathbb{W}} \\ &\quad + \langle \delta y_1, \delta v_1\rangle_{\mathbb{W}} + \langle \delta y_2, \delta v_2\rangle_{\mathbb{W}} \\ &= -\alpha_1(S_1) - \alpha_2(S_2) + \langle \delta y_1, \delta v_1\rangle_{\mathbb{W}} + \langle \delta y_2, \delta v_2\rangle_{\mathbb{W}} \\ &\leq -\overline{\alpha}(S/2) + \langle \delta y_1, \delta v_1\rangle_{\mathbb{W}} + \langle \delta y_2, \delta v_2\rangle_{\mathbb{W}}\end{aligned}$$

$$(A.1)$$

where $\overline{\alpha}(\cdot) := \min(\alpha_1(\cdot), \alpha_2(\cdot)) \in \mathcal{K}$. In fact,

$$\alpha_1(S_1) + \alpha_2(S_2) \geq \overline{\alpha}(S_1) + \overline{\alpha}(S_2) \geq \overline{\alpha}\left(\frac{S_1 + S_2}{2}\right) \quad (A.2)$$

where the first inequality follows from the definition of $\overline{\alpha}$, and the last inequality holds because (i) $\overline{\alpha}(S_1) \geq \overline{\alpha}\left(\frac{S_1+S_2}{2}\right)$ for $S_1 \geq S_2$; (ii) $\overline{\alpha}(S_2) \geq \overline{\alpha}\left(\frac{S_1+S_2}{2}\right)$ for $S_2 \geq S_1$; (iii) $\overline{\alpha}(S_1) \geq 0$ and $\overline{\alpha}(S_2) \geq 0$.

Note that $\langle \delta y_1, \delta v_1\rangle_{\mathbb{W}} + \langle \delta y_2, \delta v_2\rangle_{\mathbb{W}}$ characterizes an inner product on the product manifold $\mathcal{W}_1 \times \mathcal{W}_2$. □

Proof of Theorem 3 Define the differential storage $S := S_1 + S_2$. As in the proof of Theorem 2, α_1 and α_2 below must be set to zero for the case of differential passivity.

$$\begin{aligned}\dot{S} &\leq -\alpha_1(S_1) - \alpha_2(S_2) \\ &\quad - \langle \delta y_1, Dk_2(x_2)[\delta x_2]\rangle_{\mathbb{W}_{x_1}} + \langle \delta y_2, Dk_1(x_1)[\delta x_1]\rangle_{\mathbb{W}_{x_2}} \\ &\quad + \langle \delta y_1, \delta v_1\rangle_{\mathbb{W}_{x_1}} + \langle \delta y_2, \delta v_2\rangle_{\mathbb{W}_{x_2}} \\ &= -\overline{\alpha}(S/2) + \langle \delta y_1, \delta v_1\rangle_{\mathbb{W}_{x_1}} + \langle \delta y_2, \delta v_2\rangle_{\mathbb{W}_{x_2}},\end{aligned}$$

$$(A.3)$$

where $\overline{\alpha}(\cdot) := \min(\alpha_1(\cdot), \alpha_2(\cdot)) \in \mathcal{K}$. The last identity follows from (14) and from (A.2).

For each point (x_1, x_2) of the product manifold $\mathcal{M}_1 \times \mathcal{M}_2$, $\langle \delta y_1, \delta v_1\rangle_{\mathbb{W}_{x_1}} + \langle \delta y_2, \delta v_2\rangle_{\mathbb{W}_{x_2}}$ defines a $(0, 2)$-tensor $\mathbb{W}_{(x_1, x_2)}$ on the product manifold $\mathcal{W}_1 \times \mathcal{W}_2$. □

Proof of Theorem 4. Let $(x_1, \overline{u}, y_1)(\cdot), (x_2, \overline{u}, y_2)(\cdot)$ be any pair of C^2 curves in Σ. For each $s \in [0, 1]$, define $u(\cdot, s) = \overline{u}(\cdot)$, and consider a (parameterized) curve $(x, u, y)(\cdot, s) : \mathbb{R} \to \mathcal{M} \times \mathcal{W} \in \Sigma$ such that $(x, u, y)(\cdot, 0) = (x_1, \overline{u}, y_1)(\cdot)$ and $(x, u, y)(\cdot, 1) = (x_2, \overline{u}, y_2)(\cdot)$. We assume that $(x, u, y)(\cdot, \cdot) \in C^2$.

For each $s \in [0, 1]$, define

$$\begin{aligned}(x(\cdot, s), \delta x(\cdot, s)) &:= (x(\cdot, s), Dx(\cdot, s)[0, 1]) \\ (u(\cdot, s), \delta u(\cdot, s)) &:= (u(\cdot, s), Du(\cdot, s)[0, 1]) = (\overline{u}(\cdot), 0) \\ (y(\cdot, s), \delta y(\cdot, s)) &:= (y(\cdot, s), Dy(\cdot, s)[0, 1]).\end{aligned}$$

$$(A.4)$$

Repeating the argument of Remark 1, one can show that $(x(\cdot, s), \delta x(\cdot, s))$ is a solution to (2) from the initial condition $(x(0, s), \delta x(0, s))$ under the action of the input $(\overline{u}(\cdot), 0)$. Thus, the storage function S satisfies

$$\frac{d}{dt}S(x(t, s), \delta x(t, s)) \leq -\langle \delta y(t, s), \delta y(t, s)\rangle_{\mathbb{W}_{x(t,s)}}. \quad (A.5)$$

By boundedness of $(x_1, \overline{u}, y_1)(\cdot), (x_2, \overline{u}, y_2)(\cdot)$, define a compact set $X \subset \mathcal{M}$ such that $x(t, s) \in X$ for each $t \geq 0$ and each $s \in [0, 1]$. X depends on the range of $\overline{u}(\cdot)$, and on the range of parameterization of the curve $x(0, \cdot)$. The compactness of X guarantees the existence of a smooth $(0, 2)$-tensor field $\mathbb{W}$ such that

$$\inf_{x \in X, u \in U} \langle \delta y, \delta y\rangle_{\mathbb{W}_x} \geq \langle \delta y, \delta y\rangle_{\mathbb{W}} \qquad (A.6)$$

for each $x \in X$. Then,

$$\frac{d}{dt}S(x(t, s), \delta x(t, s)) \leq -\langle \delta y(t, s), \delta y(t, s)\rangle_{\mathbb{W}}, \qquad (A.7)$$

that is,

$$S(x(0,s), \delta x(0,s)) \geq \int_0^t \langle \delta y(\tau, s), \delta y(\tau, s) \rangle_{\mathbb{W}} d\tau \ . \quad \text{(A.8)}$$

By Barbalat's lemma, for each $s \in [0, 1]$,

$$\lim_{t \to \infty} \langle \delta y(t, s), \delta y(t, s) \rangle_{\mathbb{W}} = 0. \quad \text{(A.9)}$$

The applicability of Barbalat's lemma follows from the fact that $\delta y(\cdot, s)$ is uniformly continuous for each $s \in [0, 1]$. In fact, the range of $(y(\cdot, s), \delta y(\cdot, s))$ is bounded for each $s \in [0, 1]$. This is a consequence of the fact that each curve to Σ is bounded and that $\dot{S} \leq 0$. Therefore, for each $s \in [0, 1]$, $(y(\cdot, s), \delta y(\cdot, s))$ belongs to a compact subset of $T\mathcal{Y}$ that depends on the initial condition $(x(0, s), \delta x(0, s))$ and on the input $\overline{u}(\cdot)$. Thus, for each $s \in [0, 1]$, $(y(\cdot, s), \delta y(\cdot, s))$ is uniformly continuous, since it is a C^1 function on a compact set.

Consider now the distance d on $\mathcal{Y}$ induced by the Riemannian structure $\langle \delta y, \delta y \rangle_{\mathbb{W}}^{\frac{1}{2}}$. Then, (25) follows from (A.9) and the fact that

$$d(y_1(t), y_2(t)) \leq \int_0^1 \langle \delta y(t, s), \delta y(t, s) \rangle_{\mathbb{W}}^{\frac{1}{2}} ds \ . \quad \text{(A.10)}$$

$\square$

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