Paper 154

Proceedings of the Eighth International Conference on Computational Structures Technology, B.H.V. Topping, G. Montero and R. Montenegro, (Editors), Civil-Comp Press, Stirlingshire, Scotland.

Dual Analysis for Finite Element Solutions of Plate Bending

J.F. Debongnie†, N.X. Hung‡ and N.H. Cung*
† LTAS, University of Liège, Belgium
‡ University of Natural Sciences, VNU-HCM, Vietnam
*** University of Transport, HCM, Vietnam**

Abstract

This paper presents the application of the dual analysis concept to plate bending. In this method, a same problem is analyzed parallely by a displacement and an equilibrium model. The energetic distance between these two models is the sum of both global errors and consequently, an upper bound of each of them. After an exposition of the two models, numerical examples are presented, which illustrate the high obtainable accuracy of the method.

Keywords: plate bending, equilibrium element, conforming element, dual analysis.

1 Introduction

Today, most of engineering analyses deal with problems involving differential equations which are too difficult to be solved analytically. Currently, the most widely used method of solving these problems, especially those with irregular geometries and complex boundary conditions, is the Finite Element Method (FEM), which over time has become an indispensable tool for engineers.

The best known family of finite element models is the displacement model, in which the compatibility equations are a priori verified, from the fact that the variables are displacements. The solution of the problem then leads to weak forms of the equilibrium equations. But there exists a dual family of finite elements, which are called *equilibrium elements*. In these elements, an equilibrated stress field is used, and the result of the computation consists in weak compatibility equations.

A same problem may be treated by both methods. An interesting fact is that the comparison of the results obtained by these two approaches leads to a useful error measure. This is the so-called *dual analysis*. Its initial form, which was developed by Fraeijs de Veubeke in the 60's [1], is based on the fact that under some restrictive conditions on the data (homogeneous prescribed displacements or zero load), the two models give an upper and a lower bound of the energy, respectively. The difference between these energies is an energetic measure of the sum of the errors of both models. The abovementioned restrictions, the lack of equilibrium models in most FEM codes and the fact that at this time, performing two analyses of the same problem was considered as too time-consuming were at that time obstacles to a wide use of the method.

The huge improvement of computers in the recent times significantly modified the situation. By now, a double analysis is no more an obstacle. Moreover, newer investigations [5, 7] reformulated the dual analysis in a more general way, where energy bounds do no more play the central role. The result of this reconsideration is that in fact, dual error bounds do exist for all cases, not for the strain energy, but for the total complementary energy. This all gives to the dual analysis a renewed attractiveness.

This paper is devoted to the application of the dual analysis to plate problems. After a presentation of the problem and a recall of the dual analysis principles, an equilibrium element is presented, which allows the appliance of a pressure field. Finally, numerical examples illustrate the estimation of the error by *dual analysis* as a comparison of total energies.

2 General notations and the spaces of admissible fields

In what follows, the plate will be described as a plane domain Ω with a Lipschitz-continuous boundary Γ. A pressure p is acting within the domain. Boundary Γ is split in two parts, namely Γ_1 where kinematical conditions (u and u_n) are prescribed, and Γ_2 where loads are applied.

Using the notation σ_{ij} for the moment field, one can write the internal equilibrium equations in the following form,

$$D_{ij}\sigma_{ij} + p = 0 \qquad \text{in } \Omega \tag{1}$$

where p is the transverse load and $D_{ij} = \dfrac{\partial^2}{\partial x_i \partial x_j} = \partial_{ij}$ are the partial derivatives of second order.

The compatibility equations consist to say that the curvatures κ_{ij} derive from a transverse displacement field u ,

$$\kappa_{ij} = D_{ij}u \tag{2}$$

Finally, the constitutive equations may be written as

$$\sigma_{ij} = H_{ijkl}\kappa_{kl} \tag{3}$$

where H represents the plate Hooke matrix.

The following boundary conditions will be assumed, as they are sufficiently representative, though not the most general ones,

$$\begin{aligned} &on\ \Gamma_1,\ u=\bar{u},\ u_n=\bar{u}_n \\ &on\ \Gamma_2, \sigma_n=\bar{\sigma}_n,\ K_n=\bar{K}_n, Z_i=\bar{Z}_i \end{aligned} \tag{4}$$

In these expressions, u is the deflection, u_n is its normal derivative, σ_n is the normal moment, K_n is the Kirchhoff load, and the Z_i are the corner loads.

The purpose of problems in strong form is to find exact solutions satisfying all the equations from (1) to (4). As such a solution can only be obtained for academic problems, the practical way is to find an approximate solution of the weak problem as expressed by a variational principle. In the displacement formulation, it is the principle of minimal total energy. In the equilibrium formulation, it is the principle of minimal total complementary energy. Let us introduce the function spaces of kinematically and statically admissible fields.

Let two spaces of kinematically admissible displacements, denoted by V and V_0, respectively, be defined by

$$V=\{u\in H^2(\Omega), u=\bar{u}, u_n=\bar{u}_n \text{ on } \Gamma_1\} \tag{5}$$

$$V_0=\{u\in H^2(\Omega), u=u_n=0 \text{ on } \Gamma_1\} \tag{6}$$

Here, $H^2(\Omega)$ denotes the Sobolev space of order 2 [8]. Obviously, V_0 contains all differences between two elements of V, that is to say, it is the linear space of admissible displacement variations. These spaces lead to a bounded energy,

$$\int_\Omega H_{ijkl}\kappa_{ij}(u)\kappa_{kl}(u)d\Omega<\infty \tag{7}$$

From condition (7), both V and V_0 may be equipped by the energetic norm

$$\|u\|_V=\left(\int_\Omega H_{ijkl}\kappa_{ij}(u)\kappa_{kl}(u)d\Omega\right)^{1/2} \tag{8}$$

where H is a bounded uniformly positive definite matrix.

Similarly, the two spaces of statically admissible stress fields, E, and statically admissible stress variations, E_0, are defined by

$$E=\{\sigma_{ij}\in L^2(\Omega), i,j=1,2: D_{ij}\sigma_{ij}+p=0, K_n=\bar{K}_n, \sigma_n=\bar{\sigma}_n, Z_i=\bar{Z}_i\ on\,\Gamma_2\} \tag{9}$$

$$E_0=\{\sigma_{ij}\in L^2(\Omega), i,j=1,2: D_{ij}\sigma_{ij}=0, K_n=\sigma_n=Z_i=0\ on\ \Gamma_2\} \tag{10}$$

The complementary energy of their elements is bounded,

$$\int_\Omega H^{-1}_{ijkl}\sigma_{ij}\sigma_{kl}<\infty \tag{11}$$

Both E and E_0 may be equipped by the energetic norm

$$\|\sigma\|_E=\left(\int_\Omega H^{-1}_{ijkl}\sigma_{ij}\sigma_{kl}d\Omega\right)^{1/2} \tag{12}$$

which is physically equivalent to norm (7) when $\sigma_{ij}=H_{ijkl}\kappa_{kl}$.

3 Dual analysis

3.1 The displacement approach

The displacement approach consists in finding a displacement field $u \in V$ for which stresses are in equilibrium. The weak form of this condition is

$$\int_\Omega H_{ijkl}\kappa_{ij}(u)\kappa_{kl}(v)d\Omega = \int_\Omega pvd\Omega + \int_{\Gamma_2}(\overline{K}_n v - \overline{\sigma}_n v_n)d\Gamma + \sum_{i=1}^{n_C}\overline{Z}_i v_i, \forall v \in V_0 \qquad (13)$$

where $Z_i, i = 1,...,n_C$ are *corner loads*.

We here recognize a variational problem of the classical form,

Find $u \in V$ such that

$$a_V(u,v) = f_V(v), \forall v \in V_0 \qquad (14)$$

where

$$a_V(u,v) = \int_\Omega H_{ijkl}\kappa_{ij}(u)\kappa_{kl}(v)d\Omega,$$

$$f_V(v) = \int_\Omega pvd\Omega + \int_{\Gamma_2}(\overline{K}_n v - \overline{\sigma}_n v_n)d\Gamma + \sum_{i=1}^{n_C}\overline{Z}_i v_i, \forall v \in V_0$$

Problem (14) has a unique solution, from a classical inequality of Sobolev spaces. It may also be presented as the solution of the following minimization problem : Find $u \in V$ such that

$$\Pi(u) = \inf \Pi(v), \forall v \in V \qquad (15)$$

where

$$\Pi(u) = \frac{1}{2}a_V(v,v) - f_V(v)$$

Functional Π is called *total energy*.

3.2 The equilibrium approach

Here, equilibrium is supposed to hold a priori. The equilibrium method is to find an equilibrated stress field that verifies the so-called compatibility condition [9], which in weak form writes as

$$\int_\Omega H^{-1}_{ijkl}\sigma_{ij}\tau_{kl}d\Omega = \int_{\Gamma_1}(K_n(\tau)\overline{u} - \tau_n\overline{u}_n)d\Gamma + \sum_{i=1}^{n_C}Z_i(\tau)\overline{u}_i, \forall \tau \in E_0 \qquad (16)$$

This leads to the following variational problem,

Find $\sigma \in E$ such that

$$a_E(\sigma,\tau) = f_E(\tau), \forall \tau \in E_0 \qquad (17)$$

where

$$a(\sigma,\tau)=\int_\Omega H^{-1}_{ijkl}\sigma_{ij}\tau_{kl}d\Omega\,,\; f_E(\tau)=\int_{\Gamma_1}(K_n(\tau)\overline{u}-\tau_n\overline{u}_n)d\Gamma+\sum_{i=1}^{n_C}Z_i(\tau)\overline{u}_i$$

The solution of this variational problem exists and is unique. It is equivalent to solve the following minimization problem: Find $\sigma \in E$ that minimizes the total complementary energy

$$\Psi(\sigma)=\inf\Psi(\xi),\forall\xi\in E \tag{18}$$

where

$$\Psi(\xi)=\frac{1}{2}a_E(\xi,\xi)-f_E(\xi)$$

This principle is the basis of the equilibrium approach and solving it leads to compatibility equations.

3.3 The general dual analysis

Let $u_h \in V_h \subset V$ be the discrete solution of (15), that is the element that minimizes the total energy in some finite element subspace V_h of V. It is easy to prove [7] that the energetic norm of the error may be calculated by

$$\|\Delta u\|_V^2=\|u-u_h\|_V^2=2[\Pi(u_h)-\Pi(u)] \tag{19}$$

Similarly, let $\sigma_h \in E_h \subset E$ be a strictly admissible approximate stress field (discrete solution of (18)). The energetic norm of the stress error may be reckoned as

$$\|\Delta\sigma\|_E^2=\|\sigma-\sigma_h\|_E^2=2[\Psi(\sigma_h)-\Psi(\sigma)] \tag{20}$$

From some elementary properties of the exact solution [5], the following relationship between $\Pi(u)$ and $\Psi(\sigma)$ may be obtained, which is

$$\Pi(u)+\Psi(\sigma)=0 \tag{21}$$

Adding this result to relations (19) and (20) directly leads to the fundamental result of the general dual analysis concerning the errors,

$$\|\Delta u\|_V^2+\|\Delta\sigma\|_V^2=2[\Pi(u_h)+\Psi(\sigma_h)] \tag{22}$$

This error measure only requires very simple computations from the results. Moreover, if one considers the *generalized total complementary energy* defined as being the total complementary energy in equilibrium models, and minus the total energy for displacement models, one finds that displacement models converge to the exact solution from below, while equilibrium models converge to the exact solution from above [5]. The distance between the two curves measures the global added error of both models, see Figure 1.

Practically, it is preferable to work with the square root of (22) and to compare it with an evaluation of the energetic norm of the true solution, namely

$$\left[a_V(u,u)\right]^{1/2} = \left[a_E(\sigma,\sigma)\right]^{1/2} \approx \left[\frac{1}{2}a_V(u_h,u_h) + \frac{1}{2}a_E(\sigma_h,\sigma_h)\right]^{1/2} \tag{23}$$

from which follows a useful *relative error* measure

$$RE = \left\{\frac{\|\Delta u\|_V^2 + \|\Delta\sigma\|_E^2}{\|u\|_V^2 + \|\sigma\|_E^2}\right\}^{1/2} \approx \left\{\frac{\Pi(u_h) + \Psi(\sigma_h)}{U(u_h) + V(\sigma_h)}\right\}^{1/2} \tag{24}$$

where

$U(u_h) = \frac{1}{2}a_V(u_h,u_h), V(\sigma_h) = \frac{1}{2}a_E(\sigma_h,\sigma_h)$ is the strain energy and the complementary energy, respectively.

This relative error measure only requires very simple computations from the results. One may naturally object that *two* finite element models are necessary to obtain such an error measure. But the present proof never used the assumption that u_h and σ_h should be Rayleigh-Ritz approximations. The only requirement is that u_h and σ_h have to be admissible fields. As an example, after a displacement finite element analysis, one may construct a statically admissible σ_h field, inspired from the displacement analysis, and use the above results. This way is the *Ladevèze method* [10]. The symmetrical construction, which could be named *dual Ladevèze method,* relates to the *compatibility error* [6]. As a conclusion, Ladevèze's approach may be considered as a special form of the general dual analysis.

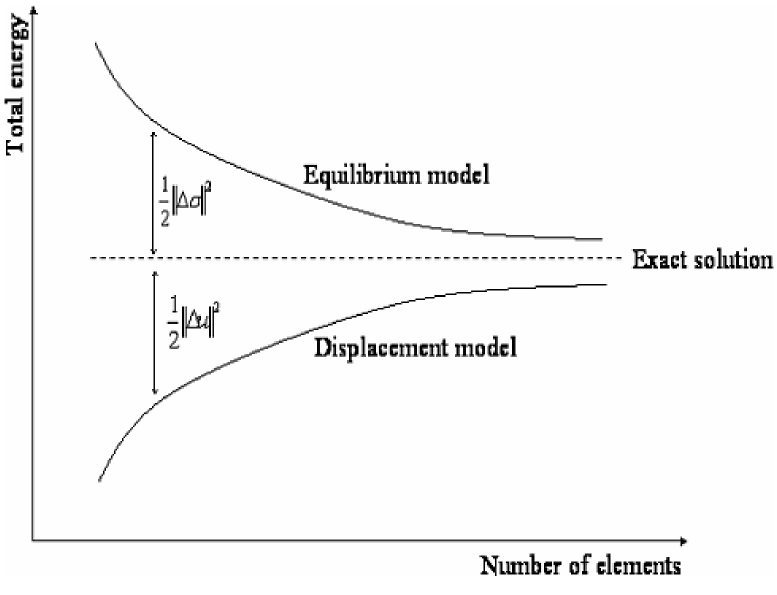


Figure 1: Convergence and error of both types of approximation

In the particular case of structures where

- one type of boundary conditions is homogeneous,

- The approximate homogeneous fields are obtained by a Rayleigh-Ritz procedure,

one can prove the existence of upper and lower bounds of the exact energy. This is the classical dual analysis as proposed by Fraeijs de Veubeke. In this case, the cumulated error is thus measured by the difference between the strain and complementary energies.

4 Choosing finite elements for plate bending

Here, we would like to choose appropriate finite elements for the displacement approach and the equilibrium approach.

In the displacement model, continuity of deflection at the interfaces is the first requirement to satisfy. The next requirement is the continuity of normal slope between adjacent elements. Thus, C^1 - continuity requirement has to be obtained. Another reason for trying to a rigorous enforcement of normal slope continuity is to guarantee that the direct influence coefficients are actually lower bounds to the true ones in case of the homogeneous displacement boundary. From the above strict conditions, the conforming elements of assembled triangle (HCT) or assembled quadrilateral (CQ) [5] should be chosen. Moreover, in reformulating these elements, the advantage of using the area coordinates is useful when assembling and calculating the fields [13]. The reliability of this way has been well tested in our package.

In the equilibrium approach, a triangular equilibrium plate element with degree zero was first introduced by L.S.D. Morley [3]. A regular family of equilibrium triangles and rectangles was still developed into a high level [4] but will not be considered here. A drawback of the constant moment field is the impossibility of obtaining exact equilibrium in the presence of a constant pressure, which is a severe limitation. In order to solve this problem, it is necessary to add a special mode in which the pressure is equilibrated by corner loads only. For this purpose, a particular system of axes will be chosen as follows. Let us call 1, 2, 3 the nodes, as taken in the counter clockwise sense. Let side 1-2 be the X axis. Y axis is perpendicular to it, passed through node 1 and orientated in such a manner that Y_3 is positive. Let $c_1(X,Y)$, $c_2(X,Y)$ and $c_3(X,Y)$ be the *three area coordinates* and $c_i = 0$ be the equation of the side which is opposite to node i. The new moment field can be expressed as follows

$$\sigma = N\beta + T\gamma \tag{25}$$

where $N = [\delta_{ij}], i, j = 1,2,3$ is a constant matrix, $\beta = \begin{bmatrix}\beta_1 & \beta_3 & \beta_3\end{bmatrix}^T$ are unknowns, amplitude γ refers to pressure, and A is the area of the triangle.

The following special mode T has been obtained by the first author

$$T_{11} = -\frac{1}{3}\left\{-\frac{X_3}{Y_3}c_1 + \frac{X_3 - X_2}{Y_3}c_2 - \frac{X_3(X_3 - X_2)}{X_2 Y_3}c_3 + \frac{1}{2A}(X^2 - X_2^2 c_2 - X_3^2 c_3)\right\}$$

$$T_{22} = -\frac{1}{3}\left\{-\frac{Y_3}{X_2}c_3 + \frac{1}{2A}(Y^2 - Y_3^2 c_3)\right\} \tag{26}$$

$$T_{12} = -\frac{1}{6}\left\{-c_1 + c_2 - \frac{2X_3 - X_2}{X_2}c_3 + \frac{1}{A}(XY - X_3 Y_3 c_3)\right\}$$

The complementary field in (26) has been added to the basic field (constant field) in order to give the general field. To specify that our element contains this special mode, we will call it *enhanced Morley* (EM) *element*. Results from this new element were reported in earlier papers [9, 12].

5 Numerical results

Problem 1

Consider a square plate, edge length L =10, thickness t = 0.1, Young's modulus E = $2.05\text{x}10^{11}$, Poisson's ratio ν = 0.3. It is clamped on all edges with a uniform distributed load, p = -1000. Only a quarter (upper-right) of the plate is modelled due to the symmetry of the geometry and the boundary condition, see Figure 2. An initial coarse mesh is created with 8 triangles. Meshes of M x M elements over one quarter are uniformly refined, with M = 4, 8, 16 and 32. The exact strain energy is U_{ex} = 10.363879 [4].

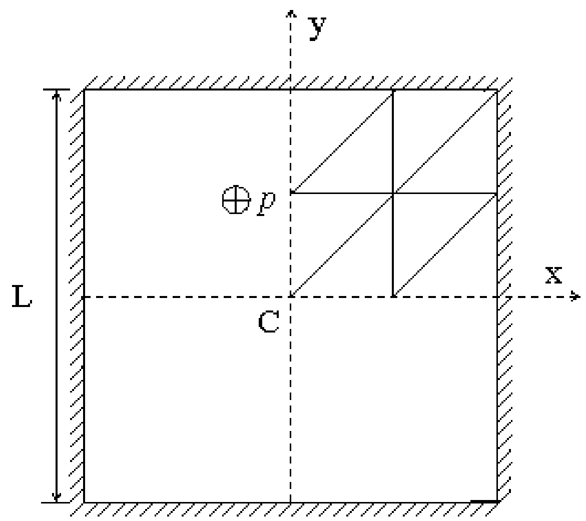


Figure 2: Square plate and initial mesh

In the case of a clamped plate, one type of boundary conditions is homogeneous (Fraeijs de Veubeke's particular case). Therefore, the relative error can be rewritten as

$$RE = \left\{ \frac{V(\sigma_h) - U(u_h)}{V(\sigma_h) + U(u_h)} \right\}^{1/2} \tag{27}$$

The error is thus measured by the difference between the two obtained values of the elastic energy.

Two different bending plate elements are used for the analyses. The first one is the HCT conforming triangle which has reformulated in [13] with three degrees of freedom (D.O.F.) per node. The other one is our *enhanced Morley equilibrium triangle* with 1 D.O.F. per node and 1 D.O.F. per edge.

Mesh	S.A.Model (EM element)	K.A. Model (HCT element)	$R.E$ (%)
	$V(\sigma_h)$	$U(u_h)$	
2x2	25.53458	8.19668	71.69
4x4	14.99747	9.77068	45.94
8x8	11.47289	10.23024	23.93
16x16	10.55812	10.33456	10.34
32x32	10.36521	10.35726	1.96

Table 1: The results on relative error of conventional dual analysis

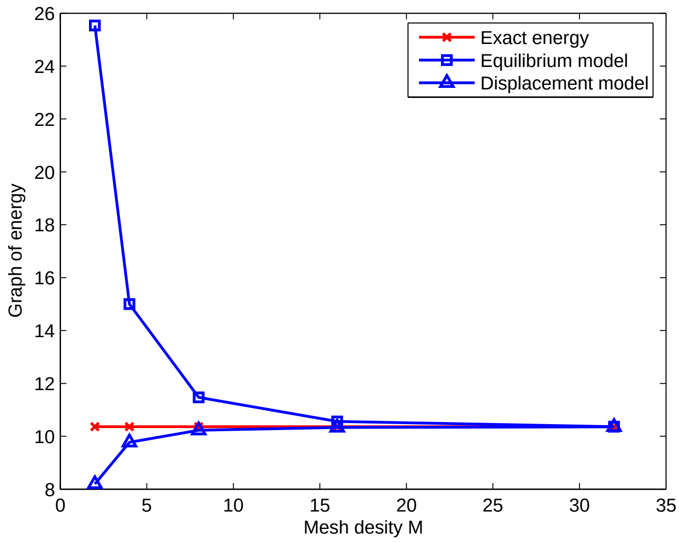
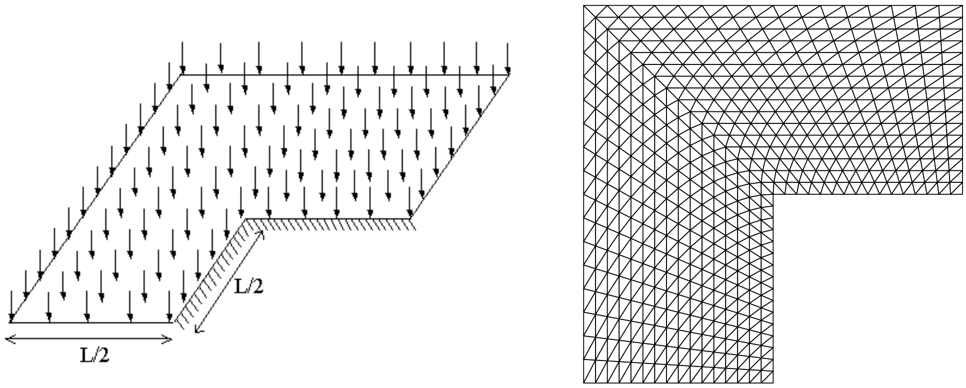


Figure 3: Convergence curve for classically dual analysis

Herein, both approaches converge when the mesh is refined. The distance between the two curves is a measure of convergence. Based on the global error (*R.E*), this method leads to a reliable estimation. The convergence behaviour of the strain energy in terms of the number of elements is illustrated in Figure 3.

Problem 2

Consider the L – shaped plate with a uniform pressure and clamped on a part of its boundary, cf. Figure 4a. Data of problem: edge length L = 5, the quantitative remainders are the same of the first problem. The meshes will generally be composed of 3 – node or 6 – node triangles with two different levels of refinement. Figure 4b is an example of a uniform mesh.



(a) (b)

Figure 4: L – shaped plate (a) and mesh of 1024 elements (b)

Mesh	Finite element model				
	EM element		HCT element		Relative error
	D.O.F	$V(\sigma_h)$	D.O.F	$U(u_h)$	*R.E* (%)
1	36	3.359811	30	1.372400	64.81
2	136	2.126611	108	1.618722	36.82
3	528	1.981689	408	1.711413	27.05
4	2080	1.865485	1584	1.751100	17.78
5	8256	1.820613	6240	1.769728	11.91

Table 2: The results on relative error of classically dual analysis

Figure 5 plots the convergence of the energy of L – shaped plate in dual analysis.

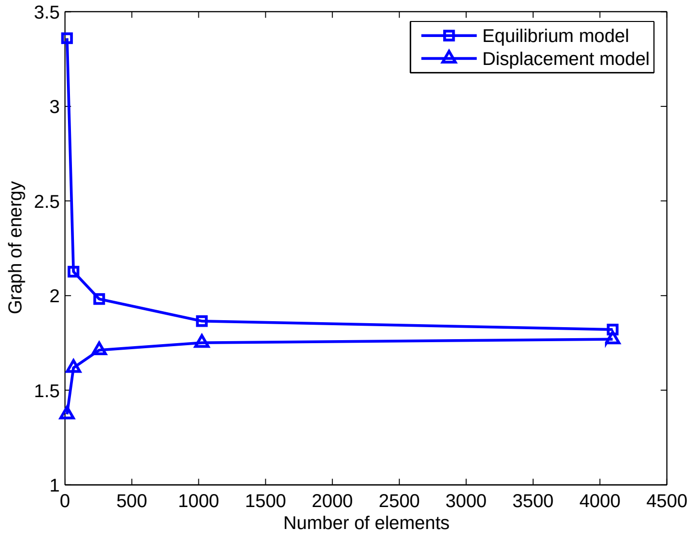


Figure 5: convergence curve for L –shaped plate

The convergence of lower bound and upper bound of strain energy in this problem is slow. This fact is not very surprising, as the problem is singular, due to the re-entrant angle. In such cases, an adaptive mesh refinement procedure would lead to better results.

Problem 3

Consider a square plate loaded at the centre, $F = -1000$, edge length L =10, thickness t = 0.1, Young's modulus E = $2.05\text{x}10^{11}$, Poisson's ratio ν = 0.3. It is clamped on one edge, the opposed edge being loaded with a prescribed non-zero transversal displacement $\bar{u} = -0.01$(Figure 6a). The finite element meshes are illustrated with triangular elements (Figure 6b) and quadrilateral elements (Figure 6c).

In this problem, the displacement element is the HCT conforming triangle as reformulated in [13] with three degrees of freedom (D.O.F.) per node. The equilibrium is the classical Morley equilibrium triangle with 1 D.O.F. per node and 1 D.O.F. per edge [3].

The calculated results from HCT element and Morley element are summarized in the following tables

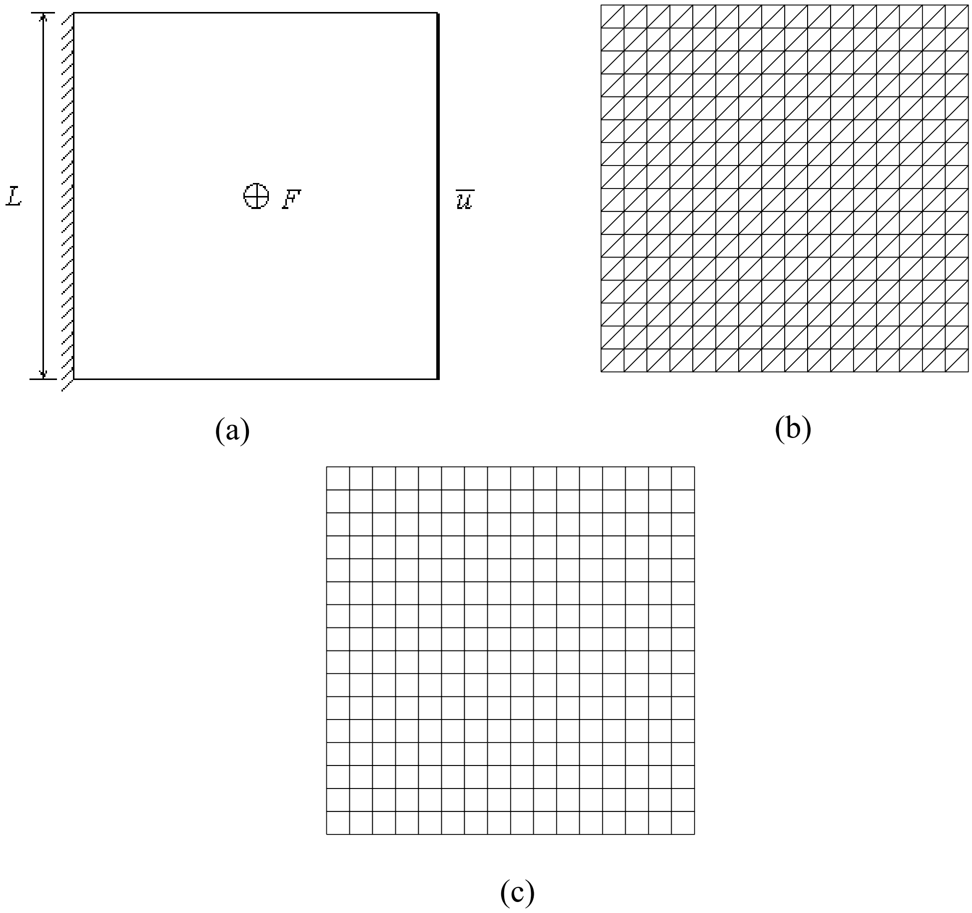


Figure 6: Square plate, boundary conditions and mesh forms

Nel	Finite element model					
	K.A. Model (HCT element)			S.A.Model (Morley element)		
	D.O.F	$U(u_h)$	$f_V(u_h)$	D.O.F	$V(\sigma_h)$	$f_E(\sigma_h)$
8	15	28.630708	3.140623	17	21.124286	38.422033
32	55	27.797573	3.199027	67	25.3217398	47.223287
128	207	27.533898	3.214918	263	26.783227	50.2852997
512	799	27.440801	3.220142	1039	27.218831	51.198337

Table 3: Convergence of the energies

Nel	$2[\Pi(u_h)+\Psi(\sigma_h)]$	$U(u_h)+V(\sigma_h)$	Relative error (%)
8	16.384676	49.754994	57.39
32	5.393998	53.119313	31.87
128	1.633815	54.659632	17.34
512	0.482306	54.317125	9.39

Table 4: Convergence of the errors in dual analysis

Results of tables 3 and 4 provide a global view based on the total energy and the upper bound of global error estimation. The relative error corresponding to the final mesh is still a large value (9.39%) [7]. Therefore, an improved solution is necessary. For this reason, we apply the CQ element to the displacement model.

D.O.F	$U(u_h)$	$f_V(u_h)$	Relative error (%)
15	27.677173	3.197004	54.25
55	27.495156	3.214167	30.02
207	27.433853	3.219642	16.21
799	27.408949	3.221671	8.72

Table 5: Convergence of the errors in dual analysis is improved significantly when using the conforming quadrilateral (CQ) element

The relative error corresponding to the final mesh of using CQ element gets a better result than the previous estimation (8.72%). The convergence behaviour of total energy in dual analysis is illustrated in Figure 7.

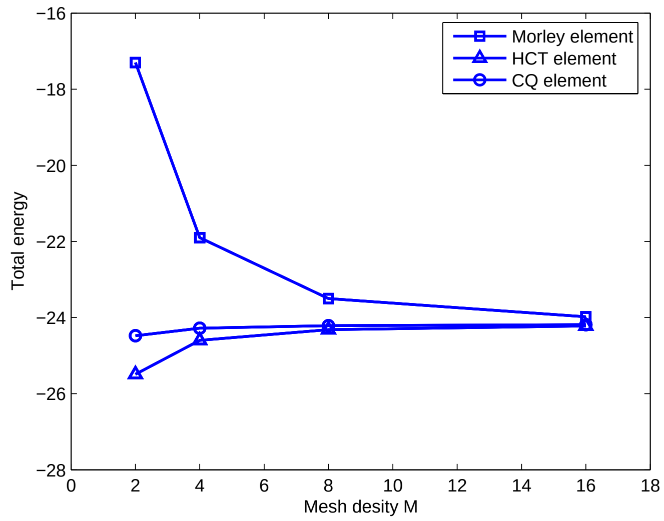


Figure 7: Convergence curve for the generally dual analysis

6 Conclusion

The application of dual analysis to plates is shown to work as an efficient error measure. Its classical form involving strain energy comparison is limited to homogeneous boundary conditions. But its more evolved version based on the total complementary energy works in any case.

The classical Morley element has been completed in order to make it able to exactly equilibrate a pressure, a fact that allows us to treat more realistic problems. This enhanced element is found to work fairly.

It has to be mentioned that, in the frame of dual analysis, there is no need to know the exact solution, as it is always comprised between two the convergence curves. So, the Richardson's extrapolation is not necessary in the error evaluation.

Our results only derive from an evolution of the energy bounds as computed by a uniform refinement of the mesh. Singular problems are more difficult to treat with uniform meshes because the convergence is slow. Thereby, a sound knowledge is necessary to generate finite element meshes based on cost-effective and accurate solutions. We thus should combine our method with an adaptive local refinement procedure in order to improve the cost effectiveness of the error bound evaluation. The result of this investigation will be shown in a further paper.

References

[1] B.M. Fraeijs de Veubeke, "*Displacement and equilibrium models in the finite element method*", in "Stress analysis", edit. Zienkiewicz O.C., Holister G.S., Wiley, London, 1965.

[2] B.M. Fraeijs de Veubeke, "*A conforming finite element for plate bending*", Int.J.Solids Structures, 4, pp.95 to 108. Pergamon Press. Printed in Great Britain, 1968.

[3] L.S.D. Morley, "*The triangular equilibrium element in the solution of plate bending problems*", Aero Quart., 19: 149-168, 1968.

[4] G. Sander, "*Applications de la methode des elements finis a la flexion des plaques*", Coll.Pub.Fac.Sc.Appli. Univ.of Liège, $N^0$15, 1969.

[5] J.F. Debongnie, "*Fundamentals of finite elements*", Les Editions de l'Université de Liège, Liège, 2003, ISBN 2-93032254-3.

[6] J.F. Debongnie and P. Beckers, "*On a general decomposition of error an approximate stress field in elasticity*", Comput. A.M.and Eng. Sciences, 8: 261-270, 2001.

[7] J.F. Debongnie, H.G. Zhong and P. Beckers, "*Dual analysis with general boundary conditions*", Comput. Methods Appl. Mech. Engrg., 122:183-192, 1995.

[8] J.F. Debongnie, "*Some aspects of the finite element errors*", Univ. of Liège,Report LMF/D42,October 2001.

[9] J.F. Debongnie, "*Applying pressures on plate equilibrium elements*", Univ. of Liège, May 2004.

[10] P. Ladevèze, Ph. Rougeot, P. Blanchard and J.P. Moreau, "*Local error estimators for finite element linear analysis*", Comput. Methods Appl. Mech. Engrg., 176: 231-246, 1999.
[11] A. Mashaie, E. Hughes and J. Goldak, "*Estimated error bounds for finite element solutions of elliptic boundary value problems*", Comput. Methods Appl.Mech.Engrg. 130 (1996) 17-31.
[12] N.X. Hung and J.F. Debongnie, "*The equilibrium finite element model and error estimation for plate bending*". International Congress "*Engineering Mechanics Today 2004*", Ho Chi Minh City, Vietnam, August 16-20, 2004.
[13] N.X. Hung, "*The conforming finite element model and error estimation for plate bending*", Proceedings of the 8th National Conference on Mechanics, Hai Phong City, Vietnam, August 27-28, 2004.