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Report SA-200

June 1999

RECENT ADVANCES IN THE DUAL ANALYSIS THEORY

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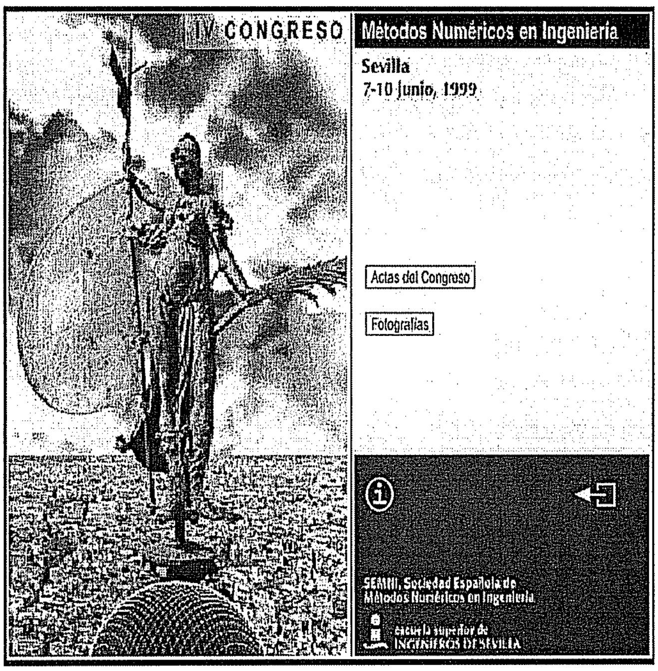

IV CONGRESO DE MÉTODOS NUMÉRICOS EN INGENIERÍA.

Organizadores: Sociedad Española de Métodos Numéricos en Ingeniería, Escuela Superior de Ingenieros de Sevilla. Celebración del congreso: 7 al 10 de junio de 1999 en Sevilla.

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Keywords : Dual analysis, error estimator, convergence.

Abstract *The dual analysis concept, which is one of the oldest methods of measuring the discretization error of finite element models, suffered by its lack of generality in its original version. In fact, severe restrictions were imposed to the boundary conditions, excluding any mixed problem.*

These restrictions, related to an analysis where stiffness plays a central role, can be avoided by using the proper reference energy, which is the total complementary energy. This leads to convergence curves which are always on the same side of the exact value and whose distance measures the quality of the analysis.

1 INTRODUCTION

During the early development of finite element analysis software, computational costs were so high that everybody was considering that it was sufficient to obtain almost any approximate solution of an elastic problem and was not motivated to evaluate the quality of the obtained solution. This situation is now completely reversed, and it is now accepted that the error analysis is an essential part of any reliable structural analysis.

However such an approach was present in very early works of Fraeijs de Veubeke and co-workers[1-6] , who proposed an error analysis based on two finite element models, one of which being of the displacement type, the second one being of the equilibrium type. The main drawback of this approach was the fact that in its initial presentation, it was restricted to two particular cases, namely, homogeneous prescribed displacements or zero applied loads. Because this frame was somewhat limited for practical applications where mixed conditions are frequently encountered, the dual analysis concept was progressively abandoned.

More recent investigations [7,8] led to the result that these limitations, due to the fact that the early dual analysis concept was developed from the point of view of upper and lower bounds of influence coefficients, are not essential, and that with a proper reformulation, the method applies with general boundary conditions.

In the early papers, however, dual bounds leading to a pair of curves converging to the same limit, i.e. the most attractive feature of dual analysis, were yet used in particular situations.

The purpose of the present paper is to present an approach where both aspects are kept in the largest frame, including completely general boundary conditions and even extending to material non linearity.

2 GENERAL NOTATIONS

In what follows, the classical frame of 3-d structural analysis will be adopted as a model, although our results are of a more general substance.

Let us consider a body with a bounded volume V, whose surface S is split in two parts S_1 and S_2. On S_1, prescribed displacements $\overline{u_i}$ are imposed. The applied loads are body forces f_i dV and surface traction t_i dS on S_2.

The equations are thus :

a) Compatibility equations

$$\varepsilon_{ij} = \frac{1}{2}(D_i u_j + D_j u_i) \text{ in V} \tag{1}$$
$$u_i = \overline{u_i} \qquad \text{on } S_1$$

b) Equilibrium equations

$$D_j \sigma_{ji} = 0 \quad \text{in V}$$
$$n_j \sigma_{ji} = t_i \quad \text{on } S_2 \tag{2}$$

c) Constitutive equations

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl} \tag{3}$$

Sufficient fixations are always assumed to be present, in order to ensure that the problem is elliptic.

Among all possible approximations of these equations, a special mention is due to pure models in which only equilibrium equations or only compatibility relations are approximated. The first case is referred as the displacement model, in which compatibility is exactly verified and the second one is known as the equilibrium model and characterized by an exact equilibrium achievement.

The dual analysis concept consists to compare results from both approaches, which have therefore to be analyzed in detail. The whole theory is based on three fundamental results that we will develop first.

3 THE DISPLACEMENT APPROACH

Admissible displacement fields are defined by the two following conditions

(i) They are of finite strain energy

$$E^U(u) = \frac{1}{2} \int_V C_{ijkl} \varepsilon_{ij}(u)\, \varepsilon_{kl}(u)\, dV < \infty \tag{4}$$

(ii) They verify the kinematic conditions on S_1

$$u_i = \overline{u_i} \ \text{on } S_1 \tag{5}$$

The set of admissible displacements will be noted U. Any difference between two admissible displacement fields will be called an admissible displacement variation,

$$\delta u_i = u_i^{(1)} - u_i^{(2)} \tag{6}$$

Consequently,

$$\delta u_i = 0 \ \text{on } S_1 \tag{7}$$

The set δU of admissible displacement variations is therefore a linear space. From condition (4), both δU and its translated U may be equipped by the energetic norm

$$\|u\| = \sqrt{2\,E^U(u)} \tag{8}$$

The displacement approach consists in finding, among all admissible displacement fields, the particular displacement u that minimises the total potential energy

$$E^E(u) = E^U(u) + E^P(u) \tag{9}$$

where the linear functional

$$E^P(u) = -\int_V f_i u_i \, dV - \int_{S_2} t_i u_i \, dS \tag{10}$$

is called *load potential.* Varying functional (9) by respect to u leads to the equilibrium equations. In other words, the exact equilibrated solution u of the elastic problem is the only one that verifies the condition

$$\delta E^E(u; \delta u) = 0 \tag{11}$$

for any $\delta u \in \delta U$

Rayleigh-Ritz approximations and, in particular, conforming finite element methods, consist to select some subset U_h of U containing displacements u_h and, consequently, a subspace δU_h of displacement variations. In a *strict* displacement model, the kinematic conditions

$$u_{hi} = \overline{u_i} \tag{12}$$

have to be verified *exactly*. In other words, u_h has to be *strictly* admissible. A Rayleigh-Ritz approximation is obtained from the condition

$$\delta\, E^E(u_h; \delta\, u_h) = 0 \tag{13}$$

for any $\delta\, u_h \in \delta\, U_h$, which is a weak form of (11). But it has to be noted that for the following developments, it is *not* necessary that u_h should be a Rayleigh-Ritz approximation. Only strict admissibility is required.

Let thus u_h be a strictly admissible approximate displacement, and let us define the *displacement error*

$$\Delta\, u = u_h - u \tag{14}$$

where u is the exact solution. Δu is obviously an admissible displacement variation, so that, in the development

$$E^E(u_h) = E^E(u + \Delta\, u) = E^E(u) + \delta\, E^E(u; \Delta\, u) + \frac{1}{2}\delta^2\, E^E(\Delta\, u) \tag{15}$$

where

$$\delta^2\, E^E(\Delta\, u) = \int_V C_{ijkl}\, \varepsilon_{ij}(\Delta\, u)\, \varepsilon_{kl}(\Delta\, u)\, dV = \|\Delta\, u\|^2 \tag{16}$$

one concludes from (11) that

$$\delta\, E^E(u; \Delta\, u) = 0 \tag{17}$$

As a consequence

$$E^E(u_h) = E^E(u) + \frac{1}{2}\|\Delta\, u\|^2, \tag{18}$$

from which the displacement error verifies

$$\|\Delta u\|^2 = 2\left[E^E(u_h) - E^E(u)\right] \tag{19}$$

This is the first fundamental result.

4 THE EQUILIBRIUM APPROACH

Statically admissible stress fields are defined by the following conditions

(i) Their complementary energy is finite

$$E^V(\sigma) = \frac{1}{2}\int_V C^{-1}_{ijkl}\sigma_{ij}\sigma_{kl}dV < \infty \tag{20}$$

(ii) They satisfy the equilibrium equations

$$\begin{cases} D_j\sigma_{ji} + \bar{f_i} = 0 \text{ in } V \\ n_j\sigma_{ji} = \bar{t_i} \quad \text{on } S_2 \end{cases} \tag{21}$$

(iii) On possible material discontinuity interfaces, if indexes + and - denote the two sides of the interface,

$$(n_j\sigma_{ji})_+ + (n_j\sigma_{ji})_- = 0 \tag{22}$$

The set of statically admissible stress fields will be noted E. Any difference between two such fields will be called a *statically admissible stress variation*,

$$\delta\,\sigma_{ij} = \sigma_{ij}^{(2)} - \sigma_{ij}^{(1)} \tag{23}$$

This implies

$$\begin{cases} D_j\delta\,\sigma_{ji} = 0 \text{ in } V \\ n_j\delta\,\sigma_{ji} = 0 \text{ on } S_2 \end{cases} \tag{24}$$

condition (22) being maintained. The space of admissible stress variations will be noted δE. Both E and δE may be equipped with the energetic norm

$$|||\sigma||| = \sqrt{2\,E^{V}(\sigma)} \tag{25}$$

The equilibrium approach consists in finding, among all admissible stress fields, the particular one that minimizes the total complementary energy

$$E^{C}(\sigma) = E^{V}(\sigma) + E^{Q}(\sigma) \tag{26}$$

where

$$E^{Q}(\sigma) = -\int_{S_1} n_j \sigma_{ji} \overline{u_i}\, d\,S \tag{27}$$

is the *potential of prescribed displacements*. This principle leads to compatibility equations. So, the exact solution σ verifies the condition

$$\delta\, E^{C}(\sigma; \delta\, \sigma) = 0 \tag{28}$$

for each $\delta\sigma \in \delta E$.

Rayleigh-Ritz approximations consist to select a subset E_h of E from which naturally derives a subspace δE_h of δE, and to find $\sigma_h \in E_h$ such that

$$\delta\, E^{C}(\sigma_h; \delta\, \sigma_h) = 0 \tag{29}$$

whatever be $\delta\sigma_h \in \delta E_h$. In a *strict* equilibrium model, equations (21) and (22) have to be satisfied exactly by each element of E_h. For displacement models, this is the only required property, and it is by no means necessary that σ_h should be Rayleigh-Ritz approximations.

Let thus σ_h be any strictly admissible approximate stress field. The *stress error*

$$\Delta\sigma = \sigma_h - \sigma \tag{30}$$

is then an admissible stress variation. From (28), the development

$$E^{C}(\sigma_h) = E^{C}(\sigma) + \delta\, E^{C}(\sigma; \Delta\sigma) + \frac{1}{2}\delta^2\, E^{C}(\Delta\sigma) \tag{31}$$

with

$$\delta^2 E^C(\Delta\sigma) = \int_V C^{-1}_{ijkl}\, \Delta\, \sigma_{kl}\, dV = |||\Delta\sigma|||^2 \tag{32}$$

reduces to

$$|||\Delta\sigma|||^2 = 2\left[E^C(\sigma_h) - E^C(\sigma)\right], \tag{33}$$

which constitutes the second fundamental result.

5 SOME PROPERTIES OF THE EXACT SOLUTION

Displacement and equilibrium approaches lead both to the same exact solution :

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}(u),$$

so that

$$E^U(u) = \frac{1}{2}\int_V C_{ijkl}\varepsilon_{ij}(u)\varepsilon_{kl}(u)\, dV = \frac{1}{2}\int_V \sigma_{ij}\varepsilon_{ij}(u)\, dV \tag{34}$$

$$= \frac{1}{2}\int_V C^{-1}_{ijkl}\sigma_{ij}\sigma_{kl}\, dV = E^V(\sigma)$$

This also implies

$$E^U(u) + E^V(\sigma) = \int_V \sigma_{ij}\varepsilon_{ij}(u)\, dV \tag{35}$$

Now, by an integration by parts,

$$\int_V \sigma_{ij}\varepsilon_{ij}(u)\, dV = \int_S n_j\sigma_{ji}u_i\, dS - \int_V u_i D_j\sigma_{ji}\, dV$$

and, from equilibrium or boundary conditions, this is equivalent to

$$\int_V \sigma_{ij}\varepsilon_{ij}(u)\, dV = \int_{S_1} n_j\sigma_{ji}\overline{u_i}\, dS + \int_{S_2} t_i u_i\, dS + \int_V f_i u_i\, dV = E^Q(\sigma) + E^P(u)$$

Using result (35), one thus obtains

$$E^{U}(u)+E^{V}(\sigma)=-(\ E^{Q}(\sigma)+E^{P}(u)\ ),$$

which is equivalent to

$$E^{C}(\sigma)=-E^{E}(u) \tag{36}$$

This is the third fundamental result.

6 UPPER AND LOWER BOUNDS OF THE TOAL COMPLEMENTARY ENERGY

Relations (19) and (33) imply that both E^{E} and E^{C} have a minimum value at the solution. Consequently, if u_h and σ_h are approximate solutions obtained from a displacement model and an equilibrium model respectively, one has from (36)

$$-E^{E}(u_h)\le -E^{E}(u)=E^{C}(\sigma)\le E^{C}(\sigma_h) \tag{37}$$

In other words, if one agrees to extend the total complementary energy to displacement approximations by setting

$$E^{C*}(u_h)=-E^{E}(u_h) \tag{38}$$

one obtains

$$E^{C*}(u_h)\le E^{C}_{exact}\le E^{C}(\sigma_h) \tag{39}$$

that is, a lower bound for any admissible displacement field and an upper bound for any statically admissible stress field.

Furthermore, suppose that successive subspaces of admissible displacements $U_{h_1}\subset U_{h_2}\subset U_{h_3}$, and successive subspaces of statically stress fields $E_{h_1}\subset E_{h_2}\subset E_{h_3}$ are chosen and that a Rayleigh-Ritz process is performed in each case, leading to approximate solutions u_{h_1},u_{h_2},u_{h_3}, and $\sigma_{h_1},\sigma_{h_2},\sigma_{h_3}$, one has

$$E^{C*}(u_{h_i}) = -\inf_{v \in U_{h_i}} E^E(v) \leq -\inf_{v \in U_{h_{i+1}}} E^E(v) = E^{C*}(u_{h_{i+1}}) \tag{40}$$

and

$$E^C(\sigma_{h_i}) = \inf_{\tau \in E_{h_i}} E^C(\tau) \geq \inf_{\tau \in E_{h_{i+1}}} E^C(\tau) = E^C(\sigma_{h_{i+1}}) \tag{41}$$

it means that an increasing sequence is defined by the displacement models, while the equilibrium models lead to a decreasing sequence, both having the same limit. Plotted on a graph, the two corresponding curves give useful results concerning the quality of both analyses.

7 ENERGETIC MEASURE OF THE CUMULATED ERROR

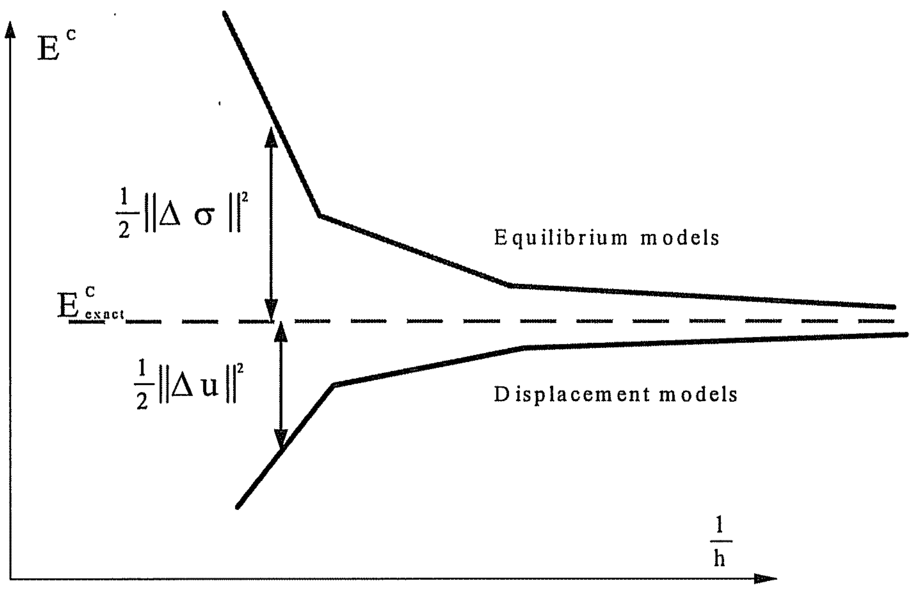

Figure 1 : Convergence and error of both types of approximations

From these curves, and assuming that the common asymptotic value E^C_{exact} can be determined, it follows from (19) and (33) that :

$$\|\Delta\, u\|^2 = 2\left[E^C_{exact} - E^{C*}(u_h)\right] \tag{42}$$

and

$$\|\Delta\, \sigma\|^2 = 2\left[E^C(\sigma_h) - E^C_{exact}\right], \tag{43}$$

so that convergence can be verified in an intuitive way (fig. 1).

Now, determining E^C_{exact} implies a sufficient number of finite elements analyses of both type, which is too demanding for an estimation error. But if one model of each type is available, an overestimation of both errors is given by

$$\left.\begin{matrix}\|\Delta\, u\| \\ \|\Delta\, \sigma\|\end{matrix}\right\} \le \left[\|\Delta\, u\|^2 + \|\Delta\, \sigma\|^2\right]^{1/2} = \left\{2\left[E^C(\sigma_h) - E^{C*}(u_h)\right]\right\}^{1/2} \tag{44}$$

This error measure only requires very simple computations from the results. If a *relative* error is need, one may compare the last member of (44) to the energetic norm of the solution

$$\left[2\,E^U(u)\right]^{1/2} = \left[2\,E^V(\sigma)\right]^{1/2} \approx \left[E^U(u_h) + E^V(\sigma_h)\right]^{1/2}$$

so as to obtain the ratio

$$R.E = \left\{\frac{2\left[E^C(\sigma_h) - E^{C*}(u_h)\right]}{E^U(u_h) + E^V(\sigma_h)}\right\}^{1/2} \tag{45}$$

Note that these results only require the admissibility of both approaches.

8 ENERGETIC DISTANCE BETWEEN DISPLACEMENT AND EQUILIBRIUM APPROXIMATIONS

Adopting the notation

$$\sigma_{ij}(v) = C_{ijkl}\varepsilon_{kl}(v) \tag{46}$$

and introducing the distance between two stress fields

$$d^2(\sigma_1,\sigma_2) = \int_V C^{-1}_{ijkl}(\sigma_1-\sigma_2)_{ij}(\sigma_1-\sigma_2)_{kl}\, dV\,, \tag{47}$$

let us consider an admissible displacement field u_h and a statically admissible stress field. One has

$$d^2(\sigma_h,\sigma(u_h)) = d^2(\sigma+\Delta\,\sigma,\sigma+\sigma(\Delta\,u)) = \left\|\left|\Delta\,\sigma-\sigma(\Delta\,u)\right\|\right|^2$$

$$= \left\|\left|\Delta\,\sigma\right\|\right|^2 + \left\|\Delta\,u\right\|^2 - 2\int_V \Delta\,\sigma_{ij}\varepsilon_{ij}(\Delta\,u)\,dV$$

The last term vanishes, as an integration by parts leads to

$$\int_V \Delta\,\sigma_{ij}\varepsilon_{ij}(\Delta\,u)\,dV = \int_{S_1} n_j\Delta\,\sigma_{ji}\Delta\,u_i\,dS + \int_{S_2} n_j\Delta\,\sigma_{ji}\Delta\,u_i dS - \int_V (D_j\Delta\,\sigma_{ji})\Delta\,u_i\,dV$$

and the admissibility conditions are

$$\Delta\,u_i = 0 \text{ on } S_1,\ n_j\Delta\,\sigma_{ji} = 0 \text{ on } S_2,\ D_j\Delta\,\sigma_{ji} = 0 \text{ in V.}$$

Finally,

$$d^2(\sigma_h,\sigma(u_h)) = \left\|\left|\Delta\,\sigma\right\|\right|^2 + \left\|\Delta\,u\right\|^2 = 2\left[E^C(\sigma_h) - E^{C^*}(u_h)\right] \tag{48}$$

So, the error measure (44) is nothing than the squared distance between the two approximations.

9 CONVENTIONAL AND UNCONVENTIONAL DUAL ANALYSIS PROCEDURES

The *conventional* dual analysis consists in treating the same problem by two ways, the first using a kinematically admissible model and the second, a statically admissible one. A Rayleigh-Ritz in performed in both cases, so that both approaches do converge when the mesh is refined. The error measure (44) then converges to zero. It is therefore possible not only to detect too coarse meshes but also to ascertain that the solution is a good one. It is the direct generalization of Fraeijs de Veubeke views to general boundary conditions. The main drawback of this approach is the necessity of a double analysis.

Owing to the fact that the above error measure holds under the only condition that *admissible* fields are used, it is possible to define *unconventional* dual analysis schemes, where the tested fields are not necessarily obtained from a Rayleigh-Ritz process. This class of methods contains the Ladevèze-Oden approach[10-13] where, after a classical displacement analysis, a statically admissible stress field is generated in a post-processor scheme. In complete agreement with the previous considerations, the distance between the displacement and the equilibrium approaches is used as an error evaluation, under the name of "error on the constitutive relations". It is of course an upper bound of the error of the displacement approach, thus on the safe side. The only drawback consists in the fact that the convergence of the equilibrium approach is not guaranteed, so that the corresponding error measure does not necessarily tend to zero when the mesh is refined. So, this measure is able to detect too coarse meshes, but not to ascertain the good quality of a solution. This property will be illustrated in example 2.

10 FURTHER GENERALIZATION

Let us now consider the case of materials exhibiting *nonlinear* constitutive equations. The strain energy density $W(\varepsilon)$ is supposed to verify the ellipticity condition

$$\frac{\partial^2 W}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} \delta \varepsilon_{ij} \delta \varepsilon_{kl} > 0 \tag{49}$$

whatever be $\delta \varepsilon_{ij} \neq 0$. The stress energy density $\Phi(\sigma)$ is now defined as

$$\Phi(\sigma) = \max_{\varepsilon} \left[\sigma_{ij} \varepsilon_{ij} - W(\varepsilon) \right] \tag{50}$$

and is also supposed to be positive definite. Displacement and equilibrium approaches still hold, with

$$E^{U}(u) = \int_V W(\varepsilon(u))\, dV, \quad E^{V}(\sigma) = \int_V \Phi(\sigma)\, dV \tag{51}$$

and for admissible u_h and σ_h,

$$E^{E}(u_h) > E^{E}(u), \quad E^{C}(\sigma_h) > E^{C}(\sigma)$$

as in the linear case. The third fundamental result (36) remains valid because from (50)

$$E^U(u)+E^V(\sigma)=\int_V[W(\varepsilon)+\Phi(\sigma)]\,dV=\int_V\sigma_{ij}\varepsilon_{ij}\,dV\,,$$

i.e. relation (35). Finally, upper and lower bounds of the (extended) total complementary energy is still valid with non linear materials. The quantity $E^C(\sigma_h)-E^{C*}(u_h)$ is then a non linear function of the errors.

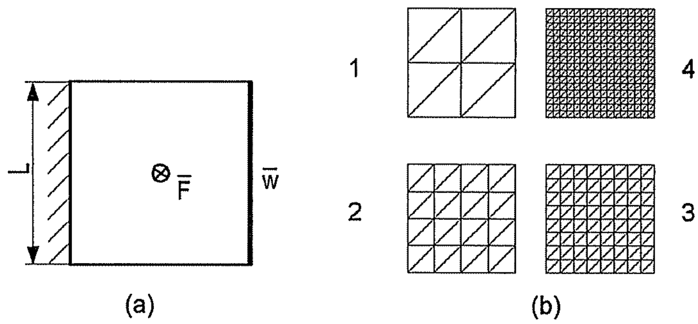

Figure 2 : Plate bending, boundary conditions (a), meshes (b)

11 NUMERICAL EXAMPLES

11.1 Plate bending problem

This first example is designed to show the conventional dual analysis of a plate problem with non homogeneous boundary conditions. In such a case, the original dual analysis, based on strain energy comparisons, does not apply.

Consider a square plate loaded at its centre. It is clamped on one edge, and on the opposite one, transversal displacement is imposed (fig. 2a). Numerical data are : edge length L = 10, thickness t = 0.1, Young's modulus E = 2.05 x 10^{11}, Poisson's ratio ν = 0,3, point load $\overline{F}$ = -1000, prescribed displacement $\overline{w}$ = -0.01.

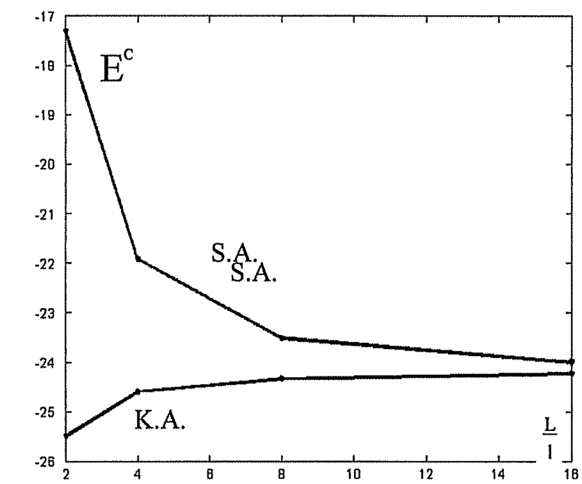

Figure 3 : Convergence curve for the plate bending problem

The displacement approach is performed with the HCT conforming triangle with three degrees of freedom (D.O.F.) per node[14]. For the equilibrium model, use is made of the Morley equilibrium triangle with 1 D.O.F. per node and 1 D.O.F. per edge[15].

As shown in fig. 2b, an initial coarse mesh is created with 8 triangles. It is uniformly refined so as to obtain meshes 2,3 and 4. Figure 3 shows the convergence curves in terms of L/h, h being the element size. In this figure, S.A. and K.A. mean statically admissible and kinematically admissible respectively. A very fair convergence is observed.

11.2 Plane stress problem

The second example illustrates the non conventional dual analysis of a plane stress problem by the Ladevèze-Oden procedure. The structure is the same as in the preceding example but is subjected to plane stress conditions (fig. 4a). The prescribed displacement are $\bar{u} = 0.01$ and the imposed traction $\bar{t} = -1.0 \times 10^8$.

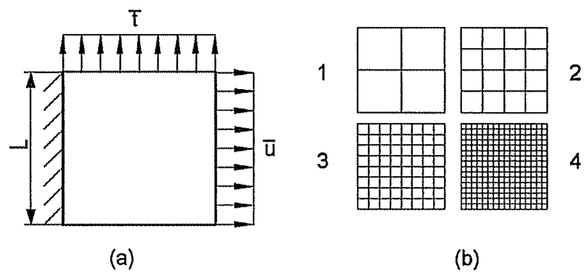

Figure 4 : Plane stress problem, boundary conditions (a), meshes (b)

The displacement approach is performed with four-nodes conforming elements. To each conforming element solution u_h is associated a statically admissible stress field σ_h, by use of the method proposed by Ladevèze[12].

An initial mesh is created with quadrilaterals. It is uniformly refined so as to obtain meshes 2,3 and 4, fig. 2b. Convergence curves are plotted on fig. 5. As can be seen, a fair convergence is obtained with the displacement approach (curve K.A.). In counterpart, the successive statically admissible stress fields (curve S.A.) are far from the quasi-converged displacement solutions. There is in fact no guarantee that these somewhat arbitrary stress fields finally converge, so that the estimated error is still great with the most refined mesh. This is always the case with non conventional dual analyses, which systematically overestimate the error.

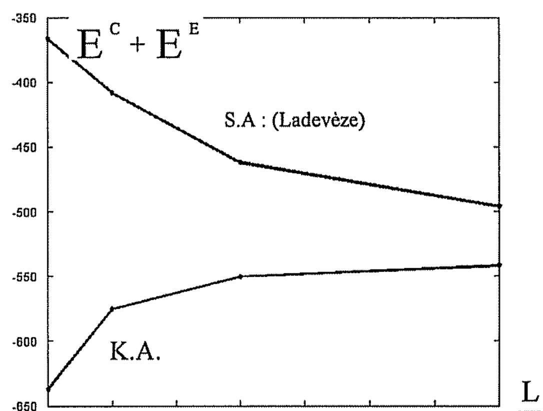

Figure 5 ; Convergence curves for the plane stress problem

12 CONCLUSIONS

The dual analysis concept, which in its original version, was restricted to problems where one type of boundary conditions are homogeneous, is now extended to the general case of *any* boundary conditions. This extension is quite useful, since it is well known that mixed problems lead to the less regular solutions and therefore are the most interesting field for error evaluations.

The Ladevèze-Oden approach, which is sometimes considered as unclassifiable[16] here finds its natural frame as a non conventional dual analysis.

Finally, dual analysis also holds with non linear materials[17], a property that could be useful in many applications such as magneto-static, as an example.

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