A New Discontinuous Galerkin Formulation for Non-Linear Elasticity

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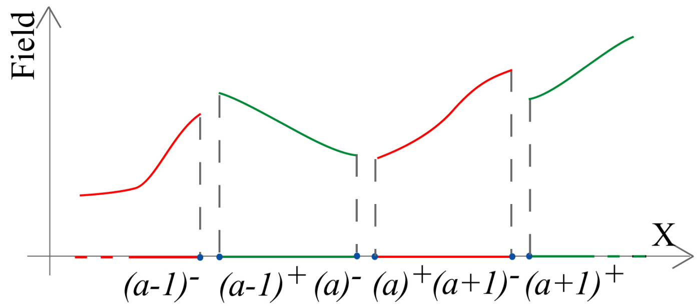
7th World Congress on Computational Mechanics
Los Angeles, California, USA, July 16-22, 2006

Introduction

- Discontinuous Galerkin methods:
  - Finite-element discretization allowing for interelement discontinuities



  - Weak enforcement of compatibility equations and continuity (C^0 or C^1, …) through interelement integrals called numerical fluxes
  - Stability is ensured with quadratic interelement integrals

Introduction

- Applications of DG to fluid dynamics:
  - High order polynomial approximations combined with discontinuous Galerkin method improve the resolution accuracy of:
    - Hyperbolic diffusion equations (Cockburn-Hou-Shu 1990, Cockburn 2003, eg)
    - Elliptic convection/diffusion equations (Bassy-Rebay 1997, Cockburn-Shu 1998)
- Applications of DG to solid mechanics:
  - Allow weak enforcement of continuity with low order polynomial approximations:
    - C^1 for beams and plates (Engel et al. 2002)
    - C^0 with reduction of locking for shells (Güzey et al. 2006)
    - Strain gradient continuity (Molari et al. 2006)

Introduction

- Motivation of the work:
  - Complex mechanics phenomena:
    - Non-local response
    - Fracture
    - Shocks
    - Strains localization
- Objectives:
  - To develop a DG framework for non-linear mechanics:
    - Large plastic deformations
    - Scalable explicit dynamics

Discontinuous Galerkin formulation

- Hu-Washizu-de Veubeke functional I_h:

$$I_h(\varphi_h, \mathbf{F}_h, \mathbf{P}_h) = \int_0^{t_f} \Big\{ \int_{B_{0h}} \Big[-\frac{\rho_0}{2}\dot{\varphi}_h^2 + W(\mathbf{C_h}) + \mathbf{P}_h : (\nabla_0 \varphi_h - \mathbf{F}_h) - \rho_0 \mathbf{B}_0 \cdot \varphi_h \Big] dV$$

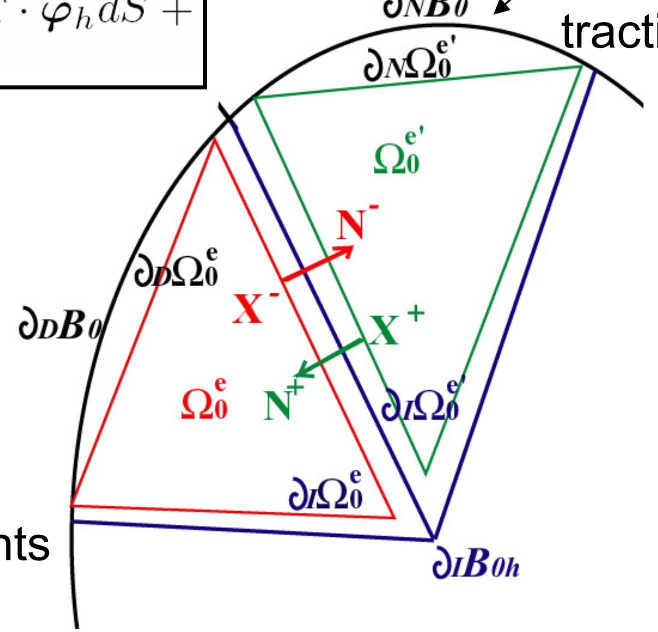
$$- \int_{\partial_D B_{0h}} (\varphi_h - \bar{\varphi}_h) \cdot \mathbf{P}_h \cdot \mathbf{N} dS - \int_{\partial_N B_{0h}} \bar{\mathbf{T}} \cdot \varphi_h dS +$$

$$\int_{\partial_I B_{0h}} [\![\varphi_h]\!] \cdot \langle \mathbf{P}_h \rangle \cdot \mathbf{N}^- dS \Big\} dt$$

Jump

Mean

Constrained tractions



Constrained displacements

Lew et al 2003,
Noels-Radovitzky IJNME 2006,
Ten Eyck-Lew IJNME 2006

Discontinuous Galerkin formulation

- Minimization of the functional I_h:
  - With respect to φ_h, leading to equilibrium equations:

$$0 = \int_0^{t_f} \{ \int_{B_{0h}} \rho_0 \ddot{\varphi}_h \cdot \delta\varphi_h + \mathbf{P}_h : \nabla_0 \delta\varphi_h dV - \int_{B_{0h}} \rho_0 \mathbf{B} \cdot \delta\varphi_h dV - \int_{\partial_N B_{0h}} \bar{\mathbf{T}} \cdot \delta\varphi dS + \int_{\partial_I B_{0h}} [\![\delta\varphi_h]\!] \cdot \langle \mathbf{P}_h \rangle \cdot \mathbf{N}^- dS \} dt$$

  - With respect to $\mathbf{F}_h$, leading to material behavior:

$$0 = \int_0^{t_f} \{ \int_{B_{0h}} \left[2\mathbf{F}_h \frac{\partial W(\mathbf{C}_h)}{\partial \mathbf{C}} - \mathbf{P}_h \right] : \delta\mathbf{F}_h dV \} dt$$

  - With respect to $\mathbf{P}_h$, leading to compatibility equations:

$$0 = \int_0^{t_f} \{ \int_{B_{0h}} \delta\mathbf{P}_h : [\nabla_0 \varphi_h - \mathbf{F}_h] dV - \int_{\partial_D B_{0h}} [\varphi_h - \bar{\varphi}_h] \cdot \delta\mathbf{P}_h \cdot \mathbf{N} dS + \int_{\partial_I B_{0h}} [\![\varphi_h]\!] \cdot \langle \delta\mathbf{P}_h \rangle \cdot \mathbf{N}^- dS \} dt$$

Expressions of interface terms arise naturally

Discontinuous Galerkin formulation

- Reduction to a one-field method:
  - Definition of the lifting operator:

$$\int_{B_{0h}} \mathbf{r}_{\mathrm{s}\bar{\varphi}_h}\left(\llbracket \mathbf{v} \rrbracket\right) : \boldsymbol{\tau} dV = \begin{cases} \int_{\mathrm{s}} \llbracket \mathbf{v} \rrbracket \cdot \langle \boldsymbol{\tau} \rangle \cdot \mathbf{N}^{-} dS & \forall \boldsymbol{\tau} \text{ and } \forall \mathrm{s} \in \partial_I B_{0h} \\ \int_{\mathrm{s}} \left[\bar{\varphi}_h \cdot \boldsymbol{\tau} \cdot \mathbf{N} - \mathbf{v} \cdot \boldsymbol{\tau} \cdot \mathbf{N} \right] dS & \forall \boldsymbol{\tau} \text{ and } \forall \mathrm{s} \in \partial_D B_{0h} \\ 0 & \forall \boldsymbol{\tau} \text{ and } \forall \mathrm{s} \in \partial_N B_{0h} \end{cases}$$

  - Compatibility equations rewritten with a stabilization parameter β:

$$\mathbf{F}_h = \boldsymbol{\nabla}_0 \boldsymbol{\varphi}_h + \sum_{\mathrm{s}}^{N_{\mathrm{s}}} \mathbf{r}_{\mathrm{s}\bar{\varphi}_h}\left(\llbracket \boldsymbol{\varphi}_h \rrbracket\right) \quad \text{and} \quad \mathbf{C}_h = \mathbf{F}_h^T \mathbf{F}_h \quad \text{in} \quad \Omega_0^e$$

$$\mathbf{F}_{\mathrm{s}} = \boldsymbol{\nabla}_0 \boldsymbol{\varphi}_h + \beta \mathbf{r}_{\mathrm{s}\bar{\varphi}_h}\left(\llbracket \boldsymbol{\varphi}_h \rrbracket\right) \quad \text{and} \quad \mathbf{C}_{\mathrm{s}} = \mathbf{F}_{\mathrm{s}}^T \mathbf{F}_{\mathrm{s}} \quad \text{on } \mathrm{s} \in \partial \Omega_0^e$$

  - Displacement-based weak form:

$$\int_{B_{0h}} \rho_0 \ddot{\boldsymbol{\varphi}}_h \cdot \delta \boldsymbol{\varphi}_h + \mathbf{P}\left(\mathbf{F}_h\right) : \boldsymbol{\nabla}_0 \delta \boldsymbol{\varphi}_h dV - \int_{B_{0h}} \rho_0 \mathbf{B} \cdot \delta \boldsymbol{\varphi}_h dV - \int_{\partial_N B_{0h}} \bar{\mathbf{T}} \cdot \delta \boldsymbol{\varphi}_h dS + \sum_{\mathrm{s}} \int_{\mathrm{s} \in \partial_I B_{0h}} \llbracket \delta \boldsymbol{\varphi}_h \rrbracket \cdot \langle \mathbf{P}\left(\mathbf{F}_s\right) \rangle \cdot \mathbf{N}^{-} dS = 0$$

Implementation as interface element

- Linearization with respect to the jump:

$$\mathbf{P}(\mathbf{F}_h) \simeq \mathbf{P}(\nabla_0 \varphi_h) + \mathbb{C}(\nabla_0 \varphi_h) : (\mathbf{F}_h - \nabla_0 \varphi_h)$$

$$\bar{\mathbf{P}} = \mathbf{P}(\nabla_0 \varphi_h) \text{ and } \mathbb{C} = \frac{\partial \mathbf{P}}{\partial \mathbf{F}^{\text{el}}}$$

- New weak formulation:

$$\begin{aligned} 0 \ = \ & \int_{B_{0h}} \rho_0 \ddot{\varphi}_h \cdot \delta\varphi_h + \bar{\mathbf{P}} : \nabla_0 \delta\varphi_h dV - \int_{B_{0h}} \rho_0 \mathbf{B} \cdot \delta\varphi_h dV - \int_{\partial_N B_{0h}} \bar{\mathbf{T}} \cdot \delta\varphi_h dS + \\ & \int_{\partial_I B_{0h}} [\![\varphi_h]\!] \cdot \langle \mathbb{C} : \nabla_0 \delta\varphi_h \rangle \cdot \mathbf{N}^- dS + \int_{\partial_I B_{0h}} [\![\delta\varphi_h]\!] \cdot \langle \bar{\mathbf{P}} \rangle \cdot \mathbf{N}^- dS + \\ & \sum_{s} \int_{s \in \partial_I B_{0h}} [\![\delta\varphi_h]\!] \otimes \mathbf{N}^- \cdot \left\langle \frac{\beta}{h_s} \mathbb{C} \right\rangle \cdot [\![\varphi_h]\!] \otimes \mathbf{N}^- dS \end{aligned}$$

  - All the discontinuity influences are isolated in the interelement integrals
  - Stress tensors are computed from the compatible deformation gradients
  - Perfectly-plastic behavior still preserves stability

Implementation as interface element

- Resulting finite element discretization:
  - Volume forces:

$$\mathbf{f}_{\mathrm{inert}\,a} \cdot \delta \mathbf{x}_a = \int_{B_{0h}} \rho_0 \ddot{\boldsymbol{\varphi}}_h \cdot \delta \boldsymbol{\varphi}_h dV$$

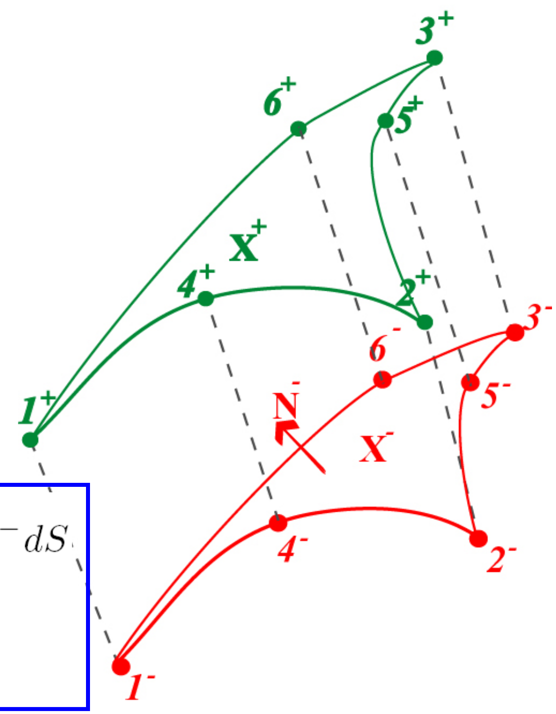
$$\mathbf{f}_{\mathrm{int}\,a} \cdot \delta \mathbf{x}_a = \int_{B_{0h}} \bar{\mathbf{P}} : \boldsymbol{\nabla}_0 \delta \boldsymbol{\varphi}_h dV$$

  - External forces

$$\mathbf{f}_{\mathrm{ext}\,a} \cdot \delta \mathbf{x}_a = \int_{B_{0h}} \rho_0 \mathbf{B} \cdot \delta \boldsymbol{\varphi}_h dV + \int_{\partial_N B_{0h}} \bar{\mathbf{T}} \cdot \delta \boldsymbol{\varphi}_h dS$$

  - Interface forces:

$$\begin{aligned}\mathbf{f}_{\mathrm{i}\,a} \cdot \delta \mathbf{x}_a \;=\; & \int_{\partial_I B_{0h}} [\![\boldsymbol{\varphi}_h]\!] \cdot \langle \mathbb{C} : \boldsymbol{\nabla}_0 \delta \boldsymbol{\varphi}_h \rangle \cdot \mathbf{N}^- dS + \int_{\partial_I B_{0h}} [\![\delta \boldsymbol{\varphi}_h]\!] \cdot \langle \bar{\mathbf{P}} \rangle \cdot \mathbf{N}^- dS \\ & + \sum_{\mathrm{s}} \int_{\mathrm{s} \in \partial_I B_{0h}} [\![\delta \boldsymbol{\varphi}_h]\!] \otimes \mathbf{N}^- \cdot \left\langle \frac{\beta}{h_{\mathrm{s}}} \mathbb{C} \right\rangle \cdot [\![\boldsymbol{\varphi}_h]\!] \otimes \mathbf{N}^- dS\end{aligned}$$



Static analysis

- Numerical properties [Noels-Radovitzky, IJNME 2006]:
  - Consistency in the non-linear range:
$$\begin{aligned} \nabla_0 \cdot \mathbf{P} + \rho_0 \mathbf{B} = & \quad \rho_0 \ddot{\varphi}_h \quad \text{in } B_{0h} \\ \mathbf{P} \cdot \mathbf{N} = & \quad \bar{\mathbf{T}} \quad \text{on } \partial_N B_{0h} \end{aligned}$$
  - Linearized equations: $a\left(\varphi_h - \mathbf{X}, \delta\varphi_h\right) = b\left(\delta\varphi_h\right)$

$$\begin{aligned} a\left(\mathbf{u}_h, \delta\varphi_h\right) &= \int_{B_h} \nabla \mathbf{u}_h : \mathcal{H} : \nabla \delta\varphi_h dV + \int_{\partial_I B_h} \mathbf{n}^- \cdot \langle \mathcal{H} : \nabla \mathbf{u}_h \rangle \cdot [\![\delta\varphi_h]\!] \, dS + \\ & \int_{\partial_I B_h} [\![\delta\varphi_h]\!] \otimes \mathbf{n}^- : \left\langle \frac{\beta}{h_s} \mathcal{H} \right\rangle : [\![\mathbf{u}_h]\!] \otimes \mathbf{n}^- dS \\ b\left(\delta\varphi_h\right) &= \int_{B_h} \rho \delta\varphi_h \cdot \mathbf{B} dV + \int_{\partial_N B_h} \bar{\mathbf{T}} \cdot \delta\varphi_h dS \end{aligned}$$

  - Stability: $b(\varphi_h - \mathbf{X}) = a\left(\varphi_h - \mathbf{X}, \varphi_h - \mathbf{X}\right) \quad \geq \quad C\left(\beta\right) |||\varphi_h - \mathbf{X}|||^2$

with $C > 0$ if $\beta > C^k$

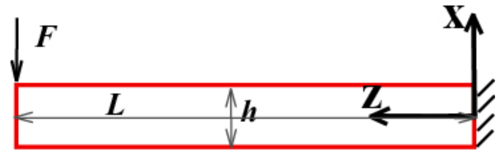
$$|||\mathbf{v}|||^2 \quad = \quad \sum_e \left\| \sqrt{\mathcal{H}} : \nabla \mathbf{v} \right\|^2_{\mathrm{L}^2(\Omega^e)} + \sum_{s \in \partial_I B_h} \left\| h_s^{-\frac{1}{2}} \sqrt{\mathcal{H}} : [\![\mathbf{v}]\!] \otimes \mathbf{n}^- \right\|^2_{\mathrm{L}^2(s)}$$ Energy norm

  - Convergence rate: $|||\varphi_h - \varphi^k||| \quad \leq \quad C h^k_{\max} \left\| \varphi - \mathbf{X} \right\|_{\mathrm{H}^{k+1}(B_h)}$ with $C > 0$

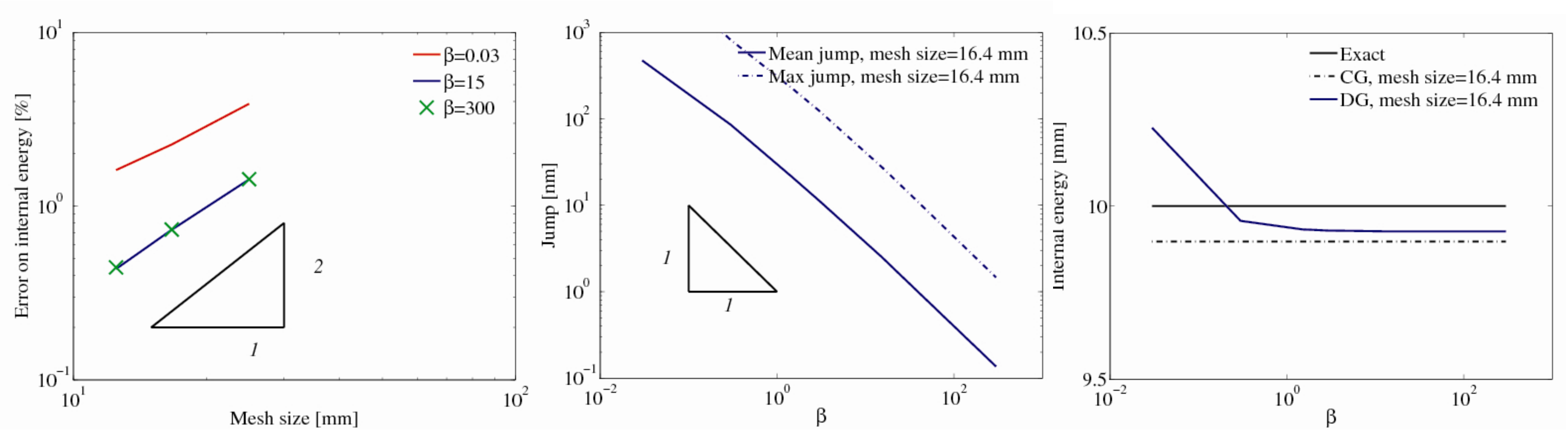
$\mathbf{e}^k = \varphi_h - \varphi^k$ Interpolation of the exact solution

Static analysis

- Cantilever beam in small deformations:

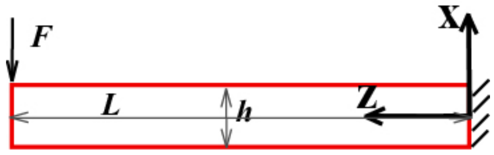


Length	1m
Thickness	0.1m
Young modulus	200 GPa
Poisson ratio	0.3
Tip force	10 kPa

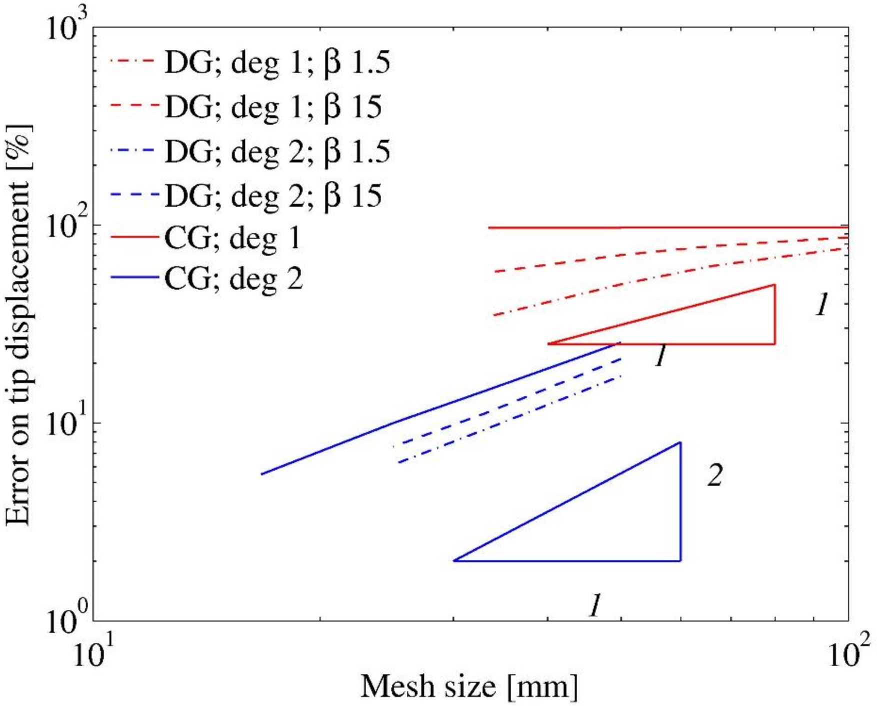


Static analysis

- Nearly incompressible cantilever beam in small deformations:



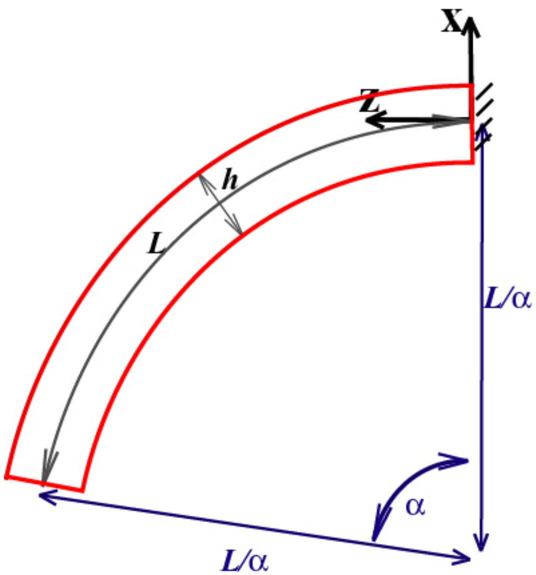
Length	1m
Thickness	0.1m
Young modulus	200 GPa
Poisson ratio	0.499
Tip force	10 kPa



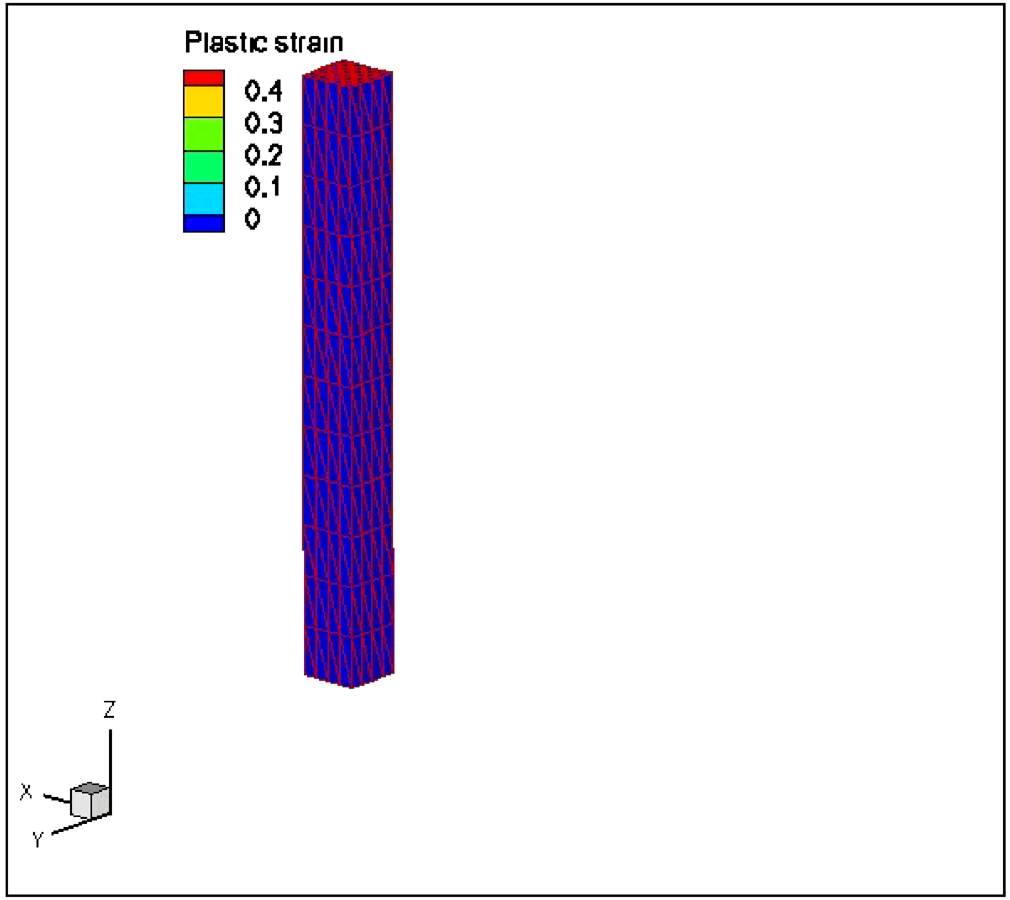
Static analysis

- Formation of a ring:

Dg with β=300



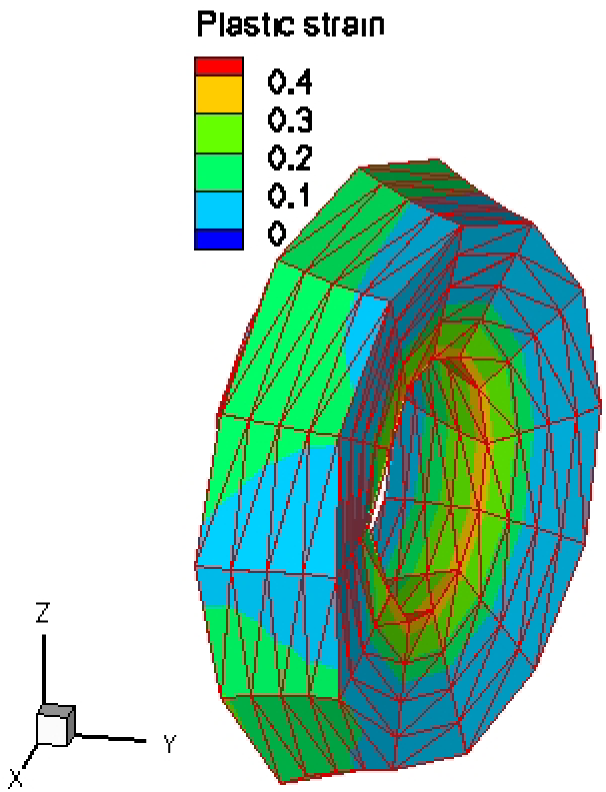
Length	1m
Thickness	0.1m
Young modulus	117 Pa
Poisson ratio	0.46
Yield stress	15 Pa



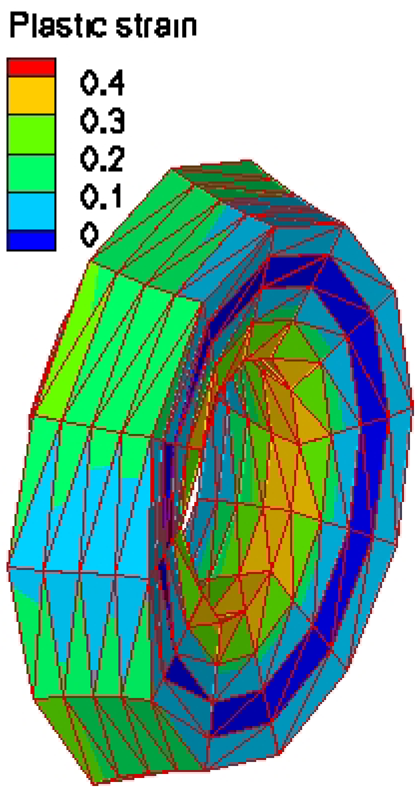
Static analysis

- Formation of a ring:

Continuous Galerkin



Discontinuous Galerkin (β = 300)

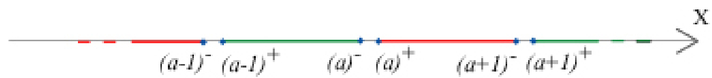


Explicit time integration

- Central difference scheme:

$$
\begin{aligned}
\mathbf{x}_a^{n+1} &= \mathbf{x}_a^{n} + \Delta t\, \dot{\mathbf{x}}_a^{n+\frac{1}{2}} \\
\dot{\mathbf{x}}_a^{n+\frac{1}{2}} &= \dot{\mathbf{x}}_a^{n-\frac{1}{2}} + \Delta t\, \ddot{\mathbf{x}}_a^{n} \\
\ddot{\mathbf{x}}_a^{n+1} &= \mathbf{M}_{ab}^{-1} \left[\mathbf{f}_{\text{ext}} - \mathbf{f}_{\text{int}} - \mathbf{f}_{\text{i}}\right]_b^{n+1}
\end{aligned}
$$

- Conditional stability: $\Delta t \leq \min_j \dfrac{2}{\sqrt{\lambda_j}}$ with $\lambda_j \mathbf{M}\Phi_j = \mathbf{K}\Phi_j \quad \forall j \in [1, N_{\text{dofs}}]$
  - Linearized system:



$$
\mathbf{M}^{-1}\mathbf{K} = \frac{E}{\rho l^2}
\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & [1+2\beta] & -1 & 0 & 0 & 0 & 0 & 0 & \cdot \\
\cdot & -1 & [1+2\beta] & [1-2\beta] & -1 & 0 & 0 & 0 & \cdot \\
\cdot & -1 & [1-2\beta] & [1+2\beta] & -1 & 0 & 0 & 0 & \cdot \\
\cdot & 0 & 0 & -1 & [1+2\beta] & [1-2\beta] & -1 & 0 & \cdot \\
\cdot & 0 & 0 & -1 & [1-2\beta] & [1+2\beta] & -1 & 0 & \cdot \\
\cdot & 0 & 0 & 0 & 0 & -1 & [1+2\beta] & [1-2\beta] & \cdot \\
\cdot & 0 & 0 & 0 & 0 & -1 & [1-2\beta] & [1+2\beta] & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}
\begin{matrix}
\\ (a-2)^+ \\ (a-1)^- \\ (a-1)^+ \\ (a)^- \\ (a)^+ \\ (a+1)^- \\ (a+1)^+ \\ \\
\end{matrix}
$$

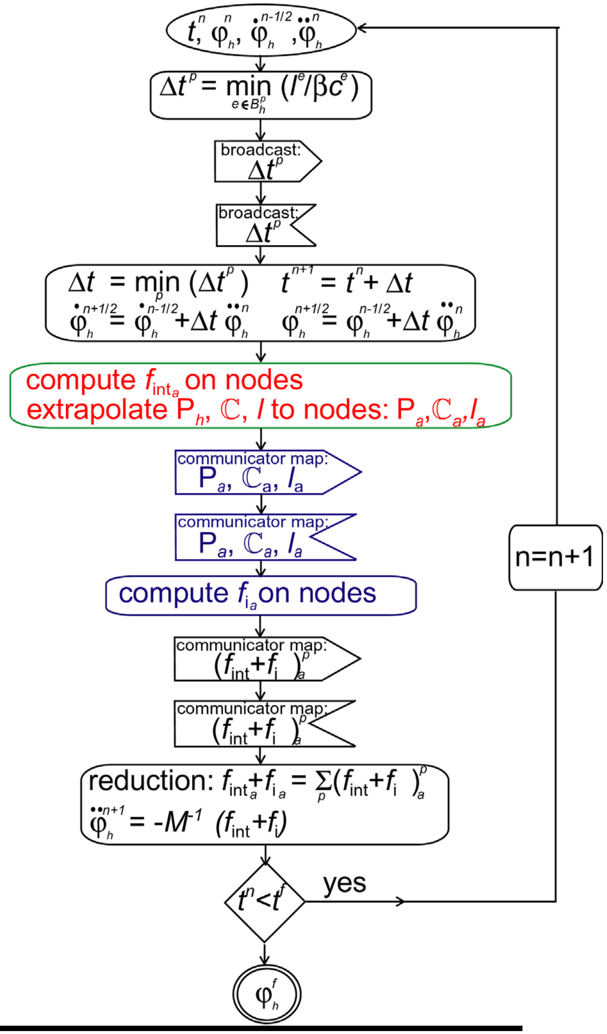
If $h_s = l$

  - New criterion:

$$
\boxed{\Delta t_{\text{crit}} = \frac{2}{\sqrt{\lambda_{\max}}} = \frac{l}{\sqrt{\beta}}\sqrt{\frac{\rho}{E}} = \frac{l}{\sqrt{\beta}c}}
$$

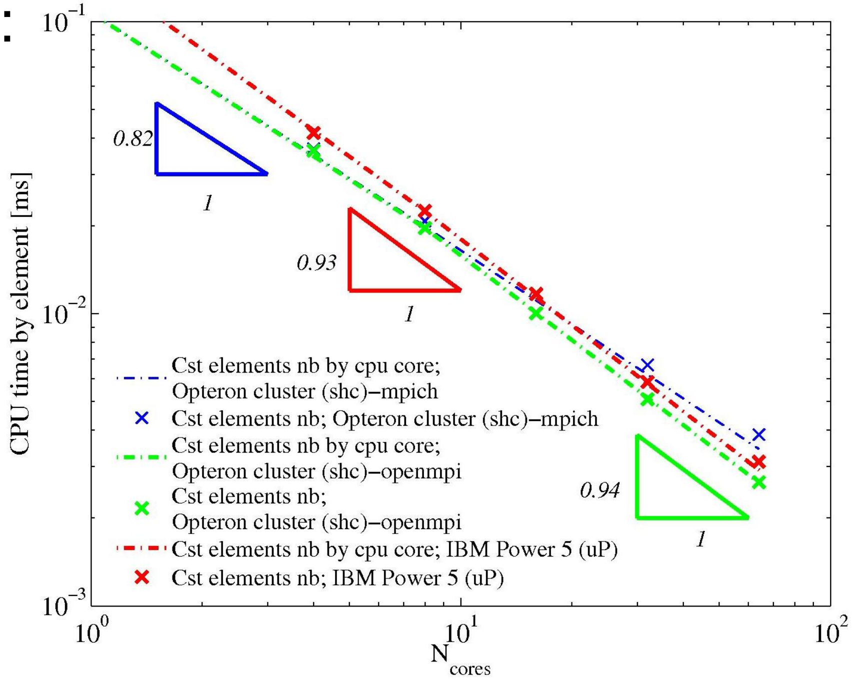
Explicit time integration

- Parallelization of the code:
  - Time step evaluations:
  - Predictions:
  - Computation of volume forces:
  - Exchange between processors of the stress fields:
  - Computation of interface forces:
  - Reduction of residual vectors:
  - Corrections:



Scaling test

- Specific problem considered:
  - Uniform tensile beam, discretized into 9984 quadratic tetrahedra
  - Neo-Hookean material model extended to compressible range
  - Tests performed:
    - Fixed element count per processor:
      - –Divide tetrahedra and conduct simulation
    - Fix total element count:
      - –Conduct simulation on range of cpu counts



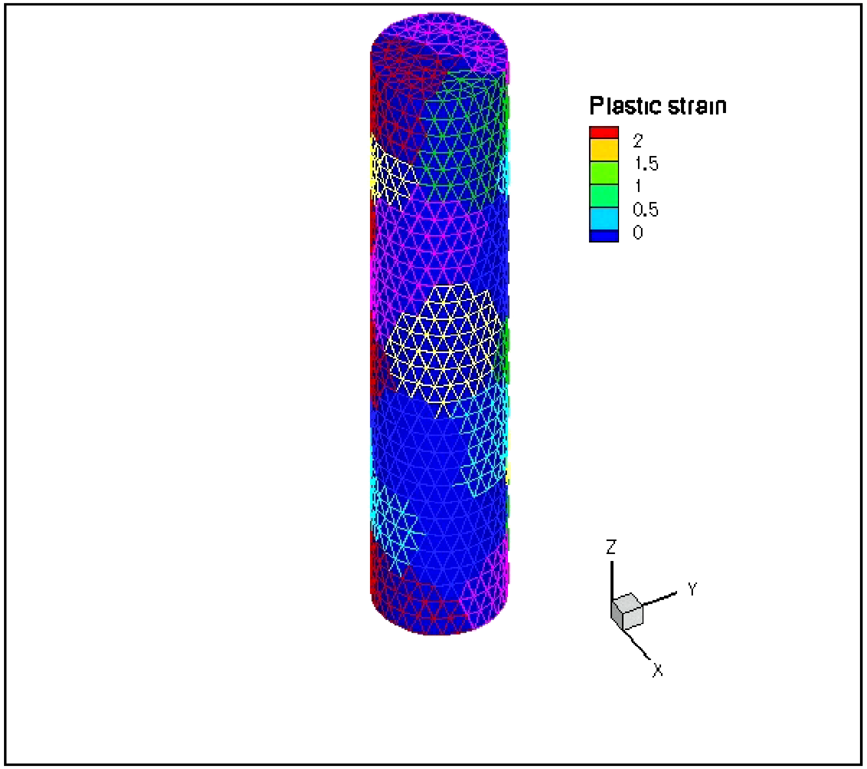
Test performed by Sharon Brunett, Caltech

Numerical dynamic examples

- Taylor's impact:

DG with β=4

Length	32.4 mm
Radius	3.2 mm
Density	8930 kg/m^3
Young modulus	117 GPa
Poisson ratio	0.35
Yield stress	400 MPa
Hardening	100 MPa
Velocity	227 m/s

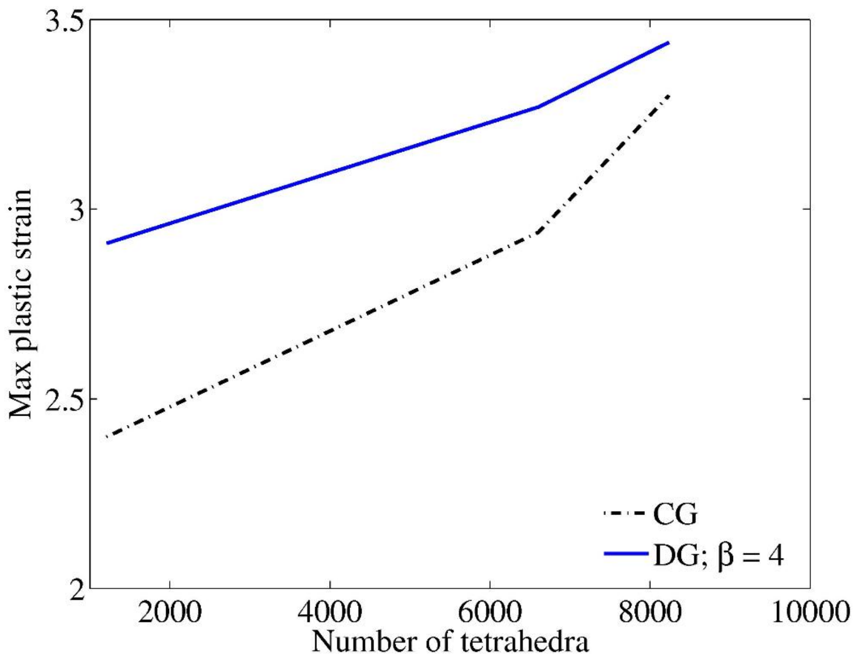


Numerical dynamic examples

- Taylor's impact:

Method	Final length [mm]	Final radius [mm]	Max ε^{pl}	Max jump [μm]
CG, coarse	21.47	6.79	2.4	-
DG, coarse , β=4	21.46	6.81	2.91	4.1
DG, coarse, β=16	21.45	6.80	2.89	1.2
DG, coarse, β=100	21.44	6.79	2.84	0.2
CG, intermediate	21.46	7.13	2.94	-
DG, intermediate, β=4	21.46	7.14	3.27	2.6
CG, fine	21.46	7.13	3.30	-
DG, fine, β=4	21.46	7.14	3.44	-

Coarse mesh of 1211 elements
Intermediate mesh of 6610 elements
Fine mesh of 8238 elements



Numerical dynamic examples

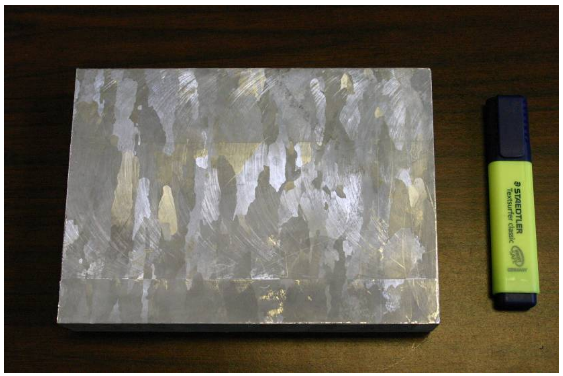
- Oligo crystal plasticity:

Oligo crystal sample preparation: MIT + Alcoa.

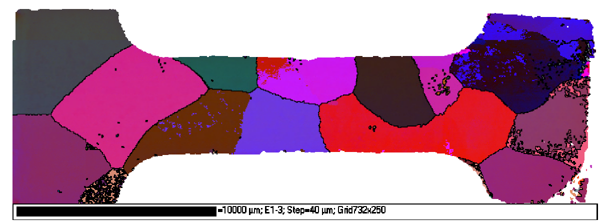
Sample cutting and polishing and EBSD: Caltech + MIT (Z. Zhao).

Tensile test + DIC: Rutgers (S. Kuchnicki, A. Cuitino) + MIT.

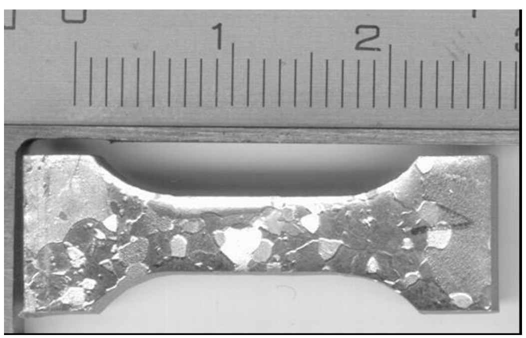
Theoretical polycrystal model: Rutgers + MIT.



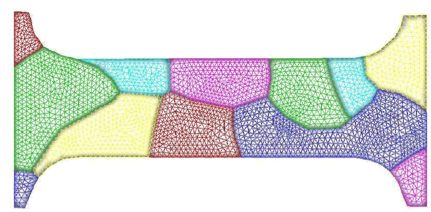
Aluminum oligo crystal sample



Grain profile from EBSD measurement



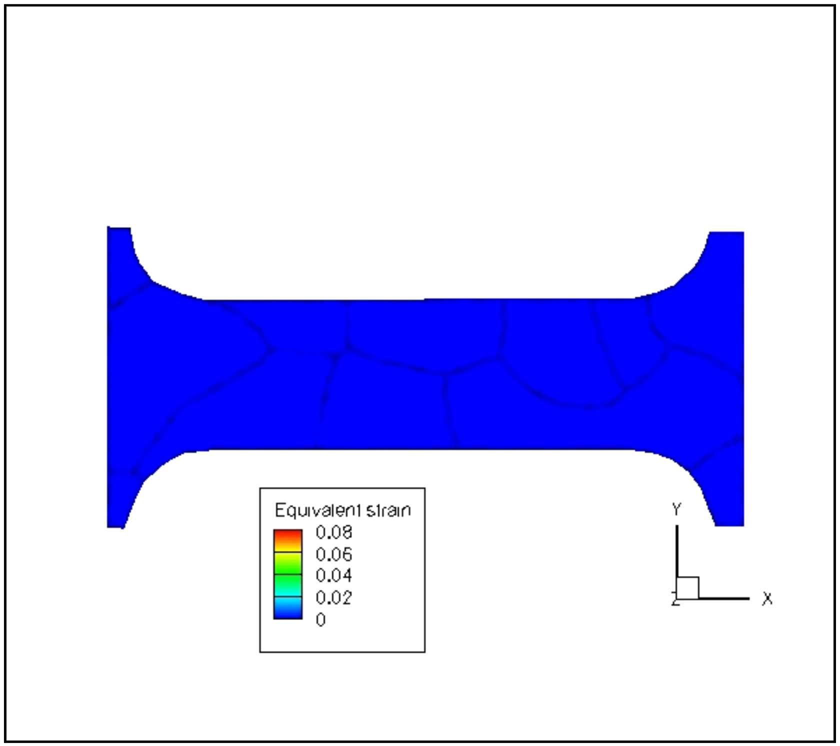
Dogbone tensile test sample

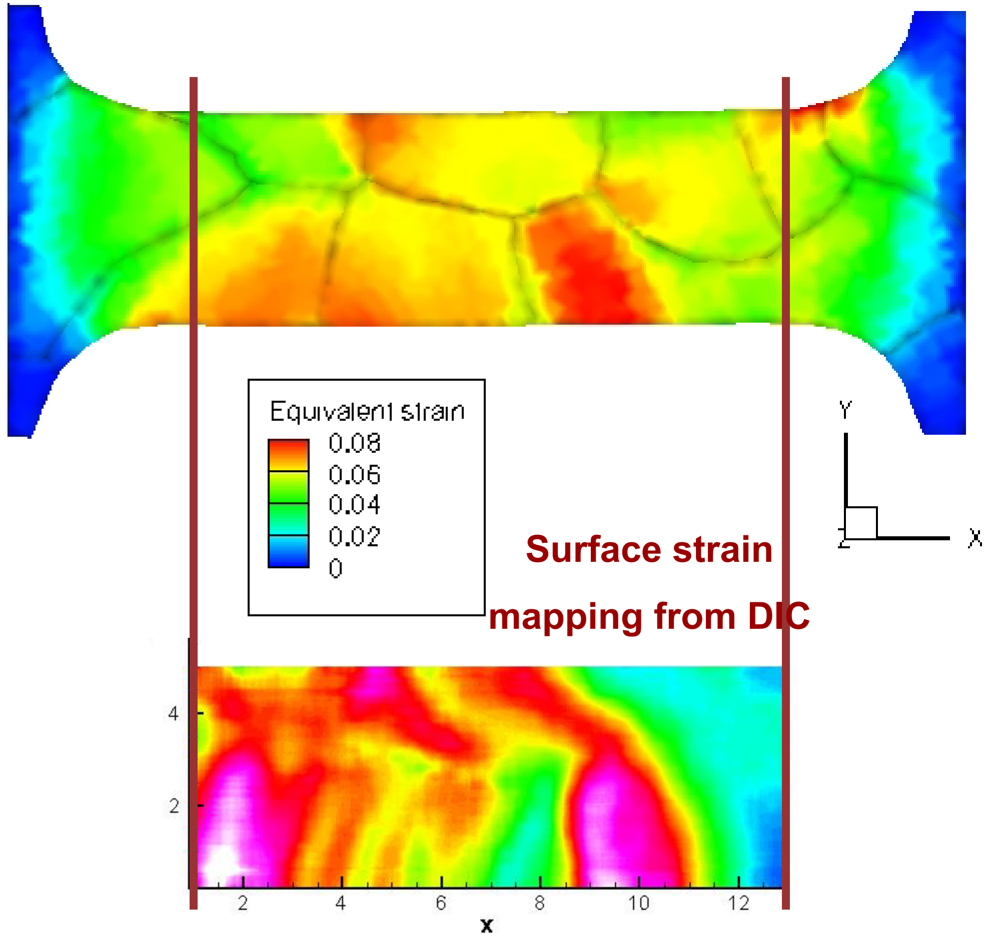


Mesh setup according to orientation mapping

Numerical dynamic examples

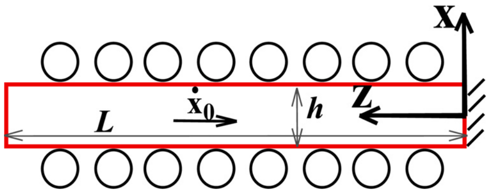
- Oligo crystal plasticity:





Numerical dynamic examples

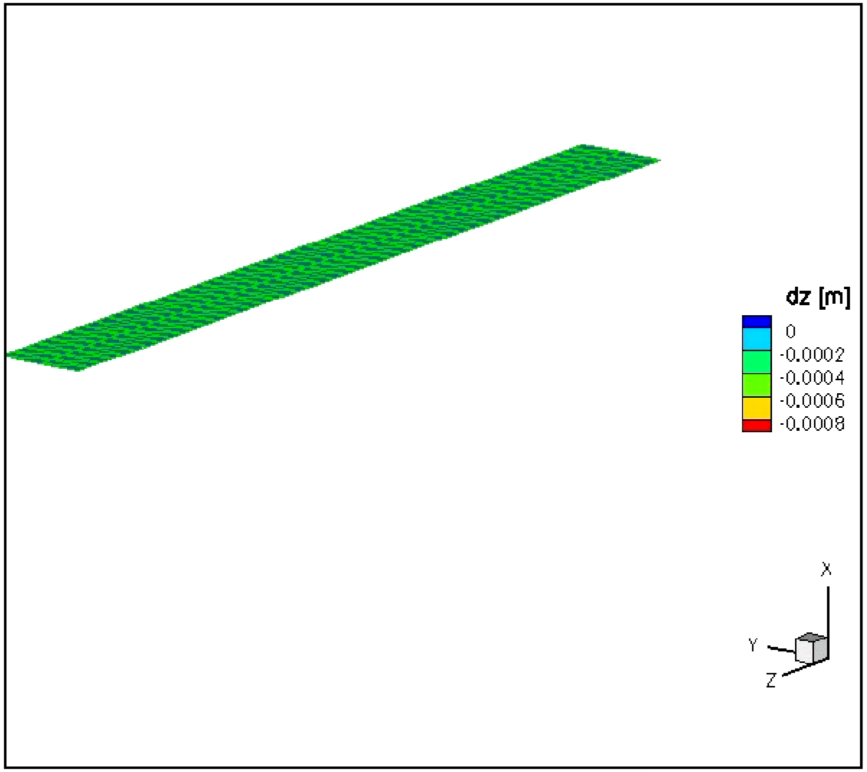
- 1D wave propagation in a beam:



Length	1m
Thickness	0.1m
Density	10000kg/m^3
Young modulus	10GPa
Poisson ratio	0
Velocity	1m/s

Displacement in the center slice (**x**=0);

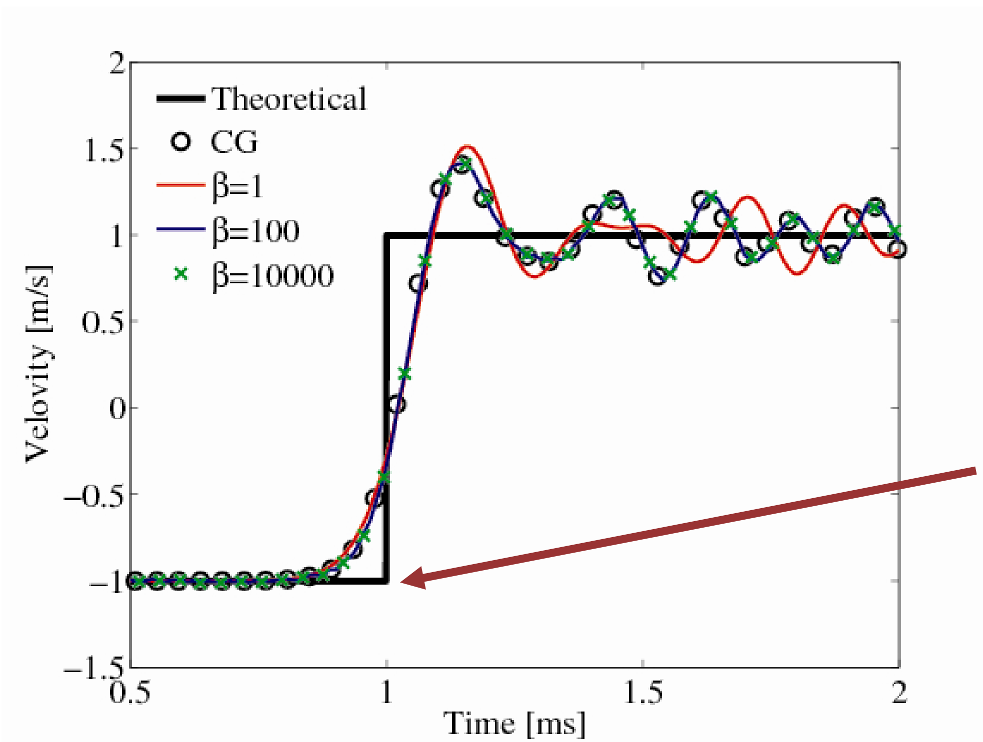
DG with β=100

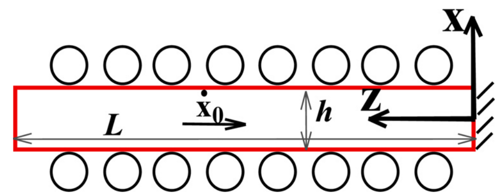


Numerical dynamic examples

- 1D wave propagation in a beam:

Time evolution of the free face velocity





The wave reaches the free face

Conclusions

- Development of a discontinuous Galerkin framework:
  - In the non-linear range
  - Allowing for complex material models (plasticity …)
- Explicit time integration:
  - New stability criterion on the time step size is dependent on the stabilization parameter
  - Efficient and easy parallelization of the code
- Stability, robustness and accuracy:
  - Demonstrated on numerical examples