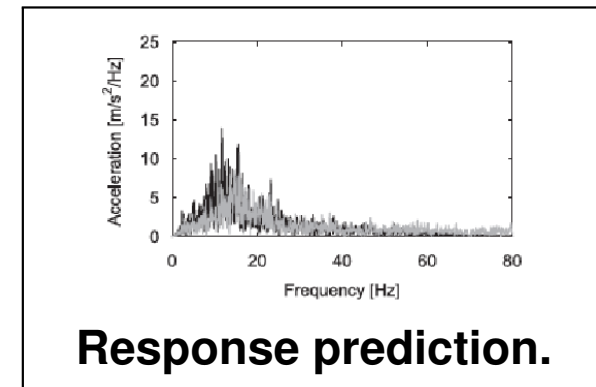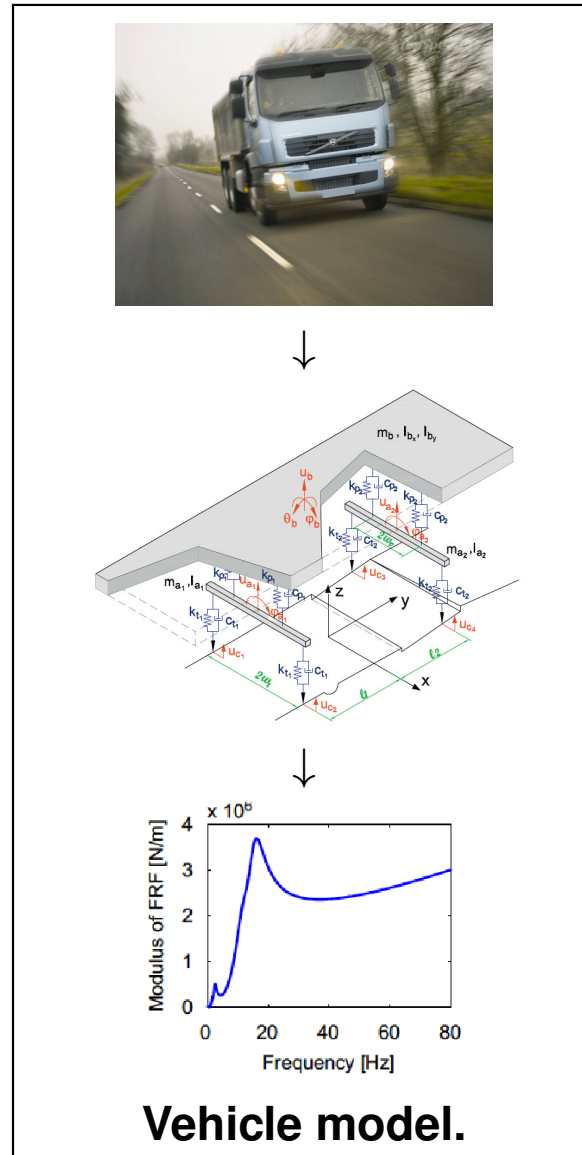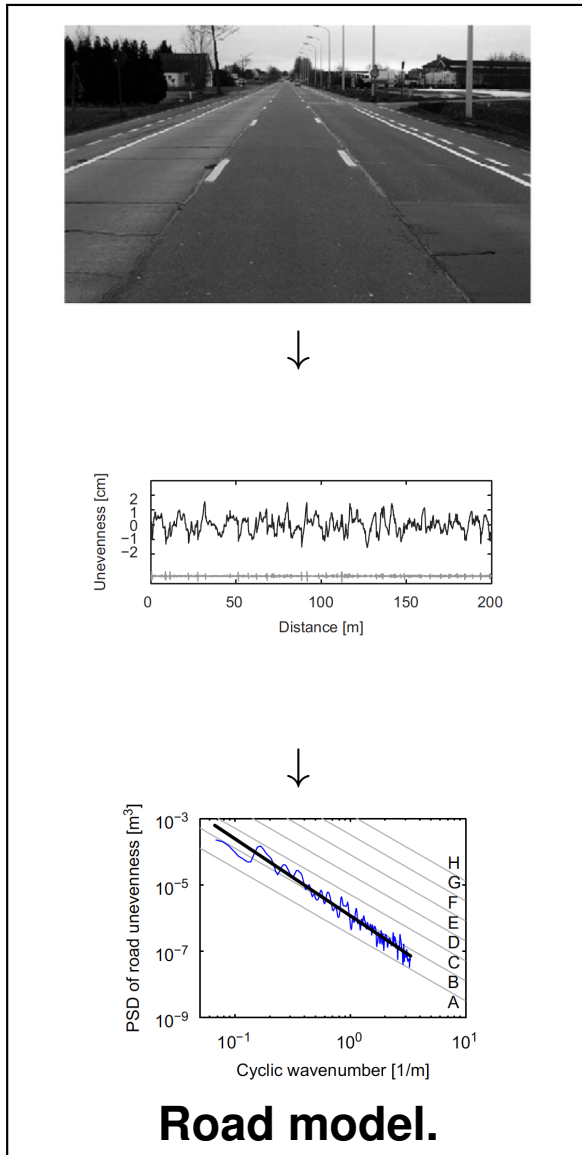# Traffic-induced Vibration
TD 2

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Road model.

Vehicle model.

Response prediction.

Modeling of traffic-induced vibration.
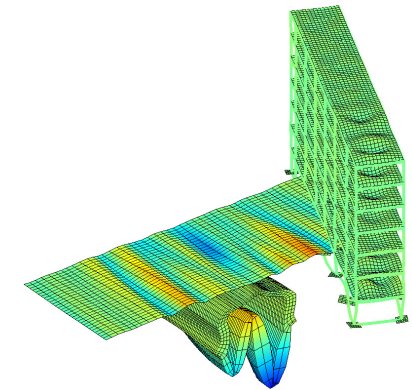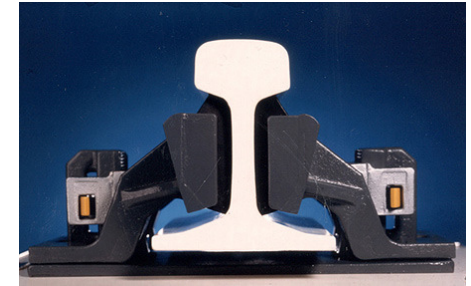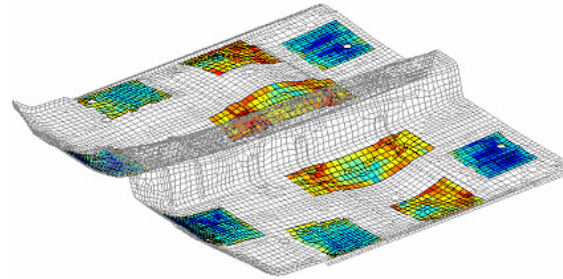
Modeling is very useful for engineering!

Let's simplify this a little bit!
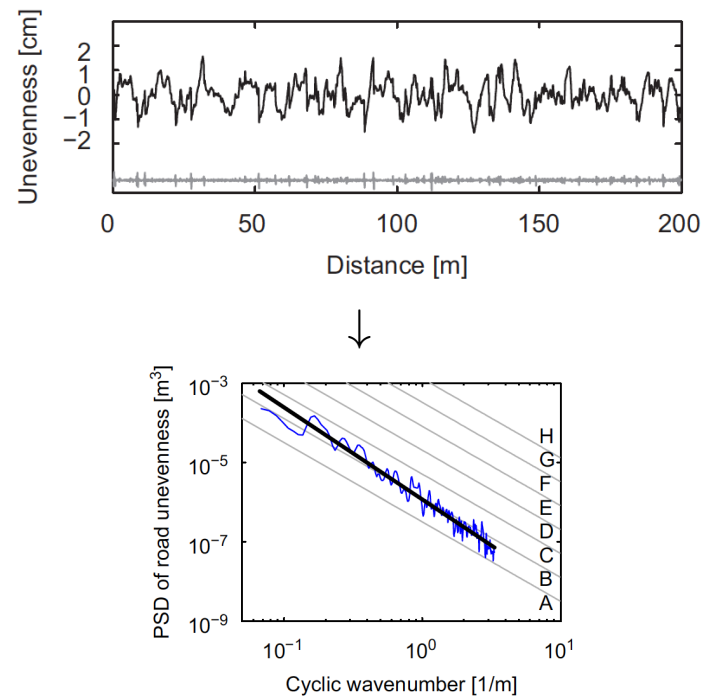
During this TD, we will study a simplified statistical method for inferring from data a power spectral density function for road/rail roughness.
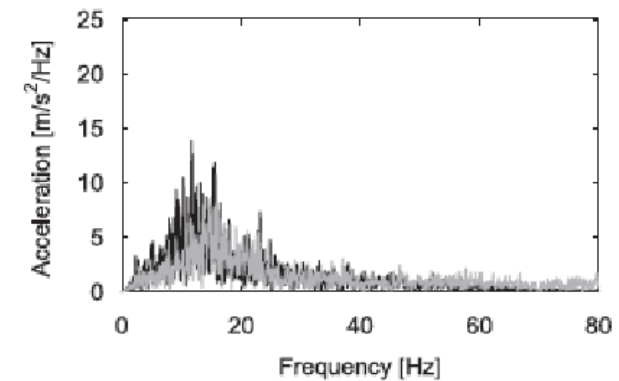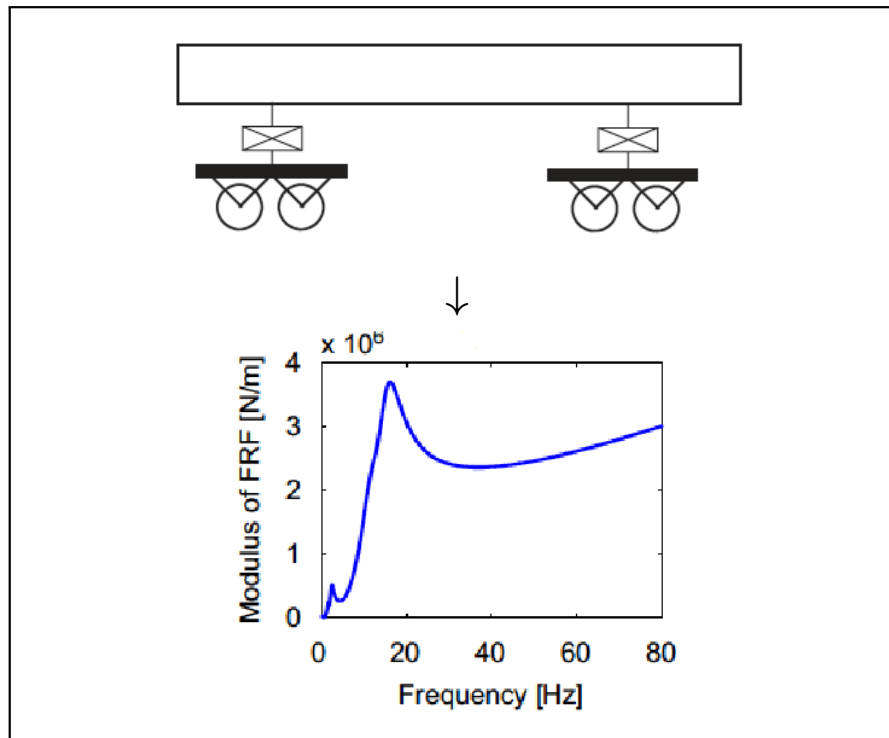
During the next TD, we will study a simplified MDOF vehicle model for obtaining FRFs, which will ultimately allow us to predict vibration.

Remember MATH0487 Elements of statistics...

We wish to extend these ideas from random variables to stochastic processes...

Remember...

- Let X be a second-order random variable defined on $(\Theta, \mathcal{F}, P)$ with values in $\mathbb{R}$, whose mean m_X and variance σ_X^2 are given by

$$m_X = \int_\Theta X \, dP,$$

$$\sigma_X^2 = \int_\Theta (X - m_X)^2 \, dP.$$

- Let $x^{(1)}, \ldots, x^{(\nu)}$ be ν independent samples of X, whose sample mean $\overline{x}^\nu$ and sample variance $(s_X^\nu)^2$ are given by

$$\overline{x}^\nu = \frac{1}{\nu} \sum_{j=1}^\nu x^{(j)},$$

$$(s_X^\nu)^2 = \frac{1}{\nu - 1} \sum_{j=1}^\nu \left(x^{(j)} - \overline{x}^\nu \right)^2.$$

- Bias and variance of the estimator $\overline{X}$:

$$\int_\Theta \overline{X}^\nu \, dP = m_X, \qquad \int_\Theta (\overline{X}^\nu - m_X)^2 \, dP = \frac{\sigma_X^2}{\nu}.$$

- Bias and variance of the estimator $(S_X^\nu)^2$:

$$\int_\Theta (S_X^\nu)^2 \, dP = \sigma_X^2, \qquad \int_\Theta \left((S_X^\nu)^2 - \sigma_X^2 \right)^2 \, dP \approx \frac{\int_\Theta (X - m_X)^4 \, dP - \sigma_X^4}{\nu}.$$

These are the main ideas!

- We will describe a statistical method for inferring from data an estimate of a power spectral density function.
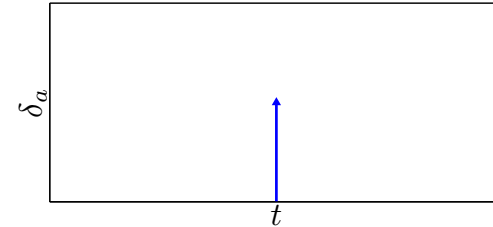
- To show the effectiveness of this statistical method, we will study the bias and variance of the corresponding estimator.

But first... Let's take a step back!
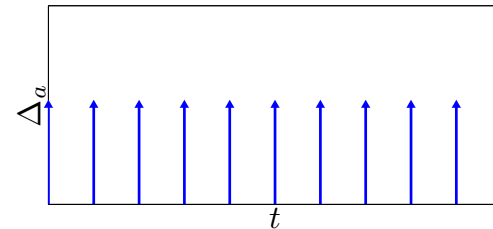
Remember SYST0002 Modélisation et analyse des systèmes...

- For any a in $\mathbb{R}$, let δ_a be the Dirac distribution located at a, which has the property that for any smooth function $f : \mathbb{R} \to \mathbb{R}$, we have

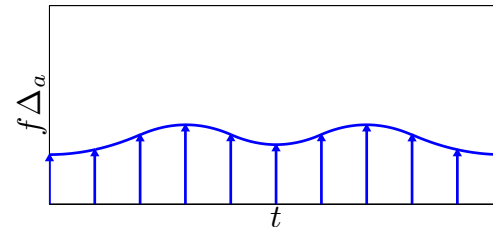$$\delta_a \quad \text{such that} \quad \int_{\mathbb{R}} f(t)\delta_a(t)dt = f(a).$$

- For any $a > 0$, let Δ_a be the Dirac comb defined by

$$\Delta_a = \sum_{k=-\infty}^{+\infty} \delta_{ka}.$$

- Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. The product of f and the Dirac comb Δ_a provides a representation of the sampling of f with period a, that is,

$$f\Delta_a = \sum_{k=-\infty}^{+\infty} f(ka)\delta_{ka}.$$

- Let f have a closed and bounded support. The convolution of f and the Dirac comb Δ_a leads to the repetition of f with period a, that is,

$$(f \star \Delta_a)(t) = \sum_{k=-\infty}^{+\infty} f(t - ka).$$
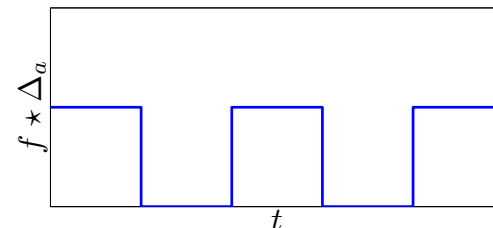
- The Fourier transform of the Dirac distribution and Dirac comb are as follows:

$$\hat{\delta}_a(\omega) = \exp(-i\omega a) \quad \text{and} \quad \widehat{\Delta}_a(\omega) = \frac{2\pi}{a}\Delta_{\frac{2\pi}{a}}(\omega).$$
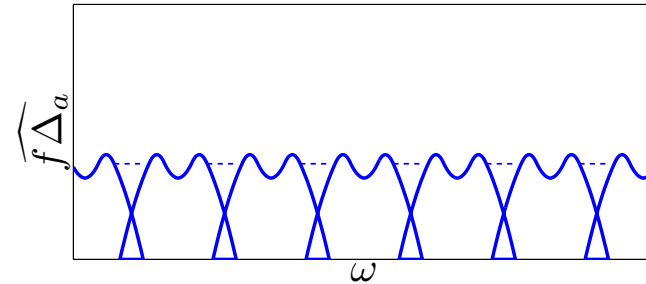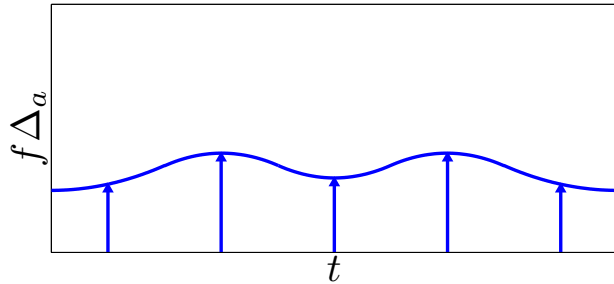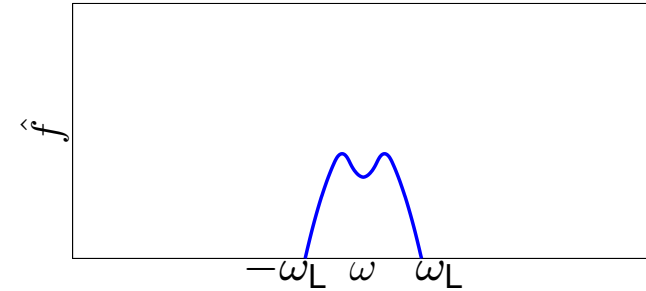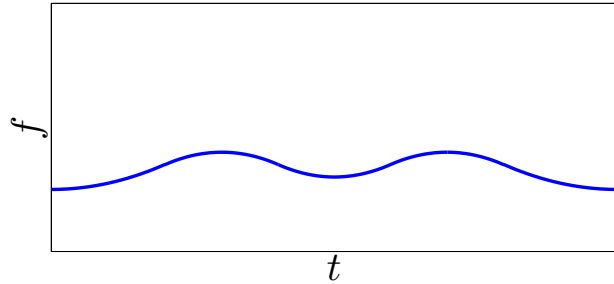
Sampling at a rate lower than the Nyquist rate: $\frac{1}{a} < \frac{\omega_{\mathsf{L}}}{\pi}$.
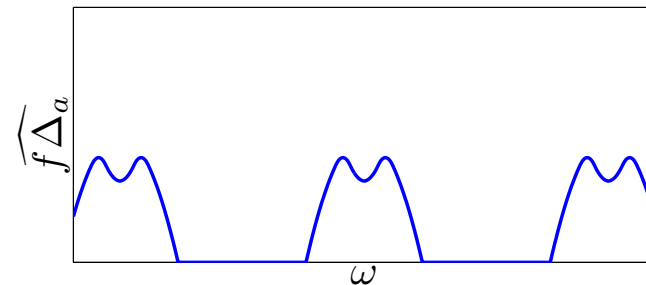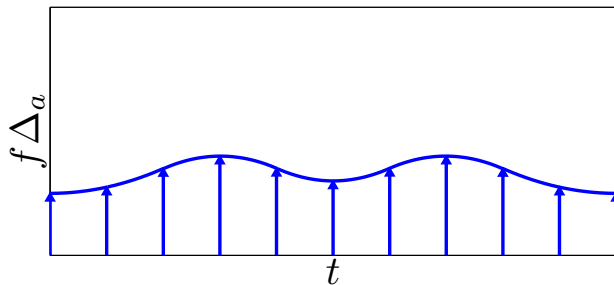
Sampling at a rate higher than the Nyquist rate: $\frac{1}{a} \geq \frac{\omega_{\mathsf{L}}}{\pi}$.

Sampling in time domain results in repetition in the frequency domain.

■ **Poisson formula**:

Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function whose Fourier transform $\hat{f}$ has a closed and bounded support. Then, we have:

$$\left(\hat{f} \star \Delta_{\frac{2\pi}{a}} \right) = \sum_{k=-\infty}^{+\infty} \hat{f}\left(\omega - k \frac{2\pi}{a} \right) = a \sum_{k=-\infty}^{+\infty} f(ka) \exp(-ika\omega),$$

$$\downarrow_{\mathcal{F}^{-1}} \qquad\qquad \uparrow^{\mathcal{F}}$$

$$\mathcal{F}^{-1}\left(\hat{f} \star \Delta_{\frac{2\pi}{a}} \right) = f \mathcal{F}^{-1}\left(\Delta_{\frac{2\pi}{a}} \right) = f a \Delta_a = a \sum_{k=-\infty}^{+\infty} f(ka) \delta_{ka}$$

■ **Shannon theorem**:

Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function whose Fourier transform $\hat{f}$ is a square-integrable function, that is, $\int_{\mathbb{R}} |\hat{f}(\omega)|^2 d\omega < +\infty$, and has a bounded support $\mathrm{supp}(\hat{f}) = [-\omega_{\mathsf{L}}, \omega_{\mathsf{L}}]$. If we sample at a rate that is higher than the Nyquist rate, that is, $\frac{1}{a} \geq \frac{\omega_{\mathsf{L}}}{\pi}$, then we have:

$$f(t) = \sum_{k=-\infty}^{+\infty} f(ka) \frac{\sin\left(\omega_{\mathsf{L}}(t - ka) \right)}{\omega_{\mathsf{L}}(t - ka)},$$

where convergence is such that $\lim_{n \to +\infty} \int_{\mathbb{R}} \left| f(t) - \sum_{k=-n}^{n} f(ka) \frac{\sin\left(\omega_{\mathsf{L}}(t-ka) \right)}{\omega_{\mathsf{L}}(t-ka)} \right|^2 dt = 0$.

**Sampling is useful only if the signal is band limited
and the sampling rate is higher than the Nyquist rate.**

Discrete Fourier transform (DFT)

- Let $\boldsymbol{f} = (f_1, , \ldots, f_m)$ be a vector in $\mathbb{C}^m$. Then, the discrete Fourier transform (DFT) of $\boldsymbol{f}$ is the vector $\hat{\boldsymbol{f}} = (\hat{f}_1, \ldots, \hat{f}_m)$ in $\mathbb{C}^m$ such that:

$$\begin{cases} f_k = \dfrac{1}{m} \displaystyle\sum_{k=1}^{m} \hat{f}_\ell \exp\left(i(k-1)\dfrac{2\pi}{m}(\ell-1) \right), \\[2em] \hat{f}_\ell = \displaystyle\sum_{\ell=1}^{m} f_k \exp\left(-i(k-1)\dfrac{2\pi}{m}(\ell-1) \right). \end{cases}$$

- It has the Parseval property that $\|\boldsymbol{f}\|^2 = \frac{1}{m}\|\hat{\boldsymbol{f}}\|^2$.

- It has the convolution property that for $\boldsymbol{f} \in \mathbb{C}^m$, $\boldsymbol{g} \in \mathbb{C}^m$, and $\boldsymbol{h} \in \mathbb{C}^m$ with $h_k = \sum_{\ell=1}^{m} f_\ell g_{k-\ell}$, $k = 1, \ldots, m$, we have $\hat{\boldsymbol{h}} \in \mathbb{C}^m$ such that $\hat{h}_\ell = \hat{f}_\ell \hat{g}_\ell$, $\ell = 1, \ldots, m$.

- The fast Fourier transform algorithm (FFT) facilitates an efficient recursive computation of the discrete Fourier transform if m is a power of two.

Computation of FT by using DFT

- We begin by expressing the Fourier transform using Poisson's formula

$$\hat{f}(\omega) = 1_{[-\omega_L, \omega_L]}(\omega) a \sum_{k'=-\infty}^{+\infty} f(k'a) \exp(-ik'a\omega),$$

which, under the conditions of Shannon's theorem, we can approximate by

$$\hat{f}^n(\omega) = 1_{[-\omega_L, \omega_L]}(\omega) a \sum_{k'=-n}^{n-1} f(k'a) \exp(-ik'a\omega),$$

where convergence is such that $\lim_{n \to +\infty} \int_{\mathbb{R}} |\hat{f}(\omega) - \hat{f}^n(\omega)|^2 d\omega = 0$.

- Using $\omega_\ell = -\omega_L + (\ell - 1)\Delta\omega$, $\ell = 1, \ldots, 2n$, where $\Delta\omega = \omega_L/n$, we obtain

$$\hat{f}^n(\omega_\ell) = a \sum_{k'=-n}^{n-1} f(k'a) \exp\left(-ik'a\left(-\omega_L + (\ell - 1)\Delta\omega\right)\right).$$

- Assuming that the sampling rate is equal to the Nyquist rate, that is, $\frac{1}{a} = \frac{\omega_L}{\pi}$, we obtain

$$\hat{f}^n(\omega_\ell) = \sum_{k'=-n}^{n-1} \left(af(k'a)\exp(ik'\pi)\right) \exp\left(-ik'\frac{\pi}{n}(\ell - 1)\right).$$

- Finally, setting $k = k' + n + 1$, we recover the discrete Fourier transform

$$\exp\left(-i\pi(\ell - 1)\right)\hat{f}^n(\omega_\ell) = \sum_{k=1}^{2n} \left(af((k - n - 1)a)\exp\left(i(k - n - 1)\pi\right)\right) \exp\left(-i(k - 1)\frac{2\pi}{2n}(\ell - 1)\right).$$

<div style="border:1px solid blue">

time step:
$$a = \frac{\pi}{\omega_L}.$$
time discretization:
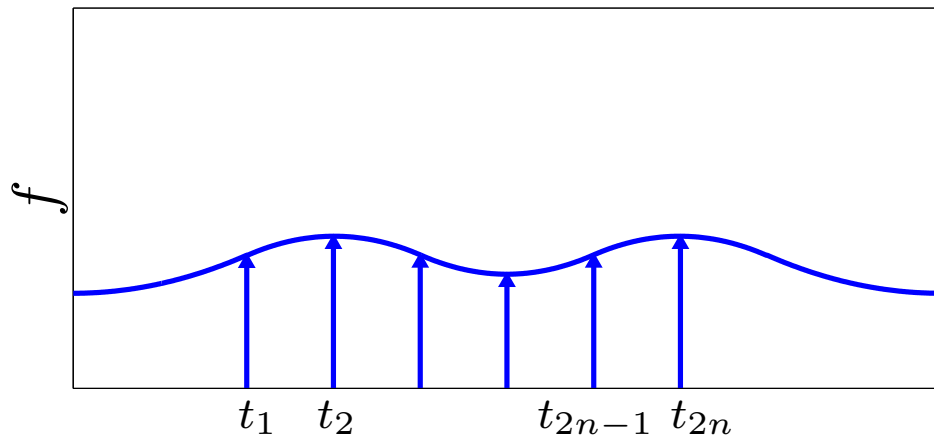$$t_k = (k - n - 1)a,$$
$$k = 1, \ldots, 2n.$$
freq. resolution:
$$\Delta\omega = \frac{\omega_L}{n}$$
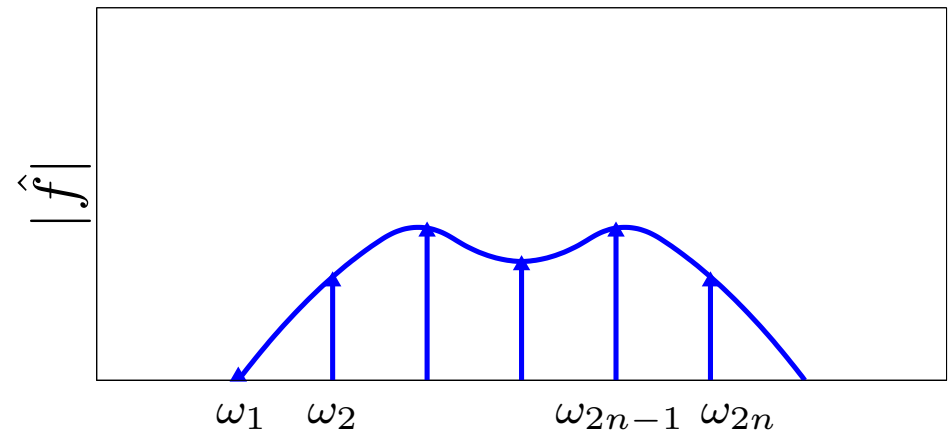freq. discretization:
$$\omega_\ell = -\omega_L + (\ell - 1)\Delta\omega,$$
$$\ell = 1, \ldots, 2n.$$

</div>

$$t_k = (k - n - 1)a, \ k = 1, \ldots, 2n.$$

Timestep $a = \frac{\pi}{\omega_{\mathsf{L}}}$.

$$\omega_\ell = -\omega_{\mathsf{L}} + (\ell - 1)\Delta\omega, \ \ell = 1, \ldots, 2n.$$

Frequency step $\Delta\omega = \frac{\omega_{\mathsf{L}}}{n}$.

$$\exp\left(-i\pi(\ell - 1)\right)\hat{f}^n(\omega_\ell) = \sum_{k=1}^{2n}\left(af\left((k-n-1)a\right)\exp\left(i(k-n-1)\pi\right)\right)\exp\left(-i(k-1)\frac{2\pi}{2n}(\ell - 1)\right).$$

Now... let's see how this extends to our stochastic processes!

Shannon's theorem for stochastic processes
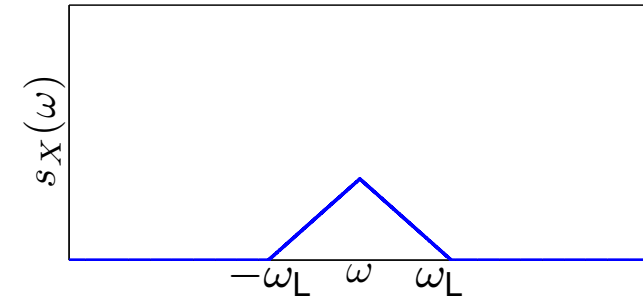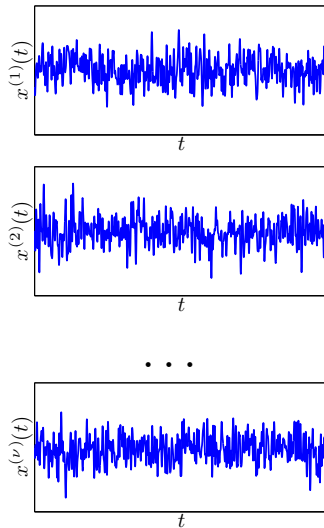
- Let $\{X(t),\ t \in \mathbb{R}\}$ be a mean-square stationary zero-mean second-order stochastic process defined on $(\Theta, \mathcal{F}, P)$ indexed by $\mathbb{R}$ with values in $\mathbb{R}$. Let $\{X(t),\ t \in \mathbb{R}\}$ admit a power spectral density function $s_X : \mathbb{R} \to \mathbb{R}$ that is bounded and has a bounded support $\mathrm{supp}(s_X) = [-\omega_\mathsf{L}, \omega_\mathsf{L}]$. If we sample at a rate that is higher than the Nyquist rate, that is, $\frac{1}{a} \geq \frac{\omega_\mathsf{L}}{\pi}$, then we have:

$$X(t) = \sum_{k=-\infty}^{+\infty} X(ka) \frac{\sin\big(\omega_\mathsf{L}(t - ka)\big)}{\omega_\mathsf{L}(t - ka)},$$

where convergence is such that

$$\lim_{n \to +\infty} \int_\Theta \left| X(t) - \sum_{k=-n}^{n} X(ka) \frac{\sin\big(\omega_\mathsf{L}(t-ka)\big)}{\omega_\mathsf{L}(t-ka)} \right|^2 dP = 0, \ \forall t \in \mathbb{R}.$$

As in the deterministic case, sampling is useful only if the stochastic process is band limited and the sampling rate is higher than the Nyquist rate.

Well... OK... but what does all of this have to do with our estimation?
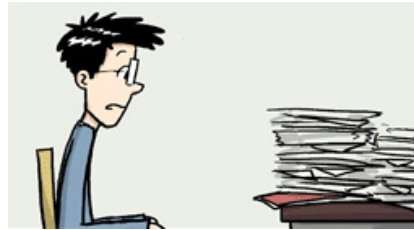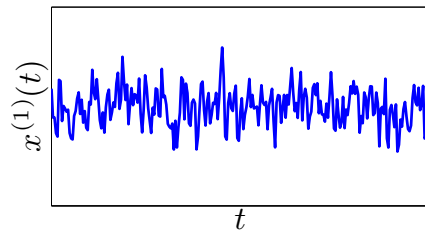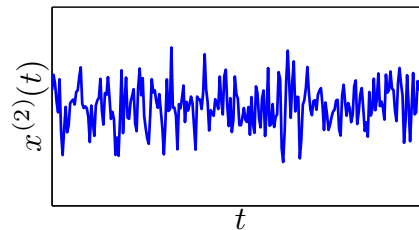
Estimation

- **Context**:
  Let $\{X(t),\ t \in \mathbb{R}\}$ be a mean-square stationary zero-mean second-order stochastic process defined on $(\Theta, \mathcal{F}, P)$ indexed by $\mathbb{R}$ with values in $\mathbb{R}$. Let $\{X(t),\ t \in \mathbb{R}\}$ admit a power spectral density function $s_X : \mathbb{R} \to \mathbb{R}$ that is continuous and has a bounded support $\mathrm{supp}(s_X) = [-\omega_\mathsf{L}, \omega_\mathsf{L}]$.

- **Available information**:
  Let the available information consist of ν trajectories $\{x^{(j)}(t),\ 0 \le t \le \tau\}$, each of signal length τ:
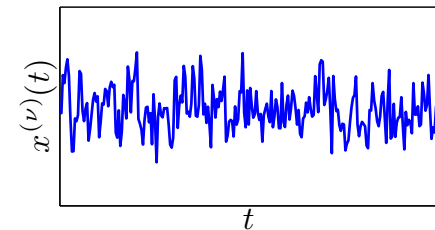
  $\{x^{(1)}(t),\ 0 \le t \le \tau\}.$ $\qquad$ $\{x^{(2)}(t),\ 0 \le t \le \tau\}.$ $\qquad \cdots \qquad$ $\{x^{(\nu)}(t),\ 0 \le t \le \tau\}.$

- **Objective of the estimation**:
 We will consider the inference of an estimate $s_X^{\tau,\nu}$ of the power spectral density function s_X from the available information $\big\{\{x^{(j)}(t),\ 0 \le t \le \tau\},\ 1 \le j \le \nu\big\}$.

- **Available estimation methods**:
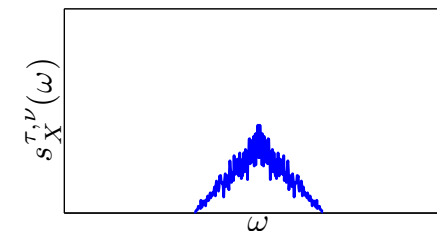 - Indirect method: estimate first the autocorrelation and then the p.s.d. function therefrom.
  - Direct method: let's look at this method in more detail!

- **Step 1:** Transform each trajectory into the frequency domain using the FT, assuming the signal vanishes outside of the interval $[0, \tau]$:
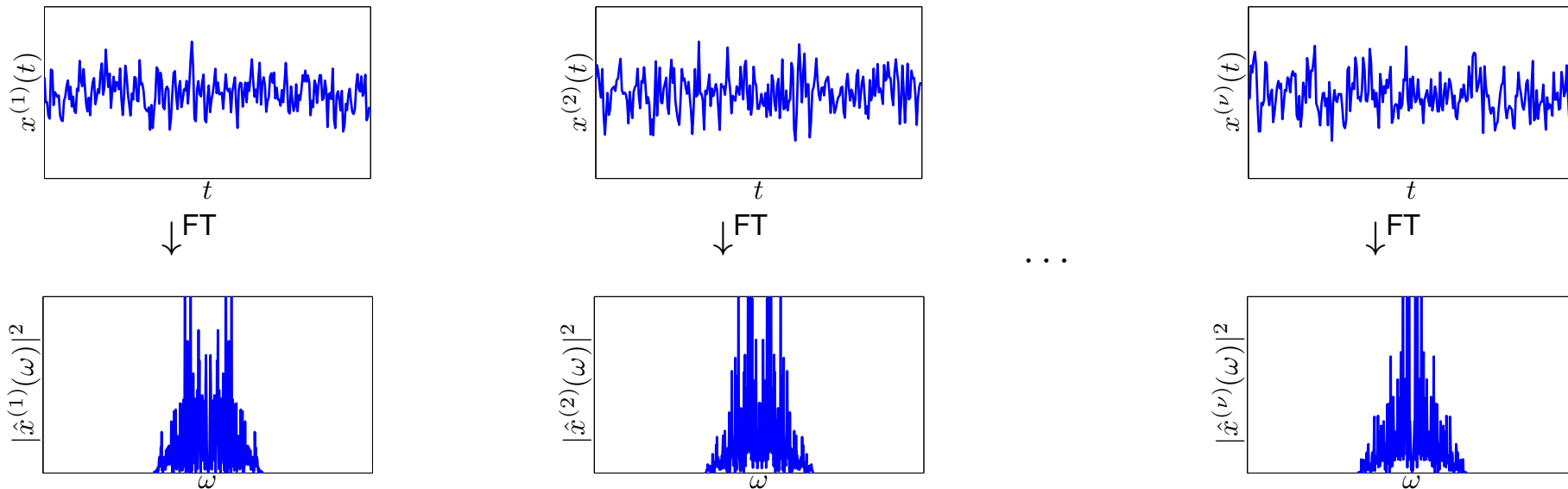
- **Step 2**: Compute the estimate $s_X^{\tau,\nu}$ of s_X as follows:

$$s_X(\omega) \approx s_X^{\tau,\nu}(\omega) = \frac{1}{\tau}\frac{1}{\nu}\sum_{j=1}^{\nu}|\hat{x}^{(j)}(\omega)|^2, \quad \forall \omega \in \mathbb{R},$$

- Here, the FT can be approximated by DFT after sampling at a rate higher than the Nyquist rate.

Why does this work?

**Let's look at convergence with respect to the signal length τ (bias)
and number of trajectories ν (variance)!**

Estimation

- The mean of the estimator of the power spectral density function reads as follows:

$$\int_\Theta S_X^{\tau,\nu}(\omega)dP = \frac{1}{\tau}\int_\Theta |\widehat{X}(\omega)|^2 dP, \quad \text{where} \quad \widehat{X}(\omega) = \int_\mathbb{R} 1_{[0,\tau]}(t)X(t)\exp(-i\omega t)dt.$$
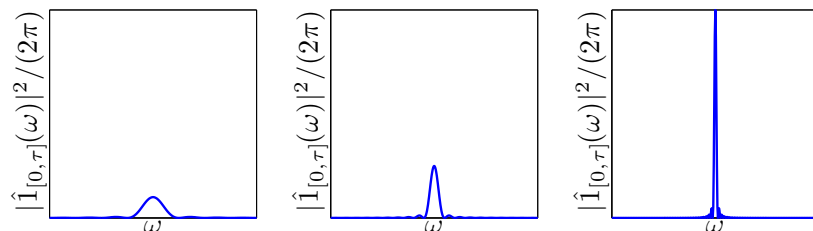
- Elaborating and then using the expression for the autocorrelation function, we obtain

$$\int_\Theta S_X^{\tau,\nu}dP = \frac{1}{\tau}\int_\mathbb{R}\int_\mathbb{R} 1_{[0,\tau]}(t)1_{[0,\tau]}(t')r_X(t-t')\exp\big(-i\omega(t-t')\big)dtdt',$$

$$= \frac{1}{\tau}\int_\mathbb{R}\int_\mathbb{R} 1_{[0,\tau]}(t)1_{[-\tau,0]}\big((t-t')-t\big)r_X(t-t')\exp\big(-i\omega(t-t')\big)dtd(t-t').$$

- Using the relationships between the convolution, product, and Fourier transform, we obtain

$$\int_\Theta S_X^{\tau,\nu}(\omega)dP = \frac{1}{\tau}\left(\frac{1}{2\pi}|\hat{1}_{[0,\tau]}|^2 \star s_X\right)(\omega).$$

- In the limit as the signal length τ of the trajectories increases to infinity, we obtain

$$\lim_{\tau\to+\infty}\frac{1}{\tau}\frac{1}{2\pi}|\hat{1}_{[0,\tau]}|^2 = \delta_0, \quad \lim_{\tau\to+\infty}\int_\Theta S_X^{\tau,\nu}(\omega)dP = s_X(\omega).$$

For finite signal length τ, there is leakage, that is, local averaging of frequency components. As the signal length τ increases, leakage decreases. The estimator is asymptotically unbiased.

Frequency resolution improves with increasing signal length τ.

Estimation

- The variance of the estimator of the power spectral density function reads as follows:

$$\int_{\Theta} \left(S_X^{\tau,\nu}(\omega) - \int_{\Theta} S_X^{\tau,\nu}(\omega)dP \right)^2 dP = \int_{\Theta} |S_X^{\tau,\nu}(\omega)|^2 dP - \left(\int_{\Theta} S_X^{\tau,\nu}(\omega)dP \right)^2.$$

- Using $S_X^{\tau,\nu}(\omega) = \frac{1}{\tau} \frac{1}{\nu} \sum_{j=1}^{\nu} |\widehat{X}^{(j)}(\omega)|^2$ where $\widehat{X}^{(j)}(\omega) = \int_{\mathbb{R}} 1_{[0,\tau]}(t) X^{(j)}(t) \exp(-i\omega t)dt$, we obtain

$$\int_{\Theta} \left(S_X^{\tau,\nu}(\omega) - \int_{\Theta} S_X^{\tau,\nu}(\omega)dP \right)^2 dP = \frac{1}{\tau^2} \frac{1}{\nu} \left(\int_{\Theta} |\widehat{X}(\omega)|^4 dP - \left| \int_{\Theta} |\widehat{X}(\omega)|^2 dP \right|^2 \right).$$

Provided that the additional condition that $\int_{\Theta} |\widehat{X}(\omega)|^4 dP < +\infty$ is fulfilled, the error introduced by the use of only a finite number of trajectories decreases with increasing number of trajectories ν.

Error introduced by the use of only a finite number of trajectories decreases with increasing number of trajectories ν.

Estimation

- What if the available information consists of only a single trajectory?

- **Step 0:** Split the trajectory into ν subtrajectories of equal signal length τ:
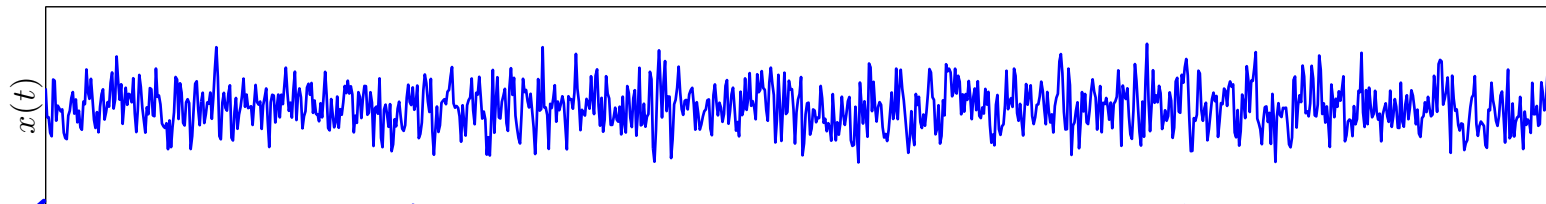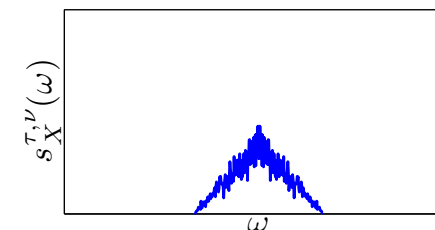

- **Step 1:** Transform each subtrajectory into the frequency domain using the FT:
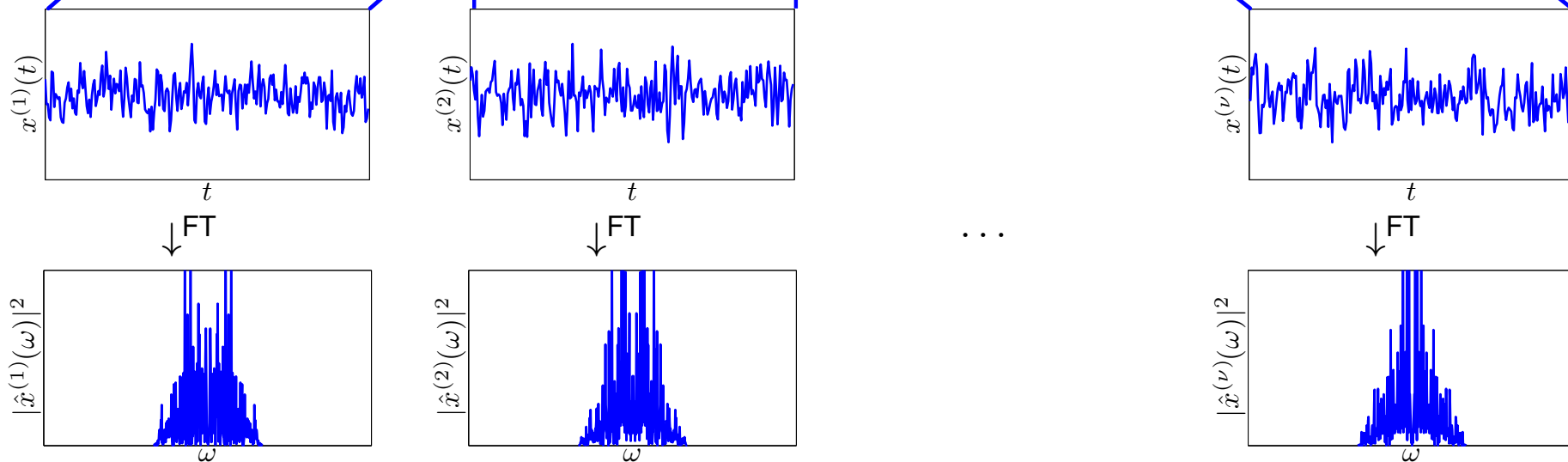
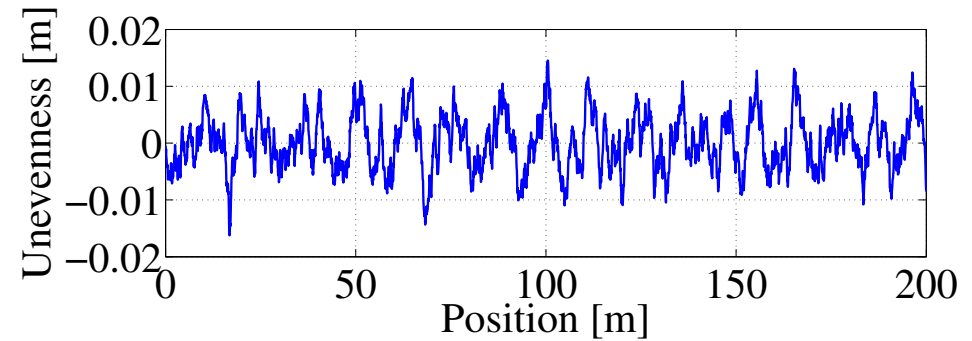- **Step 2:** Compute the estimate $s_X^{\tau,\nu}$ of s_X as follows:

$$s_X(\omega) \approx s_X^{\tau,\nu}(\omega) = \frac{1}{\tau}\frac{1}{\nu}\sum_{j=1}^{\nu}|\hat{x}^{(j)}(\omega)|^2, \quad \forall \omega \in \mathbb{R},$$

- Determine ν and τ in such a way that a suitable tradeoff between bias and variance is achieved.

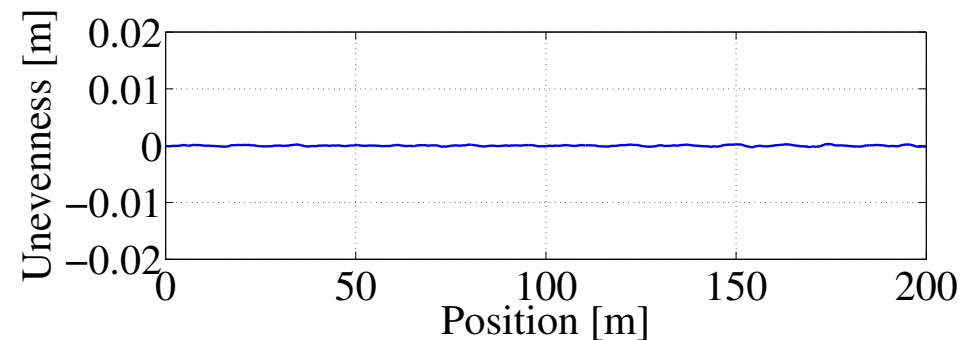|	time	space
	frequency f	
	circular frequency $\omega = 2\pi f$	wavenumber ξ
	period $\frac{1}{f} = \frac{2\pi}{\omega}$	wavelength $\frac{2\pi}{\xi}$

- **Record w1.mat (deteriorated jointed plain concrete pavement):**

- **Record w2.mat (concrete block pavement):**
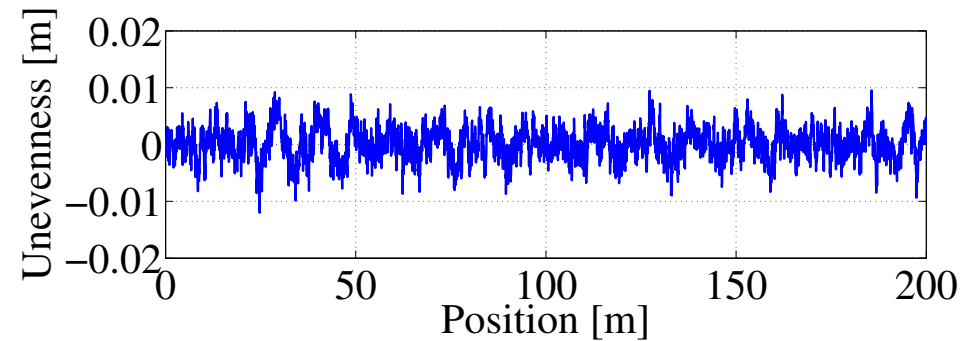
- **Record w3.mat (high speed train track):**

■ The records w1.mat, w2.mat, and w3.mat each contain a vector w of length $m = 4096$. The components $w_1, w_2, w_3, \ldots, w_m$ of w are the values taken by the unevenness at a sequence of equally spaced points $0, a, 2a, \ldots, (m-1)a$, where the sampling interval is equal to $a = 0.05\,\text{m}$.

1. For each record, please complete the following steps:
   (a) Plot the unevenness as a function of the position.
   (b) Split the record into $\nu = 8$ subrecords of equal length and transform each subrecord into the wavenumber domain by using the discrete Fourier transform. For one of the subrecords, plot the amplitude of the transformed unevenness as a function of the wavenumber, first on a linear scale and then on a loglog scale.
   (c) Estimate the power spectral density function. Plot this estimate as a function of the wavenumber, first on a linear scale and then on a loglog scale.
   (d) Consider the approximation of the power spectral density function using an expression of the form $s_W(\xi) = s_0 / \left(1 + \frac{|\xi|}{\xi_0}\right)^\alpha$, in which $\xi_0 = 0.5\,\text{m}^{-1}$. For each record, use a linear regression on a loglog scale to deduce values for s_0 and α from the estimate you obtained under (c).

2. Interpret your results.

List of references

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