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Characterizations of the elements of generalized Hölder–Zygmund spaces by means of their representation

Damien Kreit, Samuel Nicolay*

Université de Liège, Institut de Mathématique, Grande Traverse, 12, Bâtiment B37, B-4000 Liège (Sart-Tilman), Belgium

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Abstract

In this paper, we give three characterizations of the elements of generalized Hölder–Zygmund spaces. The first one, based on the Littlewood–Paley decomposition is already known, but the proof given here is much simpler. The second one, based on the wavelet decompositions generalizes a result obtained by Jaffard and Meyer. The third one uses generalized interpolation spaces. These results naturally extend the ones holding for the classical Hölder–Zygmund spaces.

Keywords: Generalized Hölder–Zygmund spaces

1. Introduction

As in [9], we define the generalized Hölder–Zygmund spaces starting from the generalized Besov spaces $B^{\sigma}_{\infty,\infty}(\mathbf{R}^d)$, where σ is an admissible sequence [5,12]. The set of natural numbers is denoted by $\mathbf{N}$ (and does not contain 0) and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$.

Definition 1. A sequence $\sigma = (\sigma_j)_{j \in \mathbf{N}_0}$ of real positive numbers is called admissible if there exists a positive constant C such that

$$C^{-1}\sigma_j \leq \sigma_{j+1} \leq C\sigma_j,$$

for any $j \in \mathbf{N}_0$.

* Corresponding author.
E-mail address: S.Nicolay@ulg.ac.be (S. Nicolay).

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If σ is such a sequence, we set

$$\underline{\Theta}_j = \inf_{k\in\mathbf{N}_0} \frac{\sigma_{j+k}}{\sigma_k} \quad \text{and} \quad \overline{\Theta}_j = \sup_{k\in\mathbf{N}_0} \frac{\sigma_{j+k}}{\sigma_k}$$

and define the lower and upper Boyd indices as follows:

$$\underline{s}(\sigma) = \lim_j \frac{\log_2 \underline{\Theta}_j}{j} \quad \text{and} \quad \overline{s}(\sigma) = \lim_j \frac{\log_2 \overline{\Theta}_j}{j}.$$

Since $(\log \overline{\Theta}_j)_{j\in\mathbf{N}_0}$ is a subadditive sequence, such limits always exist [6].

For a function $f : \mathbf{R}^d \to \mathbf{R}$ and $x, h \in \mathbf{R}^d$, the first order difference of f is

$$\Delta_h^1 f(x) = f(x+h) - f(x)$$

and the difference of order n, where n is an integer greater than 1, is iteratively defined by

$$\Delta_h^n f(x) = \Delta_h^{n-1} \Delta_h^1 f(x).$$

The generalized Hölder–Zygmund spaces can then be introduced.

Definition 2. Let $\alpha > 0$ and σ be an admissible sequence; a function $f \in L^\infty(\mathbf{R}^d)$ belongs to the space $\Lambda^{\sigma,\alpha} = \Lambda^{\sigma,\alpha}(\mathbf{R}^d)$ if there exists $C > 0$ such that

$$\sup_{|h|\leq 2^{-j}} \|\Delta_h^{[\alpha]+1} f\|_\infty \leq C\sigma_j,$$

for any $j \in \mathbf{N}_0$.

One sets $\Lambda^\sigma = \Lambda^{\sigma,\overline{s}(\sigma^{-1})}$. For example, the classical Hölder–Zygmund space Λ^α ($\alpha > 0$) is the space Λ^σ with $\sigma = (2^{-j\alpha})_j$. The basic properties of these generalized spaces have been studied in [9].

In this paper, we give three characterizations of the spaces Λ^σ, using the Littlewood–Paley theory, the discrete wavelet transform and generalized real interpolation spaces. Each result generalizes a well-known characterization of the classical Hölder–Zygmund spaces Λ^α, with $\alpha > 0$. The first one can be found in [12] with a more difficult proof, while the second characterization extends results obtained in [12] (to the case $B^\sigma_{\infty,\infty}$), using ideas introduced in [8] for the specific case of the modulus of continuity-defined Hölder–Zygmund spaces. It can also be deduced from results obtained in [14,12], but the present version is shorter and only uses classical techniques. The last characterization is new. Let us also notice that the spaces considered here are closely related to those dealt with in [13], where the continuous wavelet transform is used. Finally, let us mention that the study of such spaces takes its roots in the early Russian literature (such as [2]).

The results presented here contribute to a better understanding of the reasons why the proofs should work, even in the classical case. In particular, the notion of strong admissible sequence underlines the fundamental difference between the Hölder spaces Λ^α with $\alpha \notin \mathbf{N}$ and the Zygmund spaces Λ^n with $n \in \mathbf{N}$.

Throughout the paper, we use the letter C for a generic positive constant whose value may be different at each occurrence.

2. Previous results

We will need some of the results obtained in [9]. In this section, B denotes the open unit ball.

2.1. Two characterizations of the generalized Hölder–Zygmund spaces

The following characterizations generalize well-known results for the classical Hölder–Zygmund spaces. They will be used in the sequel.

Proposition 1. *Let σ be an admissible sequence and $f \in L^\infty(\mathbf{R}^d)$; f belongs to Λ^σ if and only if there exists a positive constant C such that*

$$\inf_{P\in\mathbf{P}} \|f - P\|_{L^\infty(2^{-j}B+x_0)} \le C\sigma_j,$$

for any $x_0 \in \mathbf{R}^d$ and any j, where the infimum is taken over all polynomials of degree at most $\overline{s}(\sigma^{-1})$.

Proposition 2. *Let σ be an admissible sequence and $n, m \in \mathbf{N}_0$ such that $n < \underline{s}(\sigma^{-1}) \le \overline{s}(\sigma^{-1}) < m$; if $f \in \Lambda^\sigma$, then f is equal almost everywhere to an element of $C^n(\mathbf{R}^d)$, $D^\alpha f \in L^\infty(\mathbf{R}^d)$ for any multi-index α such that $|\alpha| \le n$ and*

$$\sup_{|h|\le 2^{-j}} \|\Delta_h^{m-|\alpha|} D^\alpha f\|_\infty \le C 2^{j|\alpha|}\sigma_j, \tag{1}$$

for any $j \in \mathbf{N}_0$ and α such that $|\alpha| \le n$.

Conversely, if $f \in C^n(\mathbf{R}^d) \cap L^\infty(\mathbf{R}^d)$ satisfies inequality (1) for any j and any multi-index α such that $|\alpha| = n$, then $f \in \Lambda^\sigma$.

2.2. About the admissible sequences

If σ is an admissible sequence, for any $\varepsilon > 0$, then there exists a positive constant C such that

$$C^{-1}2^{j(\underline{s}(\sigma)-\varepsilon)} \le \underline{\Theta}_j \le \frac{\sigma_{j+k}}{\sigma_k} \le \overline{\Theta}_j \le C2^{j(\overline{s}(\sigma)+\varepsilon)},$$

for any $j, k \in \mathbf{N}$. The following properties will be often used (lemmata 2.4, 2.8, 2.9 and 2.10 in [9]).

Lemma 3. *Let σ be an admissible sequence.*

- *If $\underline{s}(\sigma^{-1}) > 0$, then there exists a positive constant C such that for any $J \in \mathbf{N}$,*

$$\sum_{j\ge J} \sigma_j \le C\sigma_J.$$

- *If $\overline{s}(\sigma^{-1}) < n$ with $n \in \mathbf{N}$, then there exists a positive constant C such that for any $J \in \mathbf{N}$,*

$$\sum_{j=1}^{J} 2^{jn}\sigma_j \le C2^{Jn}\sigma_J.$$

We will sometimes need to impose additional assumptions on the admissible sequences. We transpose here the concept of strong modulus of smoothness [8] to the admissible sequences [9].

Definition 3. An admissible sequence σ is a strong admissible sequence of order $n \in \mathbf{N}$ if there exists a constant C such that

$$\sum_{j=0}^{J} 2^{jn}\sigma_j \le C2^{Jn}\sigma_J \tag{2}$$

and

$$\sum_{j=J}^{\infty} 2^{j(n-1)}\sigma_j \leq C 2^{J(n-1)}\sigma_J, \tag{3}$$

for any $J \in \mathbf{N}$.

For example, the sequence $(2^{-j\alpha})_j$ defining the usual Hölder–Zygmund space Λ^α is strong (of order $[\alpha]+1$) if and only if $\alpha \notin \mathbf{N}$ (see e.g. [9]).

This notion is closely related to the Boyd indices of the inverse sequence.

Lemma 4. *A strong admissible sequence σ of order n is such that*

$$n-1 \leq \underline{s}(\sigma^{-1}) \leq \overline{s}(\sigma^{-1}) \leq n.$$

Conversely, if σ is an admissible sequence satisfying

$$n-1 < \underline{s}(\sigma^{-1}) \quad \text{and} \quad \overline{s}(\sigma^{-1}) < n$$

for some $n \in \mathbf{N}$, then σ is strong of order n.

The following easy result shows that a strong admissible sequence of order n lies in between the sequences $(2^{-jn})_{j\in\mathbf{N}}$ and $(2^{-j(n-1)})_{j\in\mathbf{N}}$.

Lemma 5. *If σ is a strong admissible sequence of order n, then there exists a positive constant C such that*

$$C^{-1}2^{-jn} \leq \sigma_j \leq C 2^{-j(n-1)},$$

for any $j \in \mathbf{N}$.

When considering strong admissible sequences, the proof of Proposition 2 (see [9]) gives the following result.

Proposition 6. *Let σ be a strong admissible sequence of order n; if $f \in \Lambda^\sigma$, then f is equal almost everywhere to an element of $C^{n-1}(\mathbf{R}^d)$, $D^\alpha f \in L^\infty(\mathbf{R}^d)$ for any multi-index α such that $|\alpha| \leq n-1$ and*

$$\sup_{|h|\leq 2^{-j}} \|\Delta_h^{n-|\alpha|} D^\alpha f\|_\infty \leq C 2^{j|\alpha|}\sigma_j, \tag{4}$$

for any $j \in \mathbf{N}_0$ and α such that $|\alpha| \leq n-1$.

Conversely, if $f \in C^{n-1}(\mathbf{R}^d) \cap L^\infty(\mathbf{R}^d)$ satisfies inequality (4) for any j and any multi-index α such that $|\alpha| = n-1$, then $f \in \Lambda^\sigma$.

Remark 1. One cannot deduce Proposition 6 from Proposition 2 and Lemma 4. Instead, one directly uses inequalities (2) and (3) in the proof of Proposition 2; no further change is necessary. This procedure allows one to obtain stronger results (than the ones obtained with Lemma 4) and will be used several times in the sequel.

3. Littlewood–Paley characterization

As usual, $\mathcal{S}(\mathbf{R}^d)$ denotes the Schwartz class and $\hat{f}$ the Fourier transform of f. In this section, $\varphi \in \mathcal{S}(\mathbf{R}^d)$ is a function such that $\hat{\varphi}(\omega) = 1$ if $|\omega| \leq 1/2$ and $\hat{\varphi}(\omega) = 0$ if $|\omega| \geq 1$ and ψ

is defined by $\psi = 2^d\varphi(2\cdot) - \varphi$. The convolution operators with $2^{dj}\varphi(2^j\cdot)$ and $2^{dj}\psi(2^j\cdot)$ are denoted by S_j and Δ_j respectively:

$$\widehat{S_j f} = \hat{\varphi}(2^{-j}\cdot)\hat{f} \quad \text{and} \quad \Delta_j f = S_{j+1}f - S_j f,$$

for any $f \in \mathcal{S}'(\mathbf{R}^d)$. The following relation, holding in $\mathcal{S}'(\mathbf{R}^d)$, is known as the Littlewood–Paley decomposition of the unity [16]:

$$I = S_0 + \sum_{j\geq 0} \Delta_j.$$

In this section, we give (under some weak additional hypothesis) a sufficient condition and a necessary condition for a function to belong to Λ^σ.

Theorem 7. *Let σ be an admissible sequence such that $\underline{s}(\sigma^{-1}) > 0$. If there exists a constant C such that $f \in L^\infty(\mathbf{R}^d)$ satisfies*

$$\|\Delta_j f\|_\infty \leq C\sigma_j, \tag{5}$$

for any $j \in \mathbf{N}_0$, then $f \in \Lambda^\sigma$.

Conversely, let σ be an admissible sequence and $n \in \mathbf{N}_0$ such that

$$n < \underline{s}(\sigma^{-1}) \leq \bar{s}(\sigma^{-1}) < n + 2.$$

If $f \in \Lambda^\sigma$, then there exists a constant C such that (5) holds.

Remark 2. The assumption $n < \underline{s}(\sigma^{-1}) \leq \bar{s}(\sigma^{-1}) < n+2$ is a technical condition and although the authors strongly believe that it is superfluous, no straightforward improvement of the proof allows to get rid of this assumption.

For the strong admissible sequences, minor changes in the proof lead to the following version.

Theorem 8. *Let σ be a strong admissible sequence of order $n \in \mathbf{N}$; $f \in L^\infty(\mathbf{R}^d)$ belongs to Λ^σ if and only if there exists a constant C such that f satisfies*

$$\|\Delta_j f\|_\infty \leq C\sigma_j,$$

for any $j \in \mathbf{N}_0$.

This result generalizes the one obtained in [8] for the Hölder–Zygmund spaces defined via a modulus of smoothness (see [9]).

Remark 3. Under the condition $S_0 f \in L^\infty(\mathbf{R}^d)$, the assumption $f \in L^\infty(\mathbf{R}^d)$ in Theorems 7 and 8 can be relaxed to consider distributions $f \in \mathcal{S}'(\mathbf{R}^d)$.

3.1. Proof of the sufficiency of the condition

Let us set $\Delta_{-1} = S_0$ and choose $n > \bar{s}(\sigma^{-1})$. Using the Bernstein inequalities, we get $\Delta_j f \in C^\infty(\mathbf{R}^d)$ and

$$\|D^\alpha \Delta_j f\|_\infty \leq C 2^{|\alpha| j}\sigma_j,$$

for any multi-index α such that $|\alpha| \leq n$ and any $j \in \mathbf{N}_0$. Therefore the series $\sum_{j\geq -1} \Delta_j f$ converges uniformly.

For $x_0 \in \mathbf{R}^d$ and $J \in \mathbf{N}_0$, let us define

$$P_j(x - x_0) = \sum_{|\alpha| \le n-1} \frac{(x - x_0)^\alpha}{|\alpha|!} D^\alpha \Delta_j f(x_0)$$

and

$$P_{x_0,J}(x - x_0) = \sum_{j=-1}^{J} P_j(x - x_0).$$

If x satisfies $|x - x_0| \le 2^{-J}$, one has

$$|f(x) - P_{x_0,J}(x - x_0)| \le \sum_{j=-1}^{J} |\Delta_j f(x) - \sum_{|\alpha| \le n-1} \frac{(x - x_0)^\alpha}{|\alpha|!} D^\alpha \Delta_j f(x_0)| + \sum_{j \ge J+1} \|\Delta_j f\|_\infty.$$

The second term on the right-hand side of the last inequality is bounded by $C\sigma_J$, while the first term is bounded by

$$\sum_{j=-1}^{J} |x - x_0|^n \sup_{|\alpha|=n} \|D^\alpha \Delta_j f\|_\infty \le C 2^{-Jn} \sum_{j=-1}^{J} 2^{jn} \sigma_j \le C\sigma_J,$$

where the constant does not depend on x or J.

3.2. *Proof of the necessity of the condition*

Using a classical argument, we can suppose that φ and thus ψ are even and using Proposition 2, we can suppose that $f \in C^n(\mathbf{R}^d)$, $D^\alpha f \in L^\infty(\mathbf{R}^d)$ and

$$\sup_{|h| \le 2^{-j}} \|\Delta_h^2 D^\alpha f\|_\infty \le C\sigma_j 2^{jn},$$

for any multi-index α such that $|\alpha| \le n$. One has, since ψ is even with a vanishing integral,

$$\begin{aligned} \Delta_j D^\alpha f(x) &= 2^{jd} \int D^\alpha f(x - y) \psi(2^j y)\, dy \\ &= 2^{jd-1} \int (D^\alpha f(x + y) - 2D^\alpha f(x) + D^\alpha f(x - y)) \psi(2^j y)\, dy. \end{aligned}$$

Moreover, the Bernstein inequalities imply

$$\|\Delta_j f\|_\infty \le C 2^{-jn} \sup_{|\alpha|=n} \|D^\alpha \Delta_j f\|_\infty$$

and therefore

$$\begin{aligned} \|\Delta_j f\|_\infty &\le C 2^{j(d-n)} \int \sup_{|\alpha|=n} \|\Delta_y^2 D^\alpha f\|_\infty |\psi(2^j y)|\, dy \\ &\le C 2^{-jn} \int \sup_{|\alpha|=n} \|\Delta_{2^{-j} y}^2 D^\alpha f\|_\infty |\psi(y)|\, dy. \end{aligned}$$

One can conclude, since we have

$$\int_{|y|\leq 1} \sup_{|\alpha|=n} \|\Delta^2_{2^{-j}y} D^\alpha f\|_\infty |\psi(y)|\, dy \leq C 2^{jn} \sigma_j$$

and

$$\begin{aligned}
&\int_{2^m\leq |y|<2^{m+1}} \sup_{|\alpha|=n} \|\Delta^2_{2^{-j}y} D^\alpha f\|_\infty |\psi(y)|\, dy \\
&\leq \int_{2^m\leq |y|<2^{m+1}} \sup_{\substack{|\alpha|=n \\ |h|\leq 2^{-j}}} \|\Delta^2_{2^{m+1}h} D^\alpha f\|_\infty |\psi(y)|\, dy \\
&\leq C 2^{2m+2} \int_{2^m\leq |y|<2^{m+1}} \sup_{\substack{|\alpha|=n \\ |h|\leq 2^{-j}}} \|\Delta^2_{h} D^\alpha f\|_\infty |\psi(y)|\, dy \\
&\leq C 2^{2m+nj} \sigma_j \int_{2^m\leq |y|<2^{m+1}} \frac{1}{(1+|y|)^k}\, dy \\
&\leq C 2^{m(d+2-k)+nj} \sigma_j,
\end{aligned}$$

for any $k \in \mathbf{N}$.

4. Wavelet characterization

Under some general assumptions (see e.g. [11,10,4]), there exist a function φ and $2^d - 1$ functions $(\psi^{(i)})_{1\leq i<2^d}$, called wavelets, such that

$$\{\varphi(x-k)\}_{k\in\mathbf{Z}^d} \cup \{\psi^{(i)}(2^j x - k) : 1 \leq i < 2^d, k \in \mathbf{Z}^d, j \in \mathbf{N}_0\}$$

form an orthogonal basis of $L^2(\mathbf{R}^d)$. Any function $f \in L^2(\mathbf{R}^d)$ can be decomposed as follows:

$$f(x) = \sum_{k\in\mathbf{Z}^d} C_k \varphi(x-k) + \sum_{j=0}^{\infty} \sum_{k\in\mathbf{Z}^d} \sum_{1\leq i<2^d} c^{(i)}_{j,k} \psi^{(i)}(2^j x - k),$$

where

$$c^{(i)}_{j,k} = 2^{dj} \int_{\mathbf{R}^d} f(x) \psi^{(i)}(2^j x - k)\, dx$$

and

$$C_k = \int_{\mathbf{R}^d} f(x) \varphi(x-k)\, dx.$$

The above formulas are still valid in more general settings; they have to be interpreted as a duality product between regular functions (the wavelets) and distributions [11,7]. In what follows, we will suppose that the wavelets are the Lemarié–Meyer wavelets [11] (φ and $\psi^{(i)}$ therefore belong to the Schwartz class $\mathcal{S}(\mathbf{R}^d)$) or the Daubechies wavelets [4] (which are compactly supported and can be chosen arbitrarily regular, let us say r-regular with $r > n$ [11], where n is the order of the strong admissible sequence).

We aim at showing the following result.

Theorem 9. *Let σ be a strong admissible sequence of order $n \in \mathbf{N}$; f belongs to Λ^σ if and only if its wavelet coefficients satisfy the following inequalities:*

$$|C_k| \leq C \quad \text{and} \quad |c_{j,k}^{(i)}| \leq C\sigma_j,$$

for some positive constant C, any $j \in \mathbf{N}$, any $k \in \mathbf{Z}^d$ and any $i \in \{1, \ldots, 2^d - 1\}$.

Remark 4. The same argument as in [7] shows that the previous result is still valid for any r-regular wavelet basis ($r \geq n$).

Remark 5. It is natural to ask whether Theorem 9 remains valid for arbitrary admissible sequences. Here again, no obvious modification of the proof gives the answer.

4.1. Proof of the necessity of the condition

The proof of the theorem is based on ideas found in [11,7]. We will need the following result (a proof is given in [9]).

Lemma 10. *If $f \in C^n(\mathbf{R}^d)$, then*

$$f(x+h) = \sum_{|\alpha|\leq n} D^\alpha f(x) \frac{h^\alpha}{|\alpha|!} + R_n(x,h)\frac{|h|^n}{n!},$$

for any $x, h \in \mathbf{R}^d$ with

$$|R_n(x,h)| \leq \sum_{|\alpha|=n} \sup_{|l|\leq|h|} \|\Delta_l^1 D^\alpha f\|_\infty.$$

Obviously, $f \in L^\infty(\mathbf{R}^d)$ implies that C_k is bounded. One has, using the fact that the wavelet has an arbitrary number of vanishing moments,

$$\begin{aligned} |c_{j,k}^{(i)}| &= 2^{jd} \left| \int f\left(\frac{k}{2^j} + \left(x - \frac{k}{2^j}\right)\right) \psi^{(i)}(2^j x - k)\, dx \right| \\ &= 2^{jd} \left| \int R_{n-1}\left(\frac{k}{2^j}, x\right) \frac{|x|^{n-1}}{(n-1)!} \psi^{(i)}(2^j x)\, dx \right| \\ &\leq C \int \sup_{\substack{|h|\leq|x|/2^j \\ |\alpha|=n-1}} \|\Delta_h^1 D^\alpha f\|_\infty |x|^{n-1} 2^{-j(n-1)} |\psi^{(i)}(x)|\, dx. \end{aligned}$$

Now, one has

$$\int_{|x|\leq 1} \sup_{\substack{|h|\leq|x|/2^j \\ |\alpha|=n-1}} \|\Delta_h^1 D^\alpha f\|_\infty |x|^{n-1} 2^{-j(n-1)} |\psi^{(i)}(x)|\, dx \leq C\sigma_j$$

and, for $m \in \mathbf{N}_0$,

$$\begin{aligned} &\int_{2^m\leq|x|\leq 2^{m+1}} \sup_{\substack{|h|\leq|x|/2^j \\ |\alpha|=n-1}} \|\Delta_h^1 D^\alpha f\|_\infty |x|^{n-1} 2^{-j(n-1)} |\psi^{(i)}(x)|\, dx \\ &\quad \leq C \int_{2^m\leq|x|\leq 2^{m+1}} \sup_{\substack{|h|\leq|x|/2^j \\ |\alpha|=n-1}} \|\Delta_h^1 D^\alpha f\|_\infty |x|^{n-1} 2^{-j(n-1)} \frac{1}{(1+|x|)^p}\, dx \end{aligned}$$

$$\leq C \int_{1/2 \leq |x| \leq 1} \sup_{\substack{|h| \leq 2^{m+1-j} \\ |\alpha| = n-1}} \| \Delta_h^1 D^\alpha f \|_\infty 2^{m(n+d-1)} |x|^{n-1} 2^{-j(n-1)} 2^{-mp} \, dx$$

$$\leq C 2^{m(n+d-p)-j(n-1)} \int_{1/2 \leq |x| \leq 1} \sup_{\substack{|h| \leq 2^{-j} \\ |\alpha| = n-1}} \| \Delta_h^1 D^\alpha f \|_\infty \, dx$$

$$\leq C 2^{m(n+d-p)} \sigma_j,$$

where $p \in \mathbf{N}$ can be chosen arbitrarily large. Putting these inequalities together gives the desired result.

4.2. Proof of the sufficiency of the condition

Here, we essentially follow the ideas given in [7]. Let us set

$$f_{-1} = \sum_k C_k \varphi(x-k) \quad \text{and} \quad f_j(x) = \sum_{i,k} c_{j,k}^{(i)} \psi^{(i)}(2^j x - k),$$

for any $j \in \mathbf{N}_0$. These series converge uniformly on any compact, thanks to the decreasing properties of φ and $\psi^{(i)}$. Now, it can be shown that the series

$$g(x) = \sum_{j \geq 1} f_j(x)$$

also converges uniformly on $\mathbf{R}^d$ to f. Using a similar argument for $D^\alpha \varphi$ and $D^\alpha \psi^{(i)}$, one gets

$$|D^\alpha f_j(x)| \leq C 2^{|\alpha| j} \sigma_j$$

for any multi-index α such that $|\alpha| \leq n$. Therefore, the series $\sum_j f_j$ can be derived term by term $n-1$ times, so that $g \in C^{n-1}(\mathbf{R}^d)$ and $|D^\alpha g(x)| \leq C$ for any $|\alpha| \leq n-1$. If α is a multi-index such that $|\alpha| \leq n-1$, for any $h \in \mathbf{R}^d$, let j_0 be such that

$$2^{-(j_0+1)} \leq |h| < 2^{-j_0}.$$

One has

$$\begin{aligned} \| \Delta_h^1 D^\alpha g \|_\infty &\leq \sum_{j \leq j_0} \| \Delta_h^1 D^\alpha f_j \|_\infty + \sum_{j > j_0} 2 \| D^\alpha f_j \|_\infty \\ &\leq \sum_{j \leq j_0} |h| \sup_{|\beta|=1} \| D^{\alpha+\beta} f_j \|_\infty + \sum_{j=j_0+1}^{\infty} 2^{(n-1)j+1} \sigma_j \\ &\leq C |h| 2^{n j_0} \sigma_{j_0} + C 2^{(n-1)(j_0+1)} \sigma_{j_0+1} \\ &\leq C 2^{(n-1) j_0} \sigma_{j_0}, \end{aligned}$$

which allows one to conclude.

5. Definition of the space Λ^σ via generalized real interpolation of Sobolev spaces

In this section, we use the generalized interpolation spaces [1,3] to define the generalized Hölder–Zygmund spaces, starting from the usual Sobolev spaces W_n^∞.

Let A and B be two Banach spaces continuously embedded into a Hausdorff topological vector space V so that $A \cap B$ and $A + B$ are well defined Banach spaces; one defines the

J-functional by

$$J(t,x) = \max\{\|x\|_A, t\|x\|_B\},$$

for any $t > 0$ and $x \in A \cap B$.

Definition 4. If $\sigma = (\sigma_j)_{j\in\mathbf{Z}}$ and $(\theta_j)_{j\in\mathbf{Z}}$ are two sequences, then the generalized interpolation space $[A,B]_{\sigma,\theta,J}$ is defined as follows: $x \in [A,B]_{\sigma,\theta,J}$ if and only if it can be written as $x = \sum_{j\in\mathbf{Z}} u_j$, where the series converges in $A + B$ with $u_j \in A \cap B$ and where the sequence $(\sigma_j J(\theta_j, u_j))_{j\in\mathbf{Z}}$ belongs to $l^\infty(\mathbf{Z})$.

Let us now define the K-functional by

$$K(t,x) = \inf\{\|x_1\|_A + t\|x_2\|_B : x = x_1 + x_2\},$$

for any $t > 0$ and $x \in A + B$.

Definition 5. If $\sigma = (\sigma_j)_{j\in\mathbf{Z}}$ and $(\theta_j)_{j\in\mathbf{Z}}$ are two sequences, then the generalized interpolation space $[A,B]_{\sigma,\theta,K}$ is defined as follows: $x \in [A,B]_{\sigma,\theta,K}$ if and only if $x \in A + B$ and the sequence $(\sigma_j K(\theta_j, x))_{j\in\mathbf{Z}}$ belongs to $l^\infty(\mathbf{Z})$.

If $\sigma_j = 2^{-j\alpha}$ and $\theta_j = 2^j$, one recovers the classical real interpolation spaces $[A,B]_{\alpha,\infty,J}$ and $[A,B]_{\alpha,\infty,K}$.

If $\sigma = (\sigma_j)_{j\in\mathbf{N}_0}$ is an admissible sequence, let us define the sequence $\sigma^{(n)} = (\sigma_j^{(n)})_{j\in\mathbf{Z}}$ by

$$\sigma_j^{(n)} = \begin{cases} 2^{jn}\sigma_{-j}^{-1} & \text{if } j \in -\mathbf{N}_0 \\ (\sigma_{-j}^{(n)})^{-1} & \text{if } j \in \mathbf{N}. \end{cases}$$

With the conditions we are going to work with, the J-method and the K-method defined above lead to the same spaces. We state the precise result here, but postpone a proof to the next section.

Proposition 11. *Let $\sigma = (\sigma_j)_{j\in\mathbf{N}}$ be an admissible sequence and $n, m \in \mathbf{N}_0$ be such that $n < \underline{s}(\sigma^{-1}) \le \overline{s}(\sigma^{-1}) < m$. If B is continuously embedded in A, one has*

$$[A,B]_{\sigma^{(n)},2^{j(m-n)},J} = [A,B]_{\sigma^{(n)},2^{j(m-n)},K}.$$

We now need some basic results about the Sobolev spaces. Let $p \in [1,\infty]$ and $n \in \mathbf{N}$; as usual, W_n^p will denote the Sobolev space

$$W_n^p = \{f \in L^p(\mathbf{R}^d) : D^\alpha f \in L^p(\mathbf{R}^d)\ \forall |\alpha| \le n\}$$

equipped with the norm

$$\|f\|_{W_n^p} = \sum_{|\alpha|\le n} \|D^\alpha f\|_p.$$

We will use the following classical result, which is a direct consequence of the Morrey inequality [15,17]: let $p \in]d,\infty]$ and $\alpha = 1 - d/p$; one has $W_1^p \subset \Lambda^\alpha$ and

$$\|f\|_{\Lambda^\alpha} \le C\|f\|_{W_1^p},$$

for any $f \in W_1^p$. We will prove the following result, showing that the generalized Hölder–Zygmund spaces can be defined via generalized interpolations of Sobolev spaces.

Theorem 12. *Let $\sigma = (\sigma_j)_{j\in\mathbf{N}}$ be an admissible sequence and $n, m \in \mathbf{N}_0$ be such that $n < \underline{s}(\sigma^{-1}) \leq \overline{s}(\sigma^{-1}) < m$. One has*

$$\Lambda^\sigma = [W_n^\infty, W_m^\infty]_{\sigma^{(n)},2^{j(m-n)},J} = [W_n^\infty, W_m^\infty]_{\sigma^{(n)},2^{j(m-n)},K}.$$

We have the following version for the strong admissible sequences.

Theorem 13. *Let $\sigma = (\sigma_j)_{j\in\mathbf{N}}$ be a strong admissible sequence of order n such that $\overline{s}(\sigma^{-1}) < n$. One has*

$$\Lambda^\sigma = [W_{n-1}^\infty, W_n^\infty]_{\sigma^{(n-1)},2^j,J} = [W_{n-1}^\infty, W_n^\infty]_{\sigma^{(n-1)},2^j,K}.$$

5.1. *Proof of Proposition 11*

Let $f \in [A, B]_{\sigma^{(n)},2^{j(m-n)},J}$. One has $f = \sum_j u_j$ with convergence in A and

$$\|u_j\|_A + 2^{j(m-n)}\|u_j\|_B \leq C(\sigma_j^{(n)})^{-1},$$

for any $j \in \mathbf{Z}$. Let

$$s_j = \sum_{k=-\infty}^{j-1} u_k \quad \text{and} \quad t_j = \sum_{k=j}^{\infty} u_k.$$

One has $s_j \in A$, $t_j \in B$; let us show that

$$\sigma_j^{(n)}(\|s_j\|_A + 2^{j(m-n)}\|t_j\|_B)$$

is bounded. For $j < 0$, one gets

$$\begin{aligned}\|s_j\|_A &\leq \sum_{k=-\infty}^{j-1} \|u_k\|_A \leq C \sum_{k=-j+1}^{\infty} (\sigma_{-k}^{(n)})^{-1} \\ &\leq C \sum_{k=-j+1}^{\infty} 2^{kn}\sigma_{-k} \leq C2^{-jn}\sigma_{-j} \\ &\leq C(\sigma_j^{(n)})^{-1}\end{aligned}$$

and

$$\begin{aligned}\|t_j\|_B &\leq \sum_{k=j}^{\infty} \|u_k\|_B \leq C \sum_{k=j}^{\infty} 2^{-k(m-n)}(\sigma_k^{(n)})^{-1} \\ &\leq C \sum_{k=1}^{-j} 2^{k(m-n)}(\sigma_{-k}^{(n)})^{-1} + C \sum_{k=0}^{\infty} 2^{-k(m-n)}(\sigma_k^{(n)})^{-1} \\ &\leq C2^{-jm}\sigma_{-j} + C \leq C2^{-j(m-n)}(\sigma_j^{(n)})^{-1}.\end{aligned}$$

Now, if $j \geq 0$,

$$\|s_j\|_A \leq \sum_{k=-\infty}^{0} \|u_k\|_A + C \sum_{k=1}^{j-1} \|u_k\|_B$$

$$\leq C + C\sum_{k=1}^{j-1} 2^{-km}\sigma_k^{-1} \leq C(\sigma_j^{(n)})^{-1}$$

and, using the fact that $\overline{s}(\sigma^{-1}) < m$,

$$\|t_j\|_B \leq \sum_{k=j}^{\infty} \|u_k\|_B \leq C\sum_{k=j}^{\infty} 2^{-km}\sigma_k^{-1}$$
$$\leq C2^{-jm}\sigma_j^{-1}.$$

Now let $f \in [A, B]_{\sigma^{(n)},2^{j(m-n)},K}$. For any $j \in \mathbf{Z}$, there exist $s_j \in A$ and $t_j \in B$ such that $f = s_j + t_j$ and

$$\|s_j\|_A + 2^{j(m-n)}\|t_j\|_B \leq C(\sigma_j^{(n)})^{-1}.$$

Let us define u_j by

$$u_j = \begin{cases} s_{j+1} - s_j & \text{if } -j \in \mathbf{N} \\ t_{j+1} - t_j & \text{if } j \in \mathbf{N}_0. \end{cases}$$

One has $f = \sum_j u_j$ in A with $u_j \in B$ for any j. Moreover,

$$\|u_j\|_A \leq C(\sigma_j^{(n)})^{-1}$$

and

$$\|u_j\|_B \leq C2^{-j(m-n)}(\sigma_j^{(n)})^{-1},$$

which leads to the conclusion.

5.2. *Proof of Theorem 12*

Let $f \in \Lambda^\sigma$ and define

$$u_j = \begin{cases} 0 & \text{if } j \in \mathbf{Z},\ j > 1 \\ S_0 f & \text{if } j = 1 \\ \Delta_{-j} f & \text{if } j \in \mathbf{Z},\ j < 1. \end{cases}$$

The Bernstein inequalities imply that the series $\sum_j u_j$ converges in W_n^∞ and $u_j \in W_m^\infty$. It is easy to check that the sequence $\sigma_j^{(n)} J(2^{j(m-n)}, u_j)$ is bounded.

Let $f \in [W_n^\infty, W_m^\infty]_{\sigma^{(n)},2^{j(m-n)},J}$, so that $f = \sum_j u_j$ with $u_j \in W_m^\infty$; we can suppose that $u_j \in C^{m-1}(\mathbf{R}^d)$. Let $|\alpha| \leq n$; we have

$$\sum_{j=0}^{\infty} \|D^\alpha u_j\|_\infty \leq C\sum_{j=0}^{\infty} 2^{-j(m-n)}(\sigma_j^{(n)})^{-1} = C\sum_{j=0}^{\infty} 2^{-jm}\sigma_j^{-1}.$$

Since $\overline{s}(\sigma^{-1}) < m$, $\sum_{j=0}^{\infty} \|D^\alpha u_j\|_\infty$ converges. Moreover,

$$\sum_{j=-\infty}^{-1} \|D^\alpha u_j\|_\infty \leq C\sum_{j=-\infty}^{-1} (\sigma_j^{(n)})^{-1} = C\sum_{j=1}^{\infty} 2^{jn}\sigma_j,$$

and so $f \in C^n(\mathbf{R}^d)$ and $D^\alpha f \in L^\infty(\mathbf{R}^d)$ for any α such that $|\alpha| \leq n$.

Let $h \in \mathbf{R}^d$ be such that $|h| \leq 2^{-j_0}$ and $|\alpha| = n$; the Morrey inequality and the mean value theorem lead to the following inequalities:

$$\begin{aligned}\sum_{j=0}^{\infty} \|\Delta_h^{m-n} D^\alpha u_j\|_\infty &\leq C|h|^{m-n} \sum_{j=0}^{\infty} \|u_j\|_{W_m^\infty} \\ &\leq C2^{-j_0(m-n)} \sum_{j=0}^{\infty} 2^{-j(m-n)} (\sigma_j^{(n)})^{-1} \\ &\leq C2^{-j_0(m-n)} \leq C2^{nj_0}\sigma_{j_0}\end{aligned}$$

and

$$\begin{aligned}\sum_{j=-\infty}^{-1} \|\Delta_h^{m-n} D^\alpha u_j\|_\infty &= \sum_{j=-\infty}^{-j_0-1} \|\Delta_h^{m-n} D^\alpha u_j\|_\infty + \sum_{j=-j_0}^{-1} \|\Delta_h^{m-n} D^\alpha u_j\|_\infty \\ &\leq C \sum_{j=-\infty}^{-j_0-1} \|u_j\|_{W_n^\infty} + |h|^{m-n} \sum_{j=-j_0}^{-1} \|u_j\|_{W_m^\infty} \\ &\leq C \sum_{j=-\infty}^{-j_0-1} (\sigma_j^{(n)})^{-1} + C2^{-j_0(m-n)} \sum_{j=-j_0}^{-1} 2^{-j(m-n)} (\sigma_j^{(n)})^{-1} \\ &\leq C \sum_{j=j_0+1}^{\infty} 2^{jn}\sigma_j + C2^{-j_0(m-n)} \sum_{j=1}^{j_0} 2^{jm}\sigma_j \\ &\leq C2^{j_0 n}\sigma_{j_0},\end{aligned}$$

which implies

$$\sup_{|h| \leq 2^{-j_0}} \|\Delta_h^{m-n} D^\alpha f\|_\infty \leq C2^{nj_0}\sigma_{j_0};$$

hence the conclusion follows.

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