Traffic-induced Vibration
TD 3

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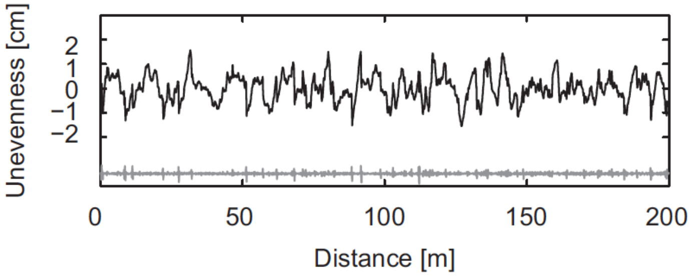
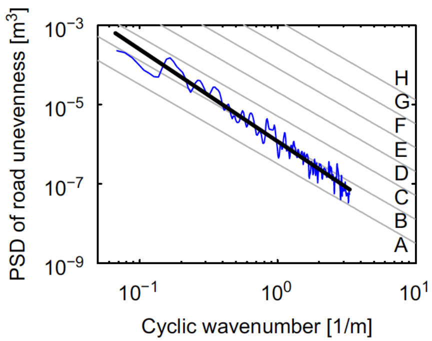
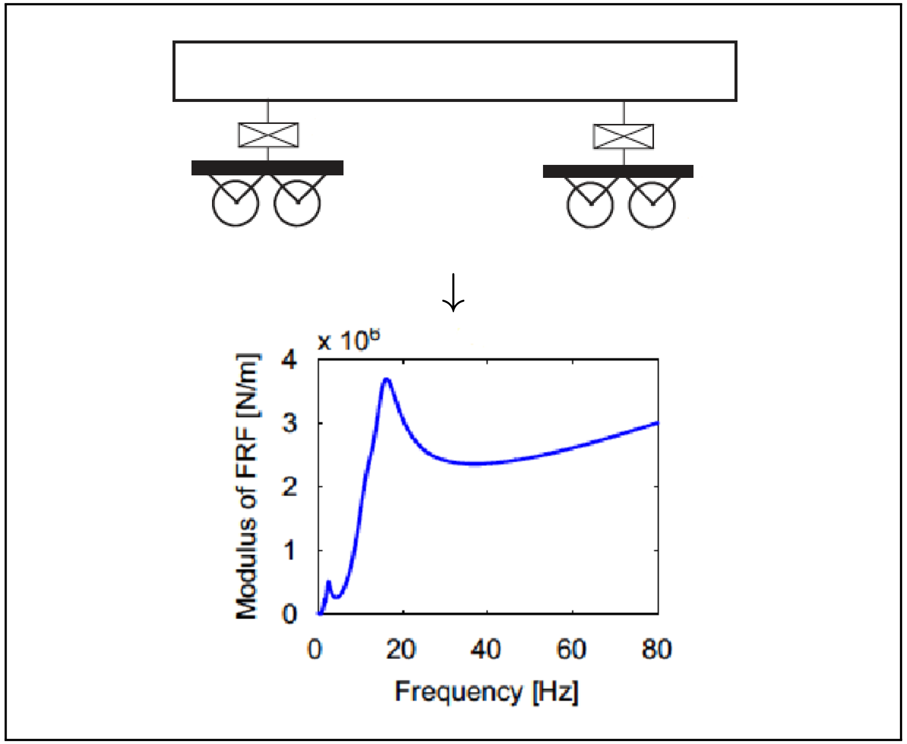
Motivation





During the previous TD, we studied a simplified statistical method for inferring from data a power spectral density function for road/rail roughness.



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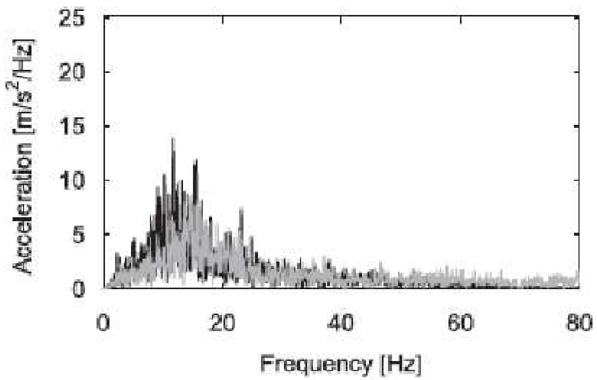
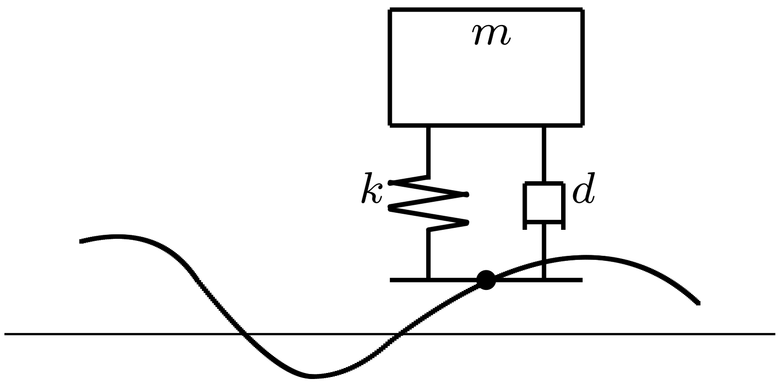
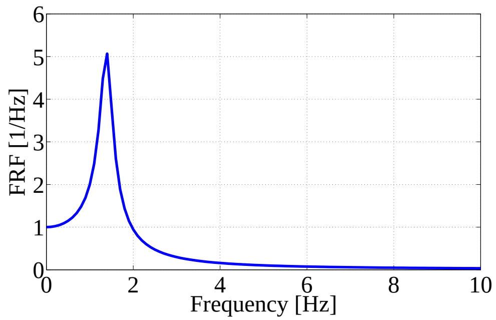


During this TD, we will study a simplified MDOF vehicle model for obtaining FRFs, which will ultimately allow us to predict vibration.

Remember...

- As part of the first TD, we obtained the following result:





$$\hat{h}(\omega) = \frac{i\omega d + k}{-m\omega^2 + i\omega d + k}$$

- **Subresonant regime** $\omega \ll \sqrt{k/m}$, where $\lim_{\omega \to 0} \hat{h}(\omega) = 1$, that is, $\hat{u}(\omega) \to \hat{u}_{\mathrm{c}}(\omega)$.

  Resonant regime $\omega \simeq \sqrt{k/m}$.

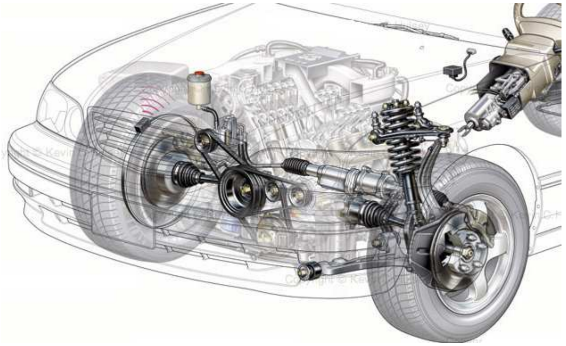
  Postresonant regime $\omega \gg \sqrt{k/m}$, where $\lim_{\omega \to +\infty} \hat{h}(\omega) = 0$, that is, $\hat{u}(\omega) \to 0$.

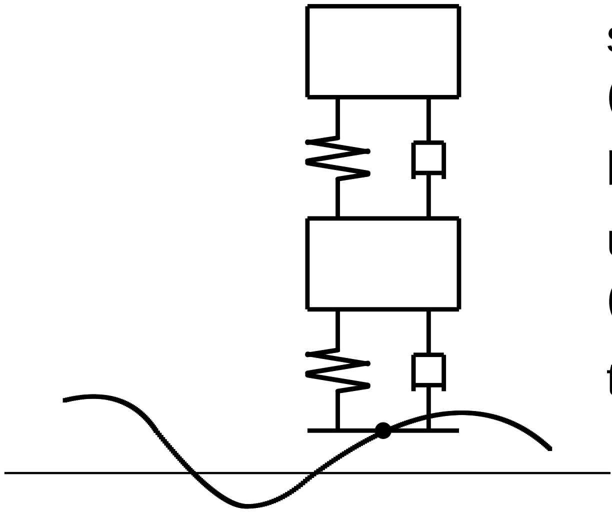
- **Vibration transmission** : if the contact point oscillates at a frequency much lower than the eigenfrequency, the mass moves along with it.

  Vibration isolation : if the contact point oscillates at a frequency much higher than the eigenfrequency, the mass is isolated from it.

Suspensions

- The suspension must address the following two opposing needs:
  - **vibration transmission**: the wheels must be able to follow bumps,
  - **vibration isolation**: the vehicle body must be isolated from vibration for passenger comfort.
- To address these needs, the suspension is designed two exhibit two principal modes of vibration:





Wheel hop mode at about 15 Hz provides bump following ability: this mode is a mode of vibration in which the unsprung mass oscillates on the tires.

Primary suspension mode at about 1.5 Hz provides vibration isolation: this mode is a mode of vibration in which the sprung mass oscillates on the primary suspension.

At least a 2-DOF model is required to represent the vibration transmission and isolation provided by the suspension.

We need to extend our theory from scalar-valued stochastic processes to vector-valued stochastic processes...

- A **second-order stochastic process** $\{\boldsymbol{X}(t),\ t \in \mathbb{R}\}$ defined on a probability triple $(\Theta, \mathcal{F}, P)$ indexed by $\mathbb{R}$ with values in $\mathbb{R}^d$ is a function that associates to any index t in $\mathbb{R}$ a second-order random variable $\boldsymbol{X}(t)$ defined on $(\Theta, \mathcal{F}, P)$ with values in $\mathbb{R}^d$, that is,

$$\sqrt{\int_\Theta \|\boldsymbol{X}(t)\|^2 dP} < +\infty, \quad \text{where} \quad \|\boldsymbol{X}(t)\|^2 = \sum_{j=1}^{d} X_j(t)^2, \quad \forall t \in \mathbb{R}.$$

  The second-order statistical descriptors of $\{\boldsymbol{X}(t),\ t \in \mathbb{R}\}$ are defined as follows:

  - **mean function** $\boldsymbol{m}_{\boldsymbol{X}}(t) = \int_\Theta \boldsymbol{X}(t) dP, \quad t \in \mathbb{R}$,
  - **autocorrelation function** $[R_{\boldsymbol{X}}(t,t')] = \int_\Theta \boldsymbol{X}(t)\boldsymbol{X}(t')^{\mathrm{T}} dP, \quad t,t' \in \mathbb{R}$,
  - **covariance function** $[C_{\boldsymbol{X}}(t,t')] = \int_\Theta \big(\boldsymbol{X}(t) - \boldsymbol{m}_{\boldsymbol{X}}(t)\big)\big(\boldsymbol{X}(t') - \boldsymbol{m}_{\boldsymbol{X}}(t')\big)^{\mathrm{T}} dP, \quad t,t' \in \mathbb{R}$.

- Properties of the autocorrelation and covariance functions:
  - $[R_{\boldsymbol{X}}(t,t')] = [R_{\boldsymbol{X}}(t',t)]^{\mathrm{T}}$ and $[C_{\boldsymbol{X}}(t,t')] = [C_{\boldsymbol{X}}(t',t)]^{\mathrm{T}}, \quad \forall t,t' \in \mathbb{R}$,
  - $[R_{\boldsymbol{X}}(t,t)]$ and $[C_{\boldsymbol{X}}(t,t)]$ are positive-definite symmetric matrices, $\quad \forall t \in \mathbb{R}$.

- The second-order stochastic process $\{\boldsymbol{X}(t),\ t \in \mathbb{R}\}$ is **mean-square stationary** if

$$\boldsymbol{m}_{\boldsymbol{X}}(t) = \boldsymbol{m}_{\boldsymbol{X}}, \qquad \text{independent of } t,$$

$$[R_{\boldsymbol{X}}(t,t')] = [R_{\boldsymbol{X}}(t-t')], \qquad \text{dependent only on lag } t-t'.$$

- The mean-square stationary second-order stochastic process $\{\boldsymbol{X}(t),\ t \in \mathbb{R}\}$ has a **power spectral density function** if there exists an integrable function $[S_{\boldsymbol{X}}] : \mathbb{R} \rightarrow \mathsf{M}_d(\mathbb{C})$ such that

$$[R_{\boldsymbol{X}}(t-t')] = \frac{1}{2\pi} \int_{\mathbb{R}} \exp\big(i\omega(t-t')\big)[S_{\boldsymbol{X}}(\omega)]d\omega, \quad \forall (t-t') \in \mathbb{R}.$$

- Properties of the power spectral density function:
  - $[S_{\boldsymbol{X}}(-\omega)] = [S_{\boldsymbol{X}}(\omega)]^{\mathrm{T}}, \quad \forall \omega \in \mathbb{R}$,
  - $[S_{\boldsymbol{X}}(\omega)]$ is a positive-definite Hermitian matrix, $\quad \forall \omega \in \mathbb{R}$,
  - the $[S_{\boldsymbol{X}}]_{jj}$ are positive real-valued, integrable, and even functions,
  - the $[S_{\boldsymbol{X}}]_{jk}$ are complex-valued, integrable functions with even real and odd complex parts.

- **Stochastic Differential Equation (SDE)**:
We consider the linear filtering of a mean-square stationary zero-mean second-order stochastic process $\{\boldsymbol{F}(t),\ t \in \mathbb{R}\}$ that we assume to admit a power spectral density function $[S_{\boldsymbol{F}}]$:

$$[M]\frac{d^2\boldsymbol{U}_{\boldsymbol{F}}}{dt^2}(t) + [D]\frac{d\boldsymbol{U}_{\boldsymbol{F}}}{dt}(t) + [K]\boldsymbol{U}_{\boldsymbol{F}} = \boldsymbol{F}(t), \quad t \in \mathbb{R}.$$

We assume that $[M]$, $[D]$, and $[K]$ are positive-definite, symmetric real matrices. Then, the FRF $\omega \mapsto [\widehat{H}(\omega)] = [P(i\omega)]^{-1} = [-\omega^2 M + i\omega D + K]^{-1}$ is such that $[P]^{-1}$ has no poles on the imaginary axis. We denote by $[H] = \mathcal{F}^{-1}([\widehat{H}])$ the impulse response function.

- **Generalized solution**:

$$\boldsymbol{U}_{\boldsymbol{F}} = [H] \star \boldsymbol{F}, \quad \text{that is,} \quad \boldsymbol{U}_{\boldsymbol{F}}(t) = \int_{\mathbb{R}} [H(s)]\boldsymbol{F}(t-s)ds.$$

The assumptions allow this convolution to be defined as a mean-square integral. Hence, $\{\boldsymbol{U}_{\boldsymbol{F}}(t),\ t \in \mathbb{R}\}$ is a second-order stochastic process. Its second-order descriptors are obtained as follows:

  - **mean function** $\boldsymbol{m}_{\boldsymbol{U}_{\boldsymbol{F}}}(t) = \int_{\mathbb{R}}[H(s)]\int_{\Theta} \boldsymbol{F}(t-s)dPds = \boldsymbol{0}$.
  - **autocorrelation function** $[R_{\boldsymbol{U}_{\boldsymbol{F}}}(t,t')] = \int_{\mathbb{R}}\int_{\mathbb{R}}[H(s)][R_{\boldsymbol{F}}(t-t'+s'-s)][H(s')]^{\mathrm{T}}dsds'$.

Hence, $\{\boldsymbol{U}_{\boldsymbol{F}}(t),\ t \in \mathbb{R}\}$ is a mean-square stationary zero-mean second-order stochastic process.

- Thus, the linear filtering of a mean-square stationary zero-mean second-order stochastic process provides in turn a mean-square stationary zero-mean second-order stochastic process.

- Substituting $[R_{\boldsymbol{F}}(t-t')] = \frac{1}{2\pi}\int_{\mathbb{R}} \exp\big(i\omega(t-t')\big)[S_{\boldsymbol{X}}(\omega)]d\omega$ in the expression for $[R_{\boldsymbol{U}_{\boldsymbol{F}}}]$, we obtain:

$$[R_{\boldsymbol{U}_{\boldsymbol{F}}}(t-t')] = \frac{1}{2\pi}\int_{\mathbb{R}} \exp\big(i\omega(t-t')\big)\left(\int_{\mathbb{R}} \exp(-i\omega s)[H(s)]ds\right)[S_{\boldsymbol{F}}(\omega)]\left(\int_{\mathbb{R}} \exp(i\omega s')[H(s')]^{\mathrm{T}} ds'\right)d\omega.$$

Hence, we obtain the **power spectral density function** of $\{\boldsymbol{U}_{\boldsymbol{F}}(t),\ t\in\mathbb{R}\}$ as follows:

$$\boxed{[S_{\boldsymbol{U}_{\boldsymbol{F}}}(\omega)] = [\widehat{H}(\omega)][S_{\boldsymbol{F}}(\omega)][\widehat{H}(\omega)]^*, \quad \forall\omega\in\mathbb{R}.}$$

- **Stochastic Initial-Value Problem (Stochastic IVP)**:

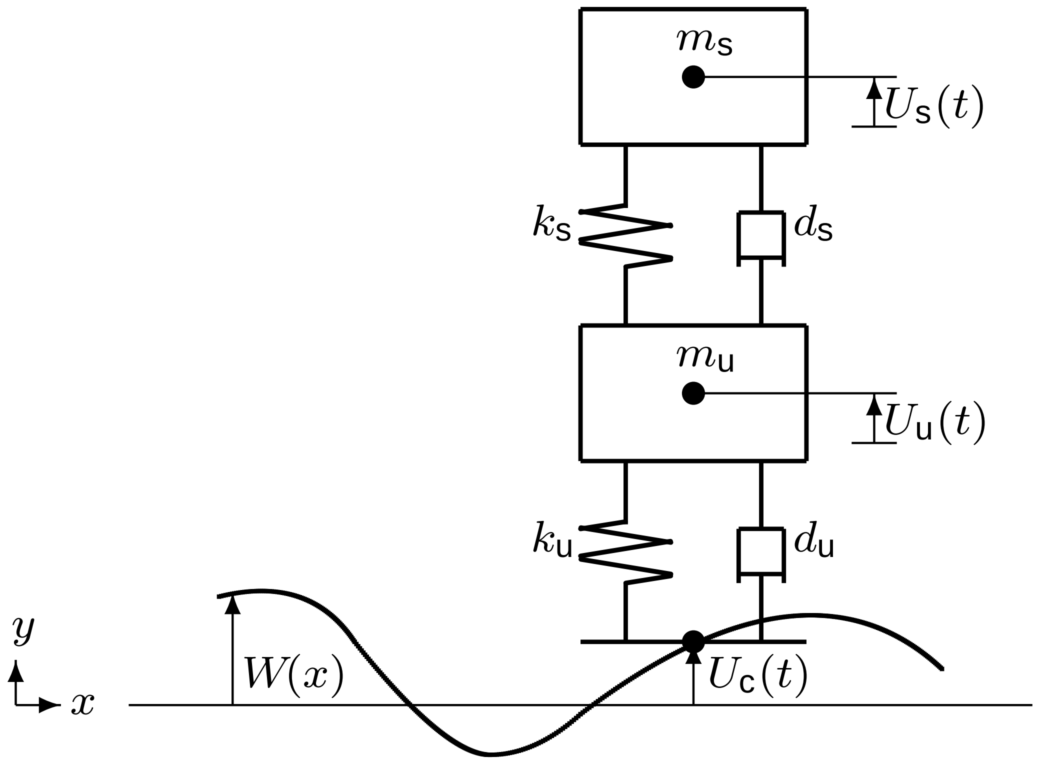
$$\begin{cases} [M]\dfrac{d^2\boldsymbol{U}_{\boldsymbol{F}}}{dt^2}(t) + [D]\dfrac{d\boldsymbol{U}_{\boldsymbol{F}}}{dt}(t) + [K]\boldsymbol{U}_{\boldsymbol{F}} = \boldsymbol{F}(t), \quad t>0, \\ \boldsymbol{U}(0) = \boldsymbol{U}_0 \quad \text{and} \quad \dfrac{d\boldsymbol{U}}{dt}(0) = \boldsymbol{U}_1, \end{cases}.$$

If $[M]$, $[D]$, and $[K]$ are positive-definite, symmetric, and real matrices, all the poles of $[P]^{-1}$ are located left of the imaginary axis, and we have

$$\boldsymbol{U}(t) = \underbrace{\boldsymbol{U}_h(t)}_{\text{transient response}} + \underbrace{\boldsymbol{U}_{\boldsymbol{F}}(t)}_{\text{forced response}} \qquad \text{with} \qquad \lim_{t\to+\infty}\sqrt{\int_{\Theta}\|\boldsymbol{U}(t)-\boldsymbol{U}_{\boldsymbol{F}}(t)\|^2 dP} = 0,$$

that is, the response $\boldsymbol{U}$ converges to the forced response $\boldsymbol{U}_{\boldsymbol{F}}$ as time advances.

Exercise



- We consider a 2-DOF model of a vehicle that moves over an uneven support at a constant horizontal velocity of v.
- Let the support unevenness be modeled by a mean-square stationary zero-mean second-order stochastic process $\{W(x),\ x \in \mathbb{R}\}$ with power spectral density function s_W.
- Let $\{U_{\mathsf{c}}(t),\ t \in \mathbb{R}\}$ be the vertical displacement of the contact point caused by the support unevenness.
- Let $\{U_{\mathsf{s}}(t),\ t \in \mathbb{R}\}$ and $\{U_{\mathsf{u}}(t),\ t \in \mathbb{R}\}$ be the vertical displacement of the sprung mass and the unsprung mass, respectively, with respect to the static equilibrium configuration on a flat support.

Exercise

1. Express dynamical equilibrium and deduce the system of two stochastic differential equations that governs the displacements $\{U_{\mathsf{s}}(t),\ t \in \mathbb{R}\}$ and $\{U_{\mathsf{u}}(t),\ t \in \mathbb{R}\}$ of the sprung mass and the unsprung mass.

2. Consider the system of two stochastic differential equations that you obtained under 1 as a convolution transformation whose input is the stochastic process $\{U_{\mathsf{c}}(t),\ t \in \mathbb{R}\}$ and whose output is the pair of stochastic processes $\{U_{\mathsf{s}}(t),\ t \in \mathbb{R}\}$ and $\{U_{\mathsf{u}}(t),\ t \in \mathbb{R}\}$. Write the expression for the FRF $\hat{\boldsymbol{h}} = (\hat{h}_{\mathsf{s}}, \hat{h}_{\mathsf{u}})$ associated with this convolution filter.

3. Write the equations that relate the mean function and the power spectral density function of $\{U_{\mathsf{s}}(t),\ t \in \mathbb{R}\}$ and $\{U_{\mathsf{u}}(t),\ t \in \mathbb{R}\}$ to those of $\{W(x),\ x \in \mathbb{R}\}$.

4. Consider $s_W(\xi) = s_0/\left(1 + \frac{|\xi|}{\xi_0}\right)^\alpha$ and the numerical values of $\xi_0 = 0.5\,\mathsf{m}^{-1}$, $m_{\mathsf{s}} = 470\,$kg, $k_{\mathsf{s}} = 36 \times 10^3$ N/m, $d_{\mathsf{s}} = 0.10 \times 2 \times \sqrt{m_{\mathsf{s}} k_{\mathsf{s}}}$, $m_{\mathsf{u}} = 39\,$kg, $k_{\mathsf{u}} = 160 \times 10^3$ N/m, $d_{\mathsf{u}} = 0.05 \times 2 \times \sqrt{m_{\mathsf{u}} k_{\mathsf{u}}}$. Plot s_W and $|\hat{h}|$, first on a linear scale and then on a loglog scale. First using the values of s_0 and α that you obtained under TD 2 question 1(d) for the deteriorated jointed plain concrete pavement (w1.mat), and then using the values of s_0 and α that you obtained under TD 2 question 1(d) for the concrete block pavement (w2.mat), plot $s_{U_{\mathsf{s}}}$ and $s_{U_{\mathsf{u}}}$ for $v = 40\,$m/s, first on a linear scale and then on a loglog scale. Interpret your results.

5. The ISO 2631-1 guide indicates that the comfort passengers perceive can be predicted on the basis of the root-mean-square value of the acceleration of the vehicle body, that is,

$$\sqrt{\int_{\Theta} |\ddot{U}_{\mathsf{s}}(t)|^2 dP} = \sqrt{\frac{1}{2\pi}\int_{\mathbb{R}} s_{\ddot{U}_{\mathsf{s}}}(\omega)d\omega} = \sqrt{\frac{1}{2\pi}\int_{\mathbb{R}} \omega^4 s_{U_{\mathsf{s}}}(\omega)d\omega}.$$

Specifically, the ISO 2631-1 guide suggests that passengers perceive comfort as follows:

root-mean-square acceleration [m/s^2]	comfort perception
Less than 0.315	not uncomfortable
0.315 to 0.63	a little uncomfortable
0.5 to 1	fairly uncomfortable
0.8 to 1.6	uncomfortable
1.25 to 2.5	very uncomfortable
greater than 2	extremely uncomfortable

Use the power spectral density function you obtained under question 4 to predict comfort perception, first when driving over the deteriorated jointed plain concrete pavement (w1.mat) and then when driving over the concrete block pavement (w2.mat).

(Note that for the sake of simplicity, we neglected here the frequency weighing suggested by the ISO 2631-1 guide to account for the fact that not all frequencies are perceived equally.)

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