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On some stability properties of the discretization of damped propagation of shallow-water inertia-gravity waves on the Arakawa B-grid

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Abstract

We investigate a problem mentioned by Deleersnijder and Campin (Deleersnijder, E., & Campin, J. -M. (1993). *Ocean Modelling 97,* 2), who looked at the problem of the discretization of inertia-gravity waves in the staggered B-grid. We generalize this study by adding diffusion. General stability conditions are found with the help of Miller's theorem, and the paradox found in Deleersnijder and Campin (Deleersnijder, E., & Campin, J. -M. (1993). *Ocean Modelling 97,* 2) is discussed. It is argued that it stems from inappropriate application of boundary conditions in conjunction with a Coriolis force treatment which could produce mechanical work.

1. Introduction

Inertia-gravity waves (e.g. Rossby, 1937) have been extensively studied (e.g. Cahn, 1945; Obukhov, 1949; Washington, 1964; Blumen, 1972; Schoenstadt, 1977; Hsieh and Gill, 1984) because they occur and participate in numerous processes: large scale atmospheric and oceanic motions reach the geostrophic equilibrium balance by means of transient inertia-gravity waves and the dynamics of tides and storm surges is dominated by the propagation of external inertia-gravity waves, which are related to the evolution of the sea surface. In strongly stratified seas, the displacement of density surfaces also leads to internal inertia-gravity waves. The propagation of inertia-gravity waves has thus been studied from a physical point of view, but it is also of paramount importance that the numerical scheme utilized to simulate these waves behaves properly. In most numerical resolutions, the horizontal grid size is determined by computer resources considerations (one should not forget that the scheme analyzed here is generally a subset of the equations solved in real applications), and a small sub-grid scale parameterization is introduced in most models by a diffusion law. We are thus confronted with external damped linear inertia-gravity waves, also called damped Poincaré waves which are governed by the following equations:

$$\frac{\partial u}{\partial t} = fv - g\frac{\partial \eta}{\partial x} + \mathscr{A}\nabla^2 u, \tag{1}$$

$$\frac{\partial v}{\partial t} = -fu - g\frac{\partial \eta}{\partial y} + \mathscr{A}\nabla^2 v, \tag{2}$$

$$\frac{\partial \eta}{\partial t} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0, \tag{3}$$

where t is the time, u and v are the horizontal velocity components in the Cartesian x and y directions, respectively, η the sea surface elevation, $\mathscr{A}$ the horizontal Laplacian diffusion coefficient, ∇ the classical horizontal nabla operator, g the gravitational acceleration, f the constant Coriolis frequency and h the unperturbed depth of the sea, which we suppose is constant here. The governing equations are generic equations, which can also be used to describe the damped internal inertia-gravity waves, provided that one interprets η, u, v and h as equivalent quantities related to the particular internal mode considered (e.g. LeBlond and Mysakm, 1978). Various discretized forms of Eqs. (l)-(3) have been examined (Schoenstadt and Williams, 1976; Arakawa and Lamb, 1977; Schoenstadt, 1977; Batteen and Han, 1981). These studies focus on the space differencing aspects. When time differencing is also considered, it is customary to restrict the analysis to pure gravity waves ($f = 0$). In this case, the stability conditions of the numerical scheme are readily obtained, but sometimes lead to practical conditions which may be wrong when an extremely small Coriolis parameter is introduced (e.g. Beckers and Deleersnijder, 1993). Since free surface effects are now taken into account in classical B-grid models (e.g. the free surface implementation in Modular Ocean Model (MOM) by Killworth et al., 1991), the stability analysis should be as complete as possible. To ensure that stability conditions are relevant, one should thus always consider the most complete set of discretized equations one is able to deal with from the stability analysis point of view. Only very few authors have performed such an analysis and sometimes found drastic changes in the overall stability behavior of the full scheme compared to the stability conditions for a subset of the scheme (e.g. Kasahara, 1965; Cushman-Roisin, 1984; Beckers and Deleersnijder, 1993; Thacker, 1977, 1978). Here we will carry out the complete analysis for a system with diffusion, Coriolis force and gravity waves.

2. The FBTCS numerical scheme

Numerical propagation of Poincaré waves strongly depends on the distribution of *u, v* and *η* over the grid points. Here, only the two most widely used numerical lattices are mentioned, namely the B and C grids, according to Arakawa's classification (Arakawa and Lamb, 1977) (Figs. 1 and 2). Using the following notations:

Fig. 1. *Arakawa B-Grid with positions of η points (●) in the center of the grid box and u and v points (◊) at the corners.*

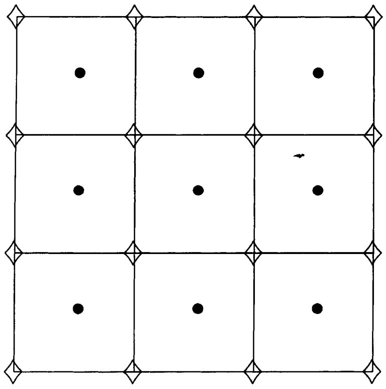
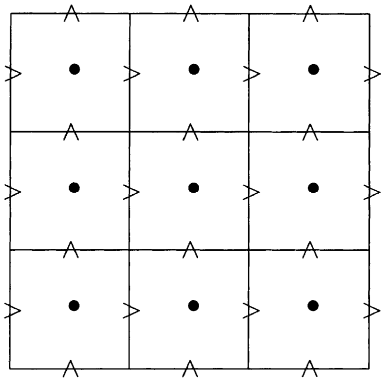


***Fig. 2**. Arakawa C-Grid with positions of η points (●) in the center of the grid box and u (>) and v points (Λ) at the interfaces.*



$$a_{n_t,n_x,n_y} = a(t,x,y) = a(n_t\,\Delta t, n_x\,\Delta x, n_y\,\Delta y), \tag{4}$$

$$(\delta_x a)_{n_t,n_x,n_y} = \frac{a_{n_t,n_x+1/2,n_y} - a_{n_t,n_x-1/2,n_y}}{\delta x}, \tag{5}$$

$$\left(\bar{a}^x\right)_{n_t,n_x,n_y} = \frac{a_{n_t,n_x+1/2,n_y} + a_{n_t,n_x-1/2,n_y}}{2}, \tag{6}$$

the FBTCS [1] scheme reads for the B-Grid:

$$(\delta_t\eta)_{n_t+1/2,n_x,n_y} + h\left[\left(\delta_x\bar{u}^y\right)_{n_t,n_x,n_y} + \left(\delta_y\bar{v}^x\right)_{n_t,n_x,n_y}\right] = 0, \tag{7}$$

$$(\delta_t u)_{n_t+1/2,n_x+1/2,n_y+1/2} - f\alpha v_{n_t+1,n_x+1/2,n_y+1/2} - f(1-\alpha)v_{n_t,n_x+1/2,n_y+1/2} = -g\left(\delta_x \bar{\eta}^y\right)_{n_t+1,n_x+1/2,n_y+1/2} + \mathscr{A}\left(\delta_x\delta_x u + \delta_y\delta_y u\right)_{n_t,n_x+1/2,n_y+1/2}, \quad (8)$$

$$(\delta_t v)_{n_t+1/2,n_x+1/2,n_y+1/2} + f\alpha u_{n_t+1,n_x+1/2,n_y+1/2} + f(1-\alpha)u_{n_t,n_x+1/2,n_y+1/2} = -g\left(\delta_y \bar{\eta}^x\right)_{n_t+1,n_x+1/2,n_y+1/2} + \mathscr{A}\left(\delta_x\delta_x v + \delta_y\delta_y v\right)_{n_t,n_x+1/2,n_y+1/2}. \quad (9)$$

The centered space differencing is the same as in Arakawa and Lamb (1977); but the time stepping of the gravity wave part is similar to the forward-backward scheme (Mesinger, 1973). The Coriolis force is treated by using a weighting between explicit and implicit treatment ($\alpha \in [0, 1]$), whereas the diffusion is discretized by an explicit scheme (because diffusion is generally small and merely a filtering process). Here we assumed furthermore that horizontal diffusion is isotropic, but the following calculations could be easily modified to account for two different diffusion coefficients $\mathscr{A}_x, \mathscr{A}_y$ in each direction of the numerical grid. In practice, this is simply achieved by replacing the parameter $\mathscr{A}\,\Delta t(\Delta x^{-2} + \Delta y^{-2})$ by $\Delta t(\mathscr{A}_x \Delta x^{-2} + \mathscr{A}_y \Delta y^{-2})$ in the following.

The von Neumann stability analysis is now applied to the discretized equations (7)-(9) by defining a spatially periodic solution

$$(\eta, u, v) = (\mathscr{E}(t), \mathscr{U}(t), \mathscr{V}(t))\,\mathrm{e}^{\mathrm{i}(k_x x + k_y y)} = \boldsymbol{x}_{n_t}\,\mathrm{e}^{\mathrm{i}(n_x 2\theta_x + n_y 2\theta_y)}, \quad (10)$$

where θ_x and θ_y are linked to the wavenumbers k_x and k_y by $0 \le 2\theta_x = k_x \Delta x \le \pi$ and $0 \le 2\theta_y = k_y \Delta y \le \pi$. The periodic solution (10) is then introduced into the system of discretized equations; for the B-grid this leads to the expression

$$\boldsymbol{A}\boldsymbol{x}_{n_t+1} + \boldsymbol{B}\boldsymbol{x}_{n_t} = 0, \quad (11)$$

from which one can compute the classical amplification matrix $\boldsymbol{H} = -\boldsymbol{A}^{-1}\boldsymbol{B}$ and its characteristic equation $\det(\boldsymbol{H} - \rho\boldsymbol{I}) = 0$ Upon defining

$$F = (f\,\Delta t)^2, \quad (12)$$

$$(c_x^2, c_y^2) = gh\left(\frac{\Delta t^2}{\Delta x^2}, \frac{\Delta t^2}{\Delta y^2}\right), \quad (13)$$

$$(d_x, d_y) = 2\mathscr{A}\left(\frac{\Delta t}{\Delta x^2}, \frac{\Delta t}{\Delta y^2}\right), \quad (14)$$

$$D = d_x \sin^2\theta_x + d_y \sin^2\theta_y, \quad (15)$$

$$G = c_x^2 \sin^2\theta_x \cos^2\theta_y + c_y^2 \sin^2\theta_y \cos^2\theta_x, \quad (16)$$

the characteristic equation therefore reads

$$a_3\rho^3 + a_2\rho^2 + a_1\rho + a_0 = 0, \quad (17)$$

where

$$a_3 = 1 + F\alpha^2, \quad (18)$$

$$a_2 = -3a_3 + 4(D + G) + 2F\alpha = -a_3(3 - b_2), \quad (19)$$

$$a_1 = 3a_3 - 4D(2 - D) - F(4\alpha - 1) - 4G(1 - 2D) = a_3(3 - b_1), \quad (20)$$

$$a_0 = -a_3 + 4D(1 - D) + F(2\alpha - 1) = -a_3(1 - b_0), \quad (21)$$

which defines certain parameters b_0, b_1, b_2 used later.

Equations describing damped inertia-gravity waves do not contain phenomena which increase in time. This implies that the von Neumann stability condition is $|p| \leq 1$.

In the following section, we assume that time integration occurs forward in time for a non-negative diffusion coefficient, which ensures that all discretization parameters are non negative, so that $G \geq 0$, $D \geq 0$, $F \geq 0$.

3. General conditions on the discretization constants

To ensure stability, one must make sure that none of the solutions of Eq. (17) has an amplitude larger than 1. This can be translated into conditions applied to the discretization constants by Miller's theorem (Miller, 1971; Beckers, 1999), but the conditions are still intricated and sometimes redundant. In any case, the coefficients a_0, a_1, a_2, a_3 vary simultaneously when changing the wavenumber, which makes analysis difficult. A necessary stability condition is readily obtained by analyzing long waves ($\theta_x = \theta_y = 0$) and using the necessary condition $|a_0| \leq |a_3|$. So we see immediately that the scheme for the Coriolis term must be at least semi-implicit:

$$F(2\alpha - 1) \geqslant 0. \tag{22}$$

In the following, we assume that Eq. (22) is satisfied when the other conditions are deduced. Since $|a_0| \leq |a_3|$ is necessary, one must have $0 \leq b_0 \leq 2$, but with the definition of b_0, the fact that $4D\ (1 - D)$ is never larger than 1 and that $\alpha \in [0, 1]$, one can show that

$$b_0 \leqslant 1. \tag{23}$$

Since $b_0 = 0$ for $a_3 = -a_0$, Miller's theorem requires in principle for two cases to be distinguished.

3.1. *Case for which* $b_0 = 0$

This case should be treated separately from the case $b_0 > 0$, but lengthy calculations show that the situation can only be present when

$$DG = 0. \tag{24}$$

So to allow b_0 to be zero, either D or G must be zero. These cases will be treated even for $b_0 > 0$ in a later section. If D and G are not zero, b_0 cannot be zero and the strict inequality

$$4D(1 - D) + F(2\alpha - 1) > 0 \tag{25}$$

must hold, which provides an upper bound for the diffusion discretization constants:

$$D < \frac{1}{2}\left(1 + \sqrt{1 + F(2\alpha - 1)}\right), \tag{26}$$

when G is not zero. It appears however that D reaches its maximum value for $\theta_x = \theta_y = \pi$, where $G = 0$. Indeed, as we will see later, this implies that the stability limit does not correspond to strict inequality but simple inequality.

3.2. *Case for which* $b_0 > 0$

We have of course the necessary stability condition $b_0 > 0$, corresponding to (26). So Miller's theorem also requires that

$$|a_3 a_1 - a_0 a_2| \leqslant a_3^2 - a_0^2, \tag{27}$$

$$|a_3 a_2 - a_0 a_1| \leqslant |a_3 a_1 - a_0 a_2| + a_3^2 - a_0^2. \tag{28}$$

In terms of b_0, b_1 and b_2 this reads

$$|3b_0 - b_1 + b_2 - b_0b_2| \leqslant b_0(2 - b_0), \tag{29}$$

$$|-3b_0 - b_1 + b_2 + b_0b_1| \leqslant |3b_0 - b_1 + b_2 - b_0b_2| + b_0(2 - b_0), \tag{30}$$

which can be further simplified by defining *x, y, z* by

$$(x, y, z) = (1 - b_0, 3 - b_1, (b_2 - b_1)/b_0). \tag{31}$$

In this case, stability is ensured if and only if

$$|y + xz| \leqslant 1 + x, \tag{32}$$

$$|z - y| \leqslant |y + xz| + 1 + x. \tag{33}$$

This delimits a portion in the *x, y, z* space for which stability is ensured. For fixed *x*, this delimits a region formed by six line segments:

$$y \leqslant 1 + x(1 - z), \tag{34}$$
$$y \geqslant -1 - x(1 + z), \tag{35}$$
$$2y \geqslant (1 - x)z - (1 + x) \text{ for } y + xz \geqslant 0, \tag{36}$$
$$z \geqslant -1 \quad \text{for } y + xz \geqslant 0, \tag{37}$$
$$z \leqslant 1 \quad \text{for } y + xz \leqslant 0, \tag{38}$$
$$2y \leqslant z(1 - x) + (1 + x) \quad \text{for } y + xz \leqslant 0. \tag{39}$$

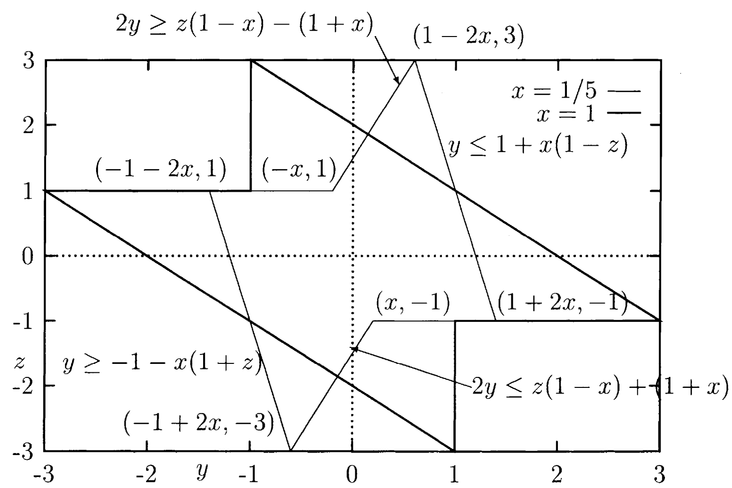
Or, equivalently, the points (y, z) inside the polygon formed by the following sequence of points, give rise to a stable scheme:

$$(-1 - 2x, 1)(-x, 1)(1 - 2x, 3)(1 + 2x, -1)(x, -1)(-1 + 2x, -3). \tag{40}$$

This is depicted in Fig. 3; since we supposed $x < 1$ ($b_0 > 0$), this domain is bounded by (y, z) ϵ]-3,3[x [-3, 3].

In principle, this allows to establish constraints on the discretization constants which ensure stability. However, such calculations are cumbersome and simpler necessary and sufficient conditions can be considered.

Fig. 3. *Stability domain for two different values of x (x = 0.2, x = 1), as functions of y and z. Points inside the polygon correspond to a stable scheme, and points outside, to an unstable scheme.*



Necessary stability conditions are readily obtained:

$$|y| \leqslant 1 + 2x, \tag{41}$$
$$|z| \leqslant 3. \tag{42}$$

In terms of the original variables and retaining only the most stringent conditions, one has the necessary conditions (22), (26) and

$$8DG \leqslant 4D^2 + F + 4G, \tag{43}$$
$$16D + 4G \leqslant 6 + 12D^2 + 8GD + F(6\alpha^2 - 8\alpha + 3), \tag{44}$$
$$16D^2 + 8DG \leqslant 16D + 4F(2\alpha - 1). \tag{45}$$

Inequality (45) is actually a stronger requirement than (26), so that the former supersedes the latter.

Sufficient stability conditions [2] are also easily obtained:

$$|y + xz| \leqslant 1 + x, \tag{46}$$
$$|z| \leqslant 1. \tag{47}$$

Practical sufficient conditions can be deduced from Eqs. (46) and (47), in conjunction with Eqs. (22) and (26) and by adequately using the fact that $b_0 \leq 1$:

$$8GD(1 + F\alpha^2) \leqslant (4D(1 - D) + F(2\alpha - 1))(4D^2 + F + 4G), \tag{48}$$
$$2D + G \leqslant 1 + 2DG + D^2 + F(\alpha - 1/2)^2, \tag{49}$$
$$2D^2 + 2DG \leqslant 2D + F(\alpha - 1/2), \tag{50}$$
$$F(2\alpha - 1) \geqslant 0. \tag{51}$$

The additional condition Eq. (26) is not needed, since it is included implicitly in the set shown here.

These inequalities could be further analyzed in terms of the discretization constants, but it can already be observed that using a more than semi-implicit scheme for the Coriolis actually enlarges both the necessary and sufficient stability domains.

4. Practical stability conditions for subsets of the equations

As shown in Beckers (1999), when one finds one root of the characteristic equation, Miller's theorem can be applied to the remaining polynomial. This approach can in practice be used to easily determine necessary and sufficient stability when at least one of the three physical processes (gravity waves, inertial oscillations and diffusion) is disregarded, as shown below.

Supposing a root $\rho = 1-\xi$ has been found, the method developed in Beckers (1999) allows to rewrite the necessary and sufficient stability conditions:

$$0 \leqslant \xi \leqslant 2, \tag{52}$$
$$b_1 - b_2 - (1 - b_2)\xi - \xi^2 \geqslant 0, \tag{53}$$
$$2 + b_2 - b_1 + (1 - b_2)\xi + \xi^2 \geqslant 0, \tag{54}$$
$$4 - b_1 + (2 - b_2)\xi + \xi^2 \geqslant 0, \tag{55}$$
$$2b_2 - b_1 - b_2\xi + \xi^2 \geqslant 0. \tag{56}$$

In terms of the discretization constants:

$$0 \leqslant \xi \leqslant 2, \tag{57}$$

$$4D(1-D) + F(2\alpha - 1) - 8GD - (1 + F\alpha^2 - 4D - 4G - 2F\alpha)\xi - \xi^2 \geqslant 0, \tag{58}$$

$$2 + 2F\left(\frac{1}{4} + (\alpha - 1/2)^2\right) - 4D(1-D) + 8GD + (1 + F\alpha^2 - 4D - 4G - 2F\alpha)\xi + \xi^2 \geqslant 0, \tag{59}$$

$$4 + F(2\alpha - 1)^2 - 4D(2-D) - 4G(1-2D) + (2 - 2F\alpha(1-\alpha) - 4D - 4G)\xi + \xi^2 \geqslant 0, \tag{60}$$

$$4G(1+2D) + F + 4D^2 - (4D + 4G + 2F\alpha)\xi + \xi^2 \geqslant 0. \tag{61}$$

4.1. Damped inertia oscillations: G = 0

Without the gravity waves, the scheme allows one root $\rho_1 = 1$, so that the application of the formulae developed (58) (61) with $\xi = 0$ immediately yields the following conditions:

$$4D(1-D) + F(2\alpha - 1) \geqslant 0, \tag{62}$$

$$4 - 8D + 4D^2 + F(2\alpha - 1)^2 \geqslant 0. \tag{63}$$

It is easily shown that (63) is always satisfied and that the necessary and sufficient stability conditions are thus (22) together with

$$D \leqslant \frac{1}{2}\left(1 + \sqrt{1 + F(2\alpha - 1)}\right). \tag{64}$$

This is merely a condition (26) which is not necessarily be a strict inequality. This is the necessary and sufficient stability condition for $G = 0$ and completes the case $b_0 = 0$.

The condition can easily be translated into conditions applied to the discretization constants by observing that inequality must hold for the maximum value D can reach, i.e.

$$d_x + d_y \leqslant \frac{1}{2}\left(1 + \sqrt{1 + F(2\alpha - 1)}\right). \tag{65}$$

We observe that compared to a pure diffusion case, the introduction of a more than semi-implicit scheme for Coriolis stabilizes the scheme by adding a real implicit damping of all modes. The necessary and sufficient stability conditions for the scheme without gravity waves are thus Eqs. (22) and (65).

4.2. Damped gravity waves F = 0

Here, without the Coriolis force, a diffusion mode $\xi = 2D$ is present and given by the exact solution $\rho_1 = 1 - 2D$, in such a way that the stability conditions are now easily found to be

$$|1 - 2D| \leqslant 1, \tag{66}$$

$$G + D \leqslant 1. \tag{67}$$

These conditions are finally translated into conditions on the discretization constants by observing that, in inequality (67), the left-hand side is a bi-linear function of $\sin^2 \theta_x$ and $\sin^2 \theta_y$, and that its maxima are thus found on the limits 0 and 1. This provides the final necessary and sufficient stability conditions (with Eq. (22))

$$d_x + d_y \leqslant 1, \tag{68}$$

$$\max(d_x + c_x^2, d_y + c_y^2) \leqslant 1. \tag{69}$$

We observe that the diffusion factor makes the stability condition more constraining, a result one does not necessarily expect a priori, but then changes in the stability conditions are continuous when adding diffusion.

4.3. Pure inertia gravity waves D = 0

Without diffusion, but with Coriolis force, the root $\rho_1 = 1$ allows the geostrophic equilibrium solution and simplifies the stability conditions as follows:

$$4 + F(1 - 2\alpha)^2 - 4G \geqslant 0. \tag{70}$$

With condition (22) the necessary and sufficient stability condition thus becomes

$$\max(c_x^2, c_y^2) \leqslant 1 + F\left(\alpha - \frac{1}{2}\right)^2, \tag{71}$$

which again shows that a more than semi-implicit treatment of the Coriolis force stabilizes discretization. This stability condition is coherent with the von Neumann stability condition of Deleersnijder and Campin (1993) where the cases $\alpha = 1/2$ and $\alpha = 1$ were calculated.

4.4. Systems with small friction

If we take a subset of an equation in which one of the three processes is absent, we obtain stability conditions which indeed show that for $\alpha = 1$, the stability domain is larger than for $\alpha = 1/2$, as predicted by the von Neumann stability analysis of Deleersnijder and Campin (1993). One might ask if the inclusion of a third process would lead to drastic changes in the stability criteria. In view of the necessary stability conditions or the sufficient stability conditions established before, this is not very likely to occur. This is due to the "predictable" behavior of these stability conditions for $D \rightarrow 0$ which always lead to an enlargement of the stability conditions for an increasing α.

If a drastic change of stability conditions were present, the problem reported in Deleersnijder and Campin (1993), who observed a decreased practical stability domain for $\alpha = 1$, might result from a singular perturbation of the stability conditions (as for example in Beckers and Deleersnijder, 1993).

In practice, ocean models include the friction by a Laplacian diffusion to parameterize sub-grid scale effects or to stabilize advection schemes. In any case, diffusion is generally kept at the lowest possible level, and one generally assumes that when introducing a small explicit diffusion into the discretization the scheme becomes more stable or at least does not lead to a drastic reduction of the stability region. We will check the case of inertia-gravity waves by assuming that the discretized diffusion operator is small compared to the others. By doing so, we can search for a root $\rho_1 = 1 - xD$ of Eq. (17), with the new unknown x equal or less than O(l). This is similar to the method described in Beckers (1999), where a solution $p = 1 - \xi$ is searched assuming $|\xi| << 1$.

$$\xi = \frac{a_0 + a_1 + a_2 + a_3}{a_1 + 2a_2 + 3a_3} = \frac{b_0 - b_1 + b_2}{2b_2 - b_1} = \frac{8GD}{F + 4G + 8GD + 4D^2}. \tag{72}$$

It is also clear that $0 \leq \xi \leq 2D$, so that for small D, $|\xi| << 1$ applies, in which case the approximate solution $\rho_1 = 1 - \xi$ is perfectly valid and corresponds to a stability condition $|p_1| \leq 1$ which is satisfied.

So the remaining stability conditions are (58)-(61); they are analyzed in Appendix A, by neglecting all higher order terms in D^2. The calculations show that for weak diffusion, one needs to satisfy (A.2) and (22), which in terms of the discretization constants is only possible when

$$F(2\alpha - 1) \geqslant 0, \tag{73}$$

$$\max(c_x^2 + d_x - G^0(1 + d_x), c_y^2 + d_y - G^0(1 + d_y)) \leqslant 0 \tag{74}$$

with $G^0 = 1 + F\,(1/2 - \alpha)^2$.

This shows that the inclusion of a small diffusion into the inertia gravity waves modifies the stability criteria in a continuous way and it cannot explain the paradox found in Deleersnijder and Campin (1993).

5. The problem of boundary conditions and practical issues

As stated in Section 1, the experiment of Deleersnijder and Campin (1993) somehow questions the practical use of the von Neumann analysis in the present case. Indeed, the authors found that in a closed squared domain, even when using a semi-implicit Coriolis treatment ($\alpha = 1/2$), the von Neumann stability criterion indeed corresponds to the limit between stability and instability. However, when they increased the implicitness of the Coriolis term ($\alpha = 1$), it was discovered that the practical stability domain was reduced instead, not increased, as the stability condition (71) would suggest. Since the stability criteria found by including diffusion show no singular perturbation problem, we must find another explanation for the destabilizing effect observed in Deleersnijder and Campin (1993). As the von Neumann stability analysis is valid only for infinite domains, we suspect boundary conditions are responsible for the destabilizing effect.

We thus repeated the experiment of Deleersnijder and Campin (1993) based on their regular grid. When using periodic boundary conditions in the same box as Deleersnijder and Campin (1993), one retrieves exactly the stability limit given by (71) for all values of α. Furthermore, in a 1D case, one also recovers the stability limit (71) numerically when using zero velocities at the boundaries. Similarly, when using grid sizes which are much larger in one direction than the other, a numerical stability which is very close to the theoretical curve is again observed. Only when the grid steps are of the same order of magnitude and one imposes zero velocity at the coasts, instability appears much sooner than predicted by the stability conditions.

In fact, this behavior is due to incorrectly imposed boundary conditions: in a linear 2D case, without any friction, one can specify only one independent boundary condition overall, and not two, as is the case when trying to impose a zero velocity vector. In a unidimensional scenario, even if one imposes the normal component plus the tangential velocity at the grid on the boundary, the latter value is never used during computations for the points in the inner domain. But in 2D, the discretization technique requires that a value is assigned to the tangential velocity at the boundary (see discretized mass conservation). However, this value is not an independent boundary condition but can be retrieved from the equations at the boundary.

Consider the example of a boundary $x = 0$. In principle one should impose $u = 0$ and calculate v from

$$\frac{\partial v}{\partial t} = -g\frac{\partial \eta}{\partial y}. \tag{75}$$

Unfortunately, the numerical discretization of this partial differential is not straightforward, since the right-hand side derivative on the staggered B-grid requires values of η outside the domain, unless a derivative is taken, based solely on the interior points. The last approach however is not sufficient to stabilize the scheme, which can be explained by the fact that when using the interior points only, one implicitly assumes

$$\frac{\partial^2 \eta}{\partial x\,\partial y} = 0 \tag{76}$$

at the boundary. This is however not coherent, because the momentum equation for u near the boundary would then read

$$\frac{\partial^2 u}{\partial y\,\partial t} = f\frac{\partial v}{\partial y} \tag{77}$$

and since the right-hand side has no particular reason to vanish when calculated by Eq. (75) in the presence of waves, the momentum equation is incoherent with the boundary condition $u = 0$.

A way to bypass this problem is to calculate virtual values for η outside the domain, and then calculate v from the v momentum equation with $u = 0$. The virtual values of η should then be such that the momentum equation for u would lead to a zero normal velocity.

In other words, at the boundaries, we would solve

$$-f\alpha v_{n_t+1,n_x+1/2,n_y+1/2} - f(1-\alpha)v_{n_t,n_x+1/2,n_y+1/2} = -g\left(\delta_x\bar{\eta}^y\right)_{n_t+1,n_x+1/2,n_y+1/2}, \tag{78}$$

$$(\delta_t v)_{n_t+1/2,n_x+1/2,n_y+1/2} = -g\left(\delta_y\bar{\eta}^x\right)_{n_t+1,n_x+1/2,n_y+1/2}, \tag{79}$$

for $n_x = 0$ to calculate $v_{n_t+1,1/2,n_y+1/2}$ and $\eta_{n_t+1,0,n_y}$ in terms of the new and old interior values of u, v and η.

Eliminating v merely yields a coherent discretized equation for the boundary condition

$$\frac{\partial^2 u}{\partial t^2} + f^2 u = -fg\frac{\partial \eta}{\partial y} - g\frac{\partial^2 \eta}{\partial t\,\partial x} = 0 \tag{80}$$

which can be rewritten as

$$\left(\delta_x\bar{\eta}^y\right)_{n_t+1,n_x+1/2,n_y+1/2} + \alpha f\Delta t\left(\delta_y\bar{\eta}^x\right)_{n_t+1,n_x+1/2,n_y+1/2} = \frac{f}{g}v_{n_t,n_x+1/2,n_y+1/2}. \tag{81}$$

The difference between applying zero velocity at the boundary and a virtual boundary condition on η is depicted in Fig. 4.

Unfortunately, this discretized equation is not very easily implemented in general circulation models, because of the recursive relationships between the virtual η values along the boundary (in practice, an iterative method along the boundary using the recurrence relation in the direction that ensures a stable recurrence relation converges rapidly, if the recurrence relation is reinitialized by the other boundary conditions). In the square box problem dealt with here, we implemented this boundary condition in $x = 0$ and $x = L$.

The verification of the technique with periodic boundary conditions applied to the y direction and the discretized virtual boundary conditions on η in $x = 0$ and $x = L$ has thus shown that the stability condition corresponds indeed the theoretical stability conditions found by the von Neumann analysis and that a more than semi-implicit scheme for Coriolis provides a larger stability domain than the semi-implicit version, in accordance with (71).

Implementing correct boundary conditions thus resolves the instability problem.

The reason why the problem of Deleersnijder and Campin (1993) appears only for $\alpha > 1/2$ and not for $\alpha = 1/2$ is related to energy generation in the model: It is easily seen that

$$\bar{u}^t\delta_t u + \bar{v}^t\delta_t v \tag{82}$$

is the measure of the kinetic energy variation and by using discrete equations, the kinetic energy production contains a term related to the Coriolis which does not vanish:

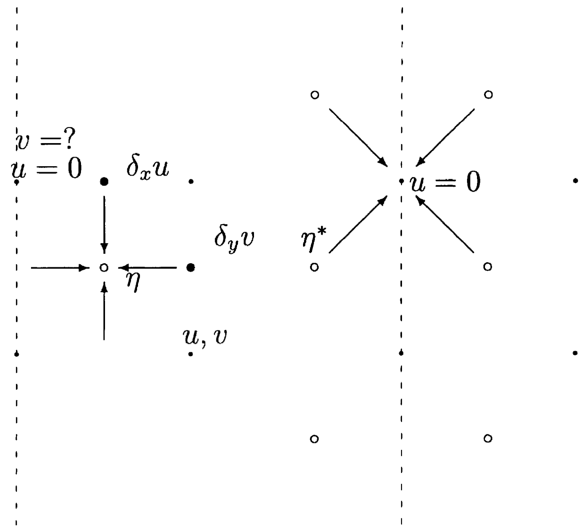
$$(1/2-\alpha)f(u_{n_t+1}v_{n_t} - u_{n_t}v_{n_t+1}). \tag{83}$$

For a pure inertial oscillation, this term can be further calculated by replacing u_{n_t+1} with the value obtained from the discrete equation, and it appears that for $\alpha = 1/2$, the Coriolis force does not produce mechanical work, and that for $\alpha > 1/2$, it dissipates energy. This is however only the case for a pure inertial oscillation. In the presence of other factors, the term (83) can actually produce energy, because the new values for u and v depend on all processes, including boundary conditions.

In the periodic domain, the energy produced or destroyed by the Coriolis force does not lead to instability if the numerical stability conditions are satisfied. On the other hand, for other boundary conditions, there appears to be a feedback of the boundary condition on the numerical work produced by the Coriolis force. This interpretation is coherent with the experiments of Deleersnijder and Campin (1993), where for $\alpha = 1/2$, numerical stability is indeed not modified by the boundary condition.

The instability therefore arose from a combined effect of inappropriate boundary conditions and a Coriolis force which generates (or dissipates) energy. This is also confirmed by observing how the system actually becomes unstable in the case of inappropriate boundary conditions. When violating the von Neumann stability criteria, instability is explosive and a few iterations lead to overflows in the computer code. As regards the instability observed when the von Neumann stability criteria is satisfied, it grows slowly and is due to a weak feedback.

Fig. 4. *Application of boundary conditions on the B-grid. On the left side, imposing the mathematically sufficient boundary condition of zero normal velocity does not provide a numerical value for v, which is however needed when computing the mass balance for η. Applying a false boundary condition on v can be avoided by the approach shown on the right, where virtual values of η* are calculated so that the normal discretizations applied to u, v at the boundaries lead to u = 0 there.*



The problem of Deleersnijder and Campin (1993) thus stems from the inappropriate application of boundary conditions combined with the Coriolis treatment. In real models however, some diffusion is always present, so that in principle, zero velocity conditions could be (and in practice are) applied in B-grids. If diffusion is very small, such boundary conditions will therefore always destabilize the inertia gravity waves scheme to some extent when $\alpha > 1/2$. In this case, B-grids with no-slip boundary conditions should include sufficiently strong horizontal diffusion which is able to dissipate the energy produced by feedback. From the point of view of discretization and precision, the no-slip condition should in any event only be applied when the grid size is small compared to the factional boundary layer. In other words, the diffusion part should be able to dissipate the energy entered into the system by setting zero velocities at the boundary and using the Coriolis force discretization which produces mechanical work.

6. Application to the C-grid

In principle, computations of the stability conditions for the case of the C-grid could be completed exactly as for the B-grid. One simply has to redefine the parameters by

$$F = (f\,\Delta t)^2 \cos^2\theta_x \cos^2\theta_y, \tag{84}$$

$$D = d_x \sin^2\theta_x + d_y \sin^2\theta_y, \tag{85}$$

$$G = c_x^2 \sin^2\theta_x + c_y^2 \sin^2\theta_y, \tag{86}$$

because the spatial averaging is applied to the Coriolis force rather than on the pressure gradient and velocity divergence. This also means that condition (22) must still hold, and that a semi-implicit treatment of the Coriolis force is still required. But on the C-grid, unlike the B-grid, this leads to a coupling between spatially adjacent points of u and v at the new time-level. This ultimately needs the inversion of sparse and banded matrixes, which is very time consuming. In fact on the C-grid, Coriolis terms are often treated by Siliecky's approach (Sielecki, 1967) using the "new velocity" only in one of the two velocity equations, so that one velocity component is computed before the second one, which in turn uses the newly computed velocity value of the other component. On the C-grid, the scheme analyzed here is thus only relevant without Coriolis force and leads then to the necessary and sufficient stability condition

$$c_x^2 + c_y^2 + d_x + d_y \leqslant 1, \tag{87}$$

where the destabilizing effect of the diffusion is again observed when the gravity waves had used a time step near the limit of stability.

It is also likely that the problem observed on the B-grid will not appear in the C-grid, because, even when treating the Coriolis as in the present paper, boundary conditions at coastlines for an inviscid ocean are readily discretized and do not involve any virtual points for η or the need to impose a boundary condition on the tangential velocity. In the C-grid, it is sufficient to set the normal velocity component to zero at the interface between a land and ocean point, while calculating the other velocity component and the sea surface elevation in the interior as usual.

7. Discussion

It has been demonstrated that the analysis of a complete scheme of damped inertia-gravity waves is much more complex than the analysis of subsets of equations, and that some counter-intuitive results may emerge. Indeed, adding diffusion leads to more constraining stability conditions, but the changes are continuous. In particular, for small diffusion, the stability of the system is not affected if one is not too close to the stability limit of pure inertia-gravity waves. The problem identified in Deleersnijder and Campin (1993) is thus not related to the stability conditions of an excessively simple subset of equations. The instability observed by the authors was created by feedback between inappropriate boundary conditions in a 2D case and the Coriolis force treatment when $\alpha > 1/2$. Stability can thus be recovered either by using $\alpha = 1/2$ or by applying correct boundary conditions.

Another way of overcoming the problem identified in Deleersnijder and Campin (1993) is the addition of diffusion which dampens the instabilities introduced by the no-slip boundary conditions. This would also be a more coherent approach in terms of physical processes and the applied boundary conditions. Since in most real applications explicit diffusion or time filtering is present, the instability problem identified by (Deleersnijder and Campin, 1993) is unlikely to arise, because it is a "slow" type of instability compared to the explosive nature of the classical numerical instability. Since in addition, in most case, Coriolis is treated as semi-implicit rather than implicit, the problem will not arise even when no diffusion is included. One should however bear in mind the possible instability mechanism introduced by inappropriate boundary conditions coupled with the treatment of a dominant term of the equations.

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Appendix A. System with small friction

For small friction, one has to ensure that stability conditions (58) (61) are satisfied with ξ given by Eq. (72).

When analyzing stability conditions, one can neglect all higher order terms in D^2, in accordance with the development of the root ρ for small D. It appears clearly that Eq. (61) is always satisfied by observing that the multiplicative factor of ξ is always negative and that the most stringent condition associated with (61) would be for $\xi = 2D$. But even then inequality is always satisfied.

Similarly, for Eq. (59), when the multiplication factor of ξ is positive, the worst case that could arise is $\xi = 0$, in which case inequality is satisfied since $4D > (1 - D) \leq 1$. If the multiplication factor of ξ is negative, the worst scenario would be $\xi = 2D$ and for small D inequality is always satisfied.

The other two inequalities are more complicated to analyze.

First we will analyze inequality (60).

By defining $G = G^0 - rD$ where G^0 corresponds to the stability limit of inertia-gravity wave case without diffusion $4G^0 = 4 + F(2\alpha - 1)^2$, we can rewrite the inequality by neglecting higher order terms in D:

$$(4(G^0 - G) - 8D + 8G^0D)(F + 4G^0 - 4rD) + 8G^0D(2 - 4G^0 - F\alpha(1 - \alpha)) \geqslant 0, \qquad \text{(A.1)}$$

which, after a few algebraic calculations, reduces to $r \geq 2 - G^0$. This means that one has to satisfy:

$$G + 2D \leqslant (1 + D)\left(1 + F(\alpha - 1/2)^2\right). \qquad \text{(A.2)}$$

If this is satisfied, one still has to verify (58), which we can also rewrite by neglecting terms in D^2:

$$(4D + F(2\alpha - 1) - 8GD)(F + 4G) - 8GD(1 + F\alpha^2 - 4G - 2F\alpha) \geqslant 0. \qquad \text{(A.3)}$$

Or equivalently, since $GF(2\alpha-1) \geq 0$ it is sufficient to have

$$4D(1 - 2G)(F + 4G) + F^2(2\alpha - 1) - 8DG(1 - (2 - \alpha)\alpha F - 4G) \geqslant 0, \qquad \text{(A.4)}$$

or

$$4DF + 8DG + F^2(2\alpha - 1) - 8(1 - \alpha)^2 DFG \geqslant 0. \qquad \text{(A.5)}$$

At the worst, the last term of the inequality is obtained when G reaches the stability limit G^0, in which case we must have finally

$$4DF(1 - 2(1 - \alpha)^2) + 8DG + 2F^2(\alpha - 1/2)(1 - 4D(1 - \alpha)^2(\alpha - 1/2)) \geqslant 0. \qquad \text{(A.6)}$$

Since $\alpha \geq 1/2$ this is always satisfied and condition (A.2) is the most stringent one.

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