MATH0074 Elements of Stochastic Processes, March 19, 2013

# Traffic-induced Vibration
# TD 1

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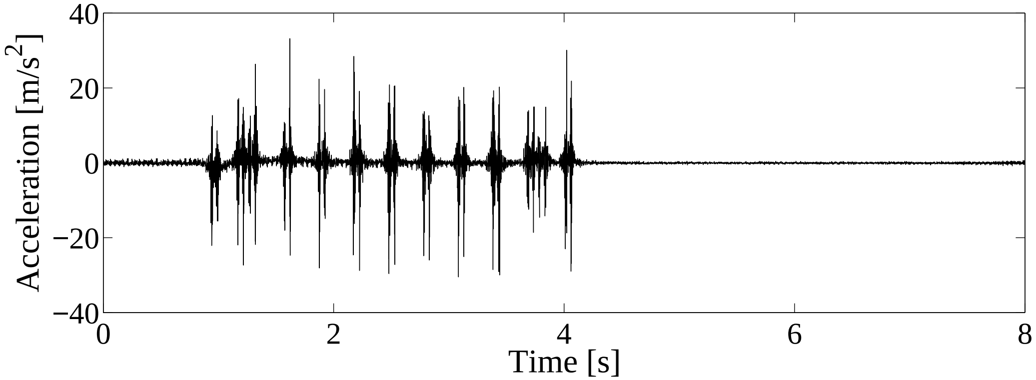
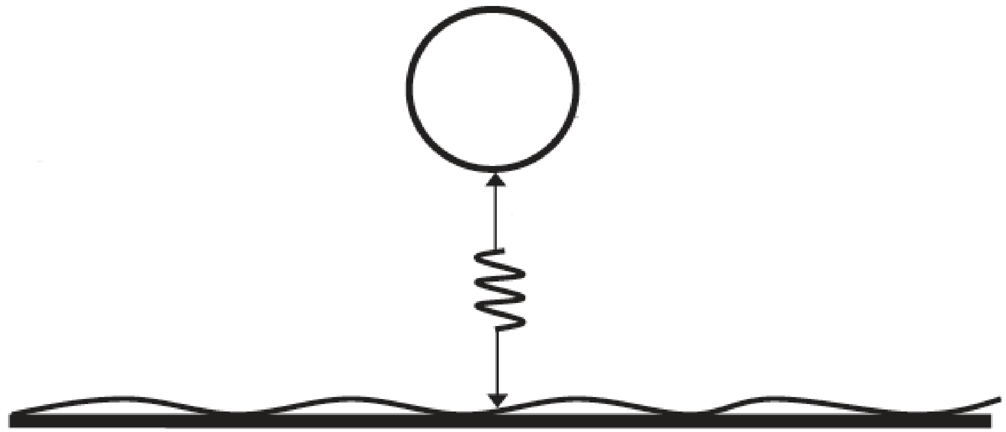



**Vibrations and noise are generated as rough wheels roll over rough supports**.

# Motivation



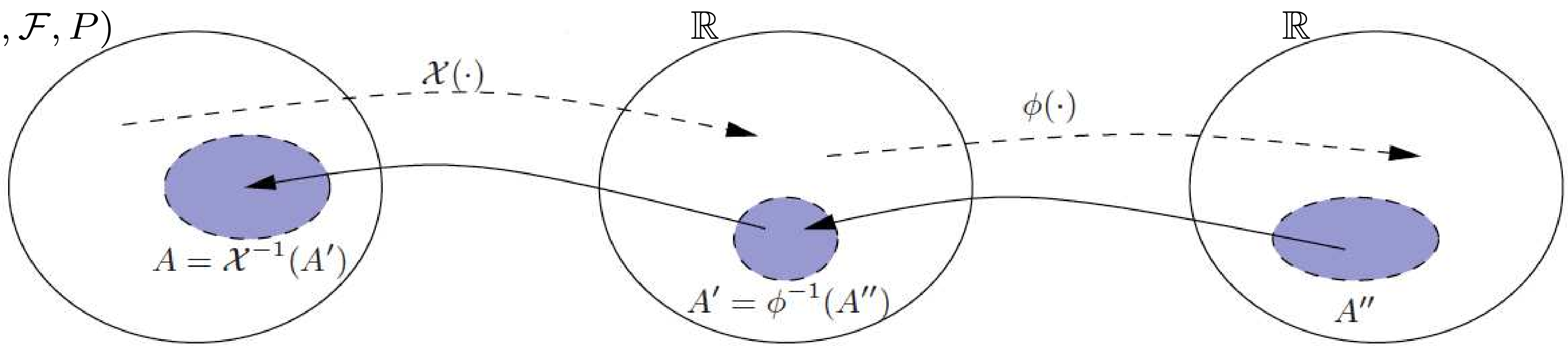
**During this TD, we will study the main principles using a 1-DOF model.**

**Remember MATH0062 Eléments du calcul des probabilités...**

**We wish to extend these ideas from random variables to stochastic processes...**

# Remember...

- $(\Theta, \mathcal{F}, P)$




Let $X$ be a random variable defined on a probability triple $(\Theta, \mathcal{F}, P)$ with values in $\mathbb{R}$.
Let $Y = \phi(X)$ be the random variable obtained by the transformation of $X$ under the mapping $\phi$.

- The **mean** and the **variance** of the random variables $X$ and $Y = \phi(X)$ are given by

$$\mu_X = \int_\Theta X(\theta) dP(\theta), \qquad \mu_Y = \int_\Theta Y(\theta) dP(\theta) = \int_\Theta \phi\big(X(\theta)\big) dP(\theta),$$

$$\sigma_X^2 = \int_\Theta \big(X(\theta) - \mu_X\big)^2 dP(\theta), \quad \sigma_Y^2 = \int_\Theta \big(Y(\theta) - \mu_Y\big)^2 dP(\theta) = \int_\Theta \Big(\phi\big(X(\theta)\big) - \mu_Y\Big)^2 dP(\theta).$$

- If the mapping $\phi$ is **linear**, that is, $Y = \phi(X) = aX + b$, we obtain

$$\mu_Y = \int_\Theta \big(aX(\theta) + b\big) dP(\theta) = a \int_\Theta X(\theta) dP(\theta) + b \int_\Theta dP(\theta) = a\mu_X + b,$$

$$\sigma_Y^2 = \int_\Theta \Big(\big(aX(\theta) + b\big) - (a\mu_X + b)\Big)^2 dP(\theta) = a^2 \int_\Theta \big(X(\theta) - \mu_X\big)^2 dP(\theta) = a^2 \sigma_X^2.$$

# These are the main ideas!

- The mean function plays for a stochastic process the role the mean plays for a random variable.

- The covariance function and the power spectral density function play for a stochastic process the role the variance plays for a random variable.

- If one stochastic process is the transformation of another stochastic process under a linear mapping, the covariance function and the power spectral density function of the former can be deduced from those of the latter by using the properties of that linear mapping.

# System of notation

- A lowercase letter, for example, $x$, is a real deterministic variable.
- A boldface lowercase letter, for example, $\boldsymbol{x} = [x_1, \ldots, x_m]^{\mathrm{T}}$, is a real deterministic column vector.
- An uppercase letter, for example, $X$, is a real random variable.
  Exceptions: $\Theta$ (sample space), $P$ (probability), and $\Gamma$ (gamma function).
- A boldface uppercase letter, for example, $\boldsymbol{X} = [X_1, \ldots, X_m]^{\mathrm{T}}$, is a real random column vector.
- An uppercase letter between square brackets, for example, $[A]$, is a real deterministic matrix.
- A boldface uppercase letter between square brackets, for example, $[\boldsymbol{A}]$, is a real random matrix.

**Let's take a step back!**

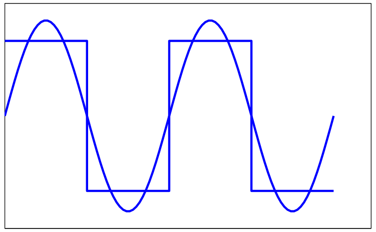
**Remember MATH0007 Analyse Mathématique II...**

# Analysis

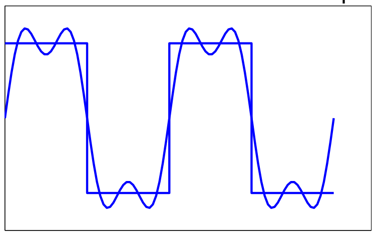
- Let $f : \mathbb{R} \to \mathbb{R}$ be a nonconstant **periodic function** that has period $a$ and is square-integrable on $[-a/2, a/2]$, that is, $\int_{-a/2}^{a/2} |f(t)|^2 dt < +\infty$. Then, the Fourier series of $f$ reads as follows:

$$\begin{cases} f(t) = \displaystyle\sum_{k=-\infty}^{+\infty} f_k \exp\left(ik\frac{2\pi}{a}t\right), \\ f_k = \displaystyle\frac{1}{a}\int_{-a/2}^{a/2} f(t) \exp\left(-ik\frac{2\pi}{a}t\right) dt. \end{cases}$$

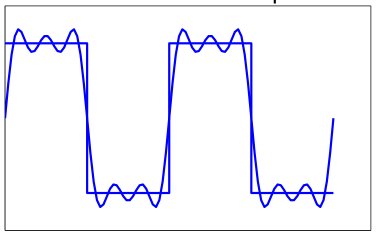
- It has the approximation property that $\lim_{n\to+\infty} \int_{-a/2}^{a/2} \left| f(t) - \sum_{k=-n}^{n} f_k \exp\left(ik\frac{2\pi}{a}t\right)\right|^2 dt = 0.$



$n = 1.$



$n = 3.$



$n = 5.$

- The more regular the function $f$, the faster the coefficients $f_k$ tend to zero:

| regularity of $f$ on $[-a/2, a/2]$ | decay of $f_k$ | proof |
|---|---|---|
| integrable | $f_k \to 0$ | Riemann-Lebesque |
| square-integrable | $\sum_{k=-\infty}^{+\infty} \|f_k\|^2 < +\infty$ | Parseval |
| continuously differentiable | $\sum_{k=-\infty}^{+\infty} \|f_k\| < +\infty$ | integration by parts |

# Analysis

- Let $f : \mathbb{R} \to \mathbb{R}$ be an **integrable function** , that is, $\int_{\mathbb{R}} |f(t)|dt < +\infty$. Then, the Fourier transform (FT) $\hat{f}$ of $f$ is the bounded, continuous function $\hat{f}$ from $\mathbb{R}$ into $\mathbb{C}$ such that
$$\hat{f}(\omega) = \mathcal{F}f(\omega) = \int_{\mathbb{R}} \exp(-i\omega t) f(t) dt.$$
The Fourier transform of an integrable function is not necessarily integrable itself.
- Let $f : \mathbb{R} \to \mathbb{R}$ be a **square-integrable function** , that is, $\int_{\mathbb{R}} |f(t)|^2 dt < +\infty$. Then, the Fourier transform $\hat{f}$ of $f$ is the square-integrable function $\hat{f}$ from $\mathbb{R}$ into $\mathbb{C}$ such that
$$\begin{cases} \hat{f}(\omega) = \mathcal{F}f(\omega) = \int_{\mathbb{R}} \exp(-i\omega t) f(t) dt, \\ f(t) = \mathcal{F}^{-1}\hat{f}(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \exp(i\omega t) \hat{f}(\omega) d\omega. \end{cases}$$
- We have the **derivation** property that $d^k \hat{f}/d\omega^k = \widehat{(-it)^k f}$ and $\widehat{d^k f/dt^k} = (i\omega)^k \hat{f}$.
- **Convolution** and Fourier transform:

| regularity of $f$ | regularity of $g$ | implied regularity of $f \star g$ | implied Fourier property |
|---|---|---|---|
| integrable | integrable | integrable | $\widehat{f \star g}(\omega) = \hat{f}(\omega)\hat{g}(\omega)$ |
| integrable | bounded | bounded and continuous | |
| square-integrable | square-integrable | bounded and continuous | $\begin{cases} f \star g(t) = \mathcal{F}^{-1}(\hat{f}\hat{g})(t) \\ \widehat{fg}(\omega) = \frac{1}{2\pi}(\hat{f} \star \hat{g})(\omega) \end{cases}$ |
| square-integrable | integrable | square-integrable | $\widehat{f \star g}(\omega) = \hat{f}(\omega)\hat{g}(\omega)$ |

**And... remember SYST0002 Modélisation et analyse des systèmes...**

- **Ordinary Differential Equation (ODE)**:

$$\sum_{k=0}^{q} b_k \frac{d^k u_f}{dt^k}(t) = f(t), \quad t \in \mathbb{R}, \quad b_q \neq 0, \quad q \geq 1.$$

- **Algebraic equation** obtained by FT (if it exists):

$$\sum_{k=0}^{q} b_k (i\omega)^k \hat{u}_f(\omega) = \hat{f}(\omega), \quad \omega \in \mathbb{R}.$$

- **Frequency Response Function (FRF)**:

$$\hat{u}_f(\omega) = \hat{h}(\omega)\hat{f}(\omega) \qquad \text{where} \qquad \hat{h}(\omega) = \frac{1}{p(i\omega)} = \frac{1}{\sum_{k=0}^{q} b_k (i\omega)^k}.$$

If $1/p$ has no poles on the imaginary axis, $\hat{h} : \mathbb{R} \to \mathbb{C}$ is a bounded, square-integrable function.

- **Impulse response function**:

$$h = \mathcal{F}^{-1}(\hat{h}).$$

If $1/p$ has no poles on the imaginary axis, $h : \mathbb{R} \to \mathbb{R}$ is an integrable, square-integrable, and bounded function that decays rapidly at infinity and is continuous (except perhaps at the origin).

- **Generalized solution** :

$$u_f = h \star f, \quad \text{that is,} \quad u_f(t) = \int_{\mathbb{R}} h(s) f(t-s)\, ds, \quad \text{(using convolution that makes sense).}$$

# System theory

- In summary, we associated to the ordinary differential equation a convolution filter, thus allowing it to be studied using the combined tools of "analysis" and "system theory:"

$$\text{input } f \longrightarrow \boxed{\text{system } h\star} \longrightarrow \text{output } u_f\ .$$

- **System theory**:

$h\star : X \to Y$ is stable $\Leftrightarrow$ $\exists c > 0,\ \forall f \in X$ with ess.sup$|f| < +\infty$ : ess.sup$|h \star f| \le c\,$ess.sup$|f|$ $\Leftarrow$ $1/p$ has no poles on the imaginary axis

$h\star : X \to Y$ is causal $\Leftrightarrow$ supp$(h) \subset [0, +\infty[$ $\Leftarrow$ poles of $1/p$ are located left of the imaginary axis

- **Initial-Value Problem (IVP)**:

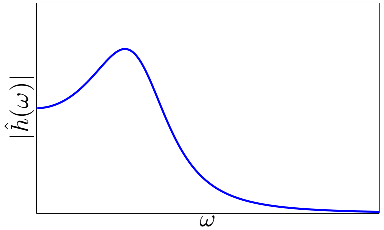
$$\begin{cases} \displaystyle\sum_{k=0}^{q} b_k \frac{d^k u}{dt^k}(t) = f(t), \quad t > 0, \quad b_q \neq 0, \quad q \geq 0, \\ \displaystyle u(0) = u_0, \ldots, \frac{d^{q-1}u}{dt^{q-1}}(0) = u_{q-1}, \end{cases} .$$

If poles of $1/p$ are located left of imaginary axis, we have

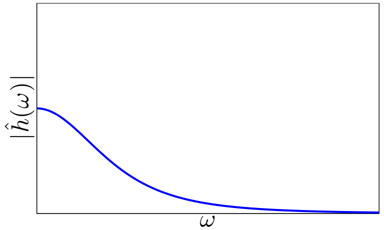
$$u(t) = \underbrace{u_h(t)}_{\text{transient response}} + \underbrace{u_f(t)}_{\text{forced response}} \qquad \text{with} \qquad \lim_{t\to+\infty} |u(t) - u_f(t)| = 0,$$

that is, the response $u$ converges to the forced response $u_f$ as time advances.

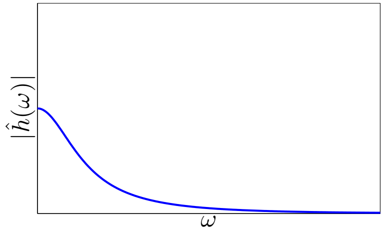
- **ODE** $m\frac{d^2u_f}{dt^2}(t) + d\frac{du_f}{dt}(t) + ku_f(t) = f(t), \quad t \in \mathbb{R}, \quad m, d, k > 0.$
- **FRF** $\hat{h}(\omega) = \frac{1}{p(i\omega)} = \frac{1}{-\omega^2 m + i\omega d + k}$.




Underdamped: $d < 2\sqrt{km}$.

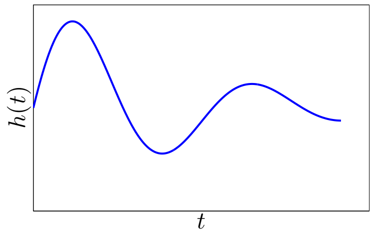



Critically damped: $d = 2\sqrt{km}$.

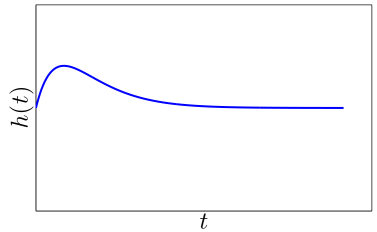



Overdamped: $d > 2\sqrt{km}$.

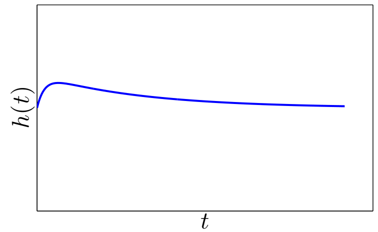
- **Impulse response function** $h = \mathcal{F}^{-1}(\hat{h})$.




Underdamped: $d < 2\sqrt{km}$.




Critically damped: $d = 2\sqrt{km}$.




Overdamped: $d > 2\sqrt{km}$.

- **Generalized solution** $u_f = h \star f$, that is, $u_f(t) = \int_{\mathbb{R}} h(s)f(t-s)ds$.
- **IVP** $m\frac{d^2u}{dt^2}(t) + d\frac{du}{dt}(t) + ku(t) = f(t), \quad t > 0, \quad m, d, k > 0, \quad u(0) = u_0, \quad \frac{du}{dt}(0) = u_1.$
  We have $\lim_{t\to+\infty} |u(t) - u_f(t)| = 0$, that is, $u$ converges to $u_f$ as time advances.

**Now... let's apply this to our stochastic processes!**

# Stochastic processes

- A **second-order stochastic process** $\{X(t),\ t \in \mathbb{R}\}$ defined on a probability triple $(\Theta, \mathcal{F}, P)$ indexed by $\mathbb{R}$ with values in $\mathbb{R}$ is a function that associates to any index $t$ in $\mathbb{R}$ a second-order random variable $X(t)$ defined on $(\Theta, \mathcal{F}, P)$ with values in $\mathbb{R}$, that is,

$$\sqrt{\int_\Theta |X(t)|^2 dP} < +\infty, \quad \forall t \in \mathbb{R}.$$

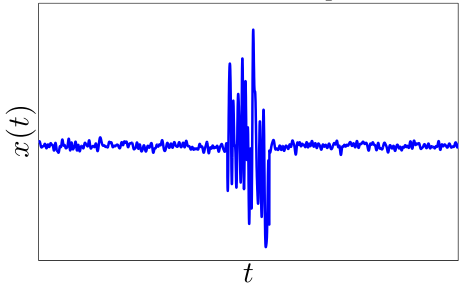
  The second-order statistical descriptors of $\{X(t),\ t \in \mathbb{R}\}$ are defined as follows:
  - **mean function** $m_X(t) = \int_\Theta X(t)dP, \quad t \in \mathbb{R}$,
  - **autocorrelation function** $r_X(t,t') = \int_\Theta X(t)X(t')dP, \quad t,t' \in \mathbb{R}$,
  - **covariance function** $c_X(t,t') = \int_\Theta \big(X(t) - m_X(t)\big)\big(X(t') - m_X(t')\big)dP, \quad t,t' \in \mathbb{R}$,

- The second-order stochastic process $\{X(t),\ t \in \mathbb{R}\}$ is **mean-square stationary** if

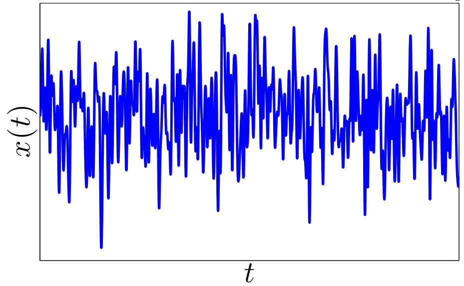
$$m_X(t) = m_X \qquad \text{independent of } t,$$

$$r_X(t,t') = r_X(t-t') \qquad \text{dependent only on lag } t-t'.$$

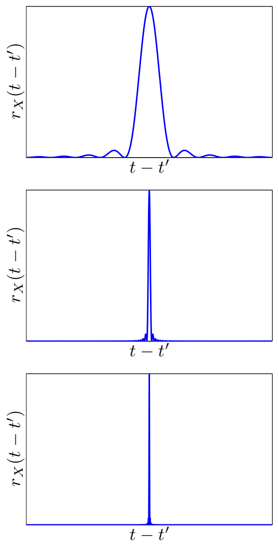
- Trajectory of a nonstationary and mean-square stationary second-order stochastic process:



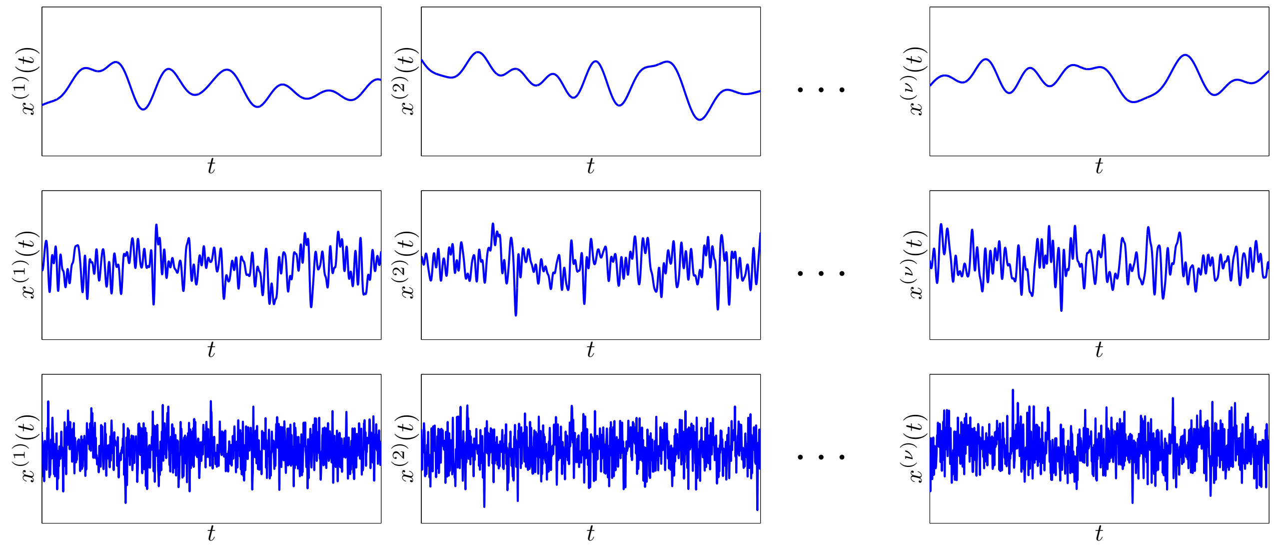

Nonstationary.




Mean-square stationary.




autocorrelation
function




trajectories

autocorrelation function and a few trajectories
of a mean-square stationary zero-mean second-order stochastic process

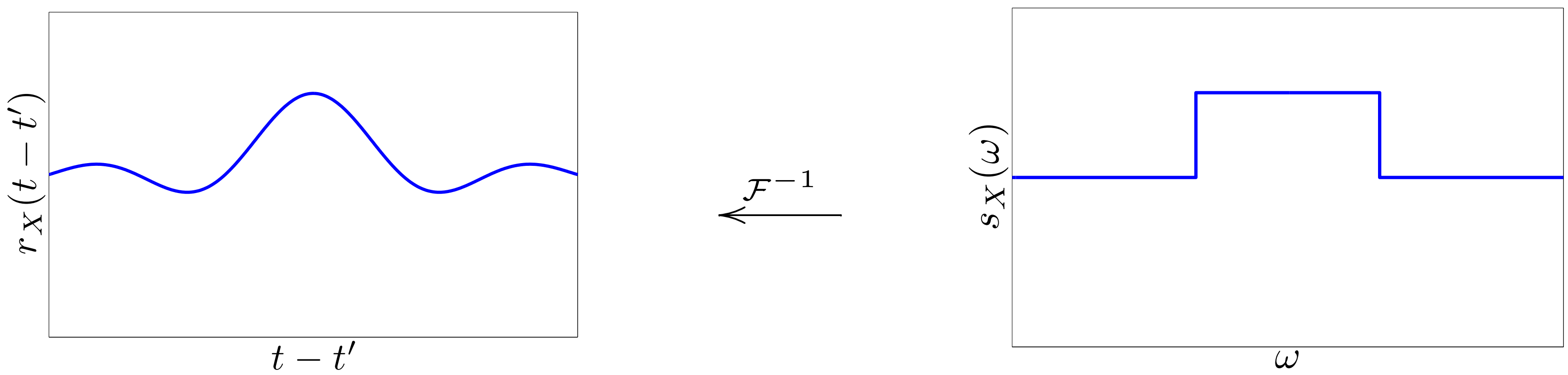
- We defined the Fourier transform only for integrable and for square-integrable functions. The trajectories of a mean-square stationary second-order stochastic process are not in general integrable or square-integrable. Thus, we cannot in general take their Fourier transform.

- To circumvent this difficulty, the frequency-domain analysis of stochastic processes involves the Fourier transform of the autocorrelation function, which often admits a Fourier transform.

- The mean-square stationary second-order stochastic process $\{X(t),\ t \in \mathbb{R}\}$ has a **power spectral density function** if there exists an integrable function $s_X : \mathbb{R} \to \mathbb{R}$ such that

$$r_X(t-t') = \frac{1}{2\pi} \int_{\mathbb{R}} \exp\big(i\omega(t-t')\big) s_X(\omega) d\omega, \quad \forall (t-t') \in \mathbb{R}.$$

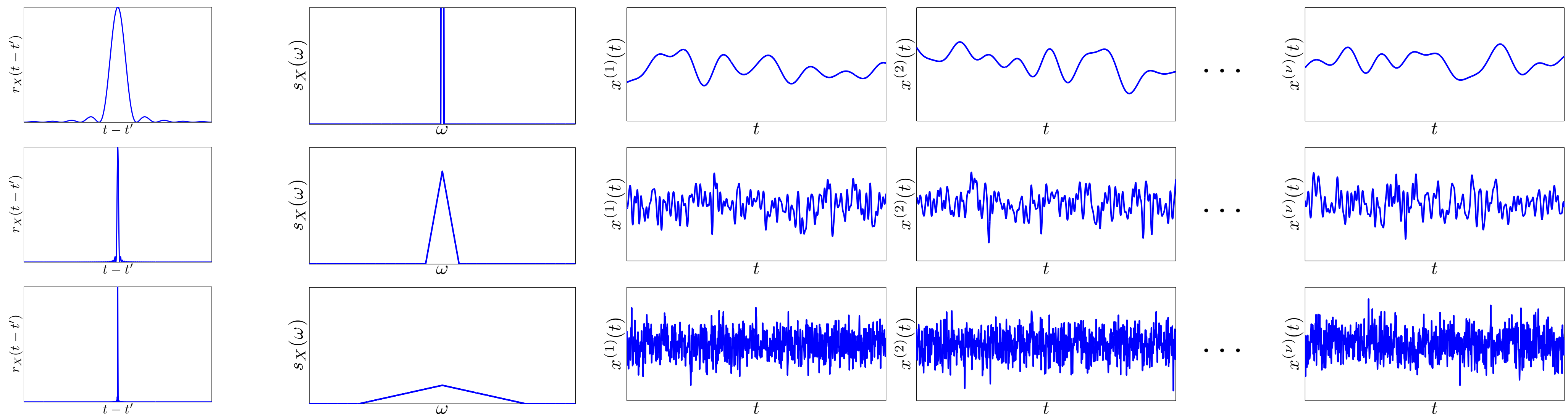
In fact, $s_X$ must be even because of the evenness of $r_X$, positive owing to Bochner's theorem, and integrable because $\int_\Theta |X(t)|^2 dP = r_X(0) = \frac{1}{2\pi} \int_{\mathbb{R}} s_X(\omega) d\omega < +\infty, \quad \forall t \in \mathbb{R}$.




Autocorrelation function.

Power spectral density function.

# Stochastic processes




autocorrelation function

power spectral density function

trajectories

autocorrelation function, power spectral density function, and a few trajectories of a mean-square stationary zero-mean second-order stochastic process

# Stochastic processes

- **Stochastic Differential Equation (SDE)**:

$$\sum_{k=0}^{q} b_k \frac{d^k U_F}{dt^k}(t) = F(t), \quad t \in \mathbb{R}, \quad b_q \neq 0, \quad q \geq 1.$$

What is meant by the derivative (and the integral) of a second-order stochastic process?

- **Mean-square derivative and integral**:
The derivative and the integral of a second-order stochastic process are defined in a manner that is consistent with its definition as a function that associates to any time instant a second-order r.v.,

$$\dot{X}(t) = \lim_{\tau \to 0} \frac{X(t+\tau) - X(t)}{\tau}, \qquad Z = \int_{\mathbb{R}} g\big(X(t), t\big)\, dt, \qquad \text{(provided that they exist)},$$

that is, the limits involved in the definition of the derivative and integral are defined in the sense of the mean-square convergence of second-order random variables (here, $g$ is a function from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$; refer to the next slide for an example).

- This is quite technical. You may find detailed information in:
  - E. Hille and R. Phillips. Functional analysis and semi-groups. AMS, 1957.
  - C. Soize. The Fokker-Planck equation for stochastic dynamical systems and its explicit steady state solutions. World Scientific, 1994.

- **Stochastic Differential Equation (SDE)**:
  We consider the linear filtering of a mean-square stationary zero-mean second-order stochastic process $\{F(t),\ t \in \mathbb{R}\}$ that we assume to admit a power spectral density function $s_F$:

$$\sum_{k=0}^{q} b_k \frac{d^k U_F}{dt^k}(t) = F(t), \quad t \in \mathbb{R}, \quad b_q \neq 0, \quad q \geq 1.$$

  Further, we assume that the FRF $\omega \mapsto \hat{h}(\omega) = 1/p(i\omega) = 1/\sum_{k=0}^{q} b_k (i\omega)^k$ is such that $1/p$ has no poles on the imaginary axis. We denote by $h = \mathcal{F}^{-1}(\hat{h})$ the impulse response function.

- **Generalized solution**:

$$U_F = h \star F, \quad \text{that is,} \quad U_F(t) = \int_{\mathbb{R}} h(s)F(t-s)ds.$$

  The assumptions allow this convolution to be defined as a mean-square integral. Hence, $\{U_F(t),\ t \in \mathbb{R}\}$ is a second-order stochastic process. Its second-order descriptors are obtained as follows:
  - **mean function** $m_{U_F}(t) = \int_\Theta U_F(t)dP = \int_{\mathbb{R}} h(s) \int_\Theta F(t-s)dPds = 0.$
  - **autocorrelation function** $r_{U_F}(t,t') = \int_\Theta U_F(t)U_F(t')dP = \int_{\mathbb{R}} \int_{\mathbb{R}} h(s) r_F(t-t'+s'-s) h(s') ds ds'.$

  Hence, $\{U_F(t),\ t \in \mathbb{R}\}$ is a mean-square stationary zero-mean second-order stochastic process.

- Thus, the linear filtering of a mean-square stationary zero-mean second-order stochastic process provides in turn a mean-square stationary zero-mean second-order stochastic process.

- Substituting $r_F(t-t') = \frac{1}{2\pi}\int_{\mathbb{R}} \exp\big(i\omega(t-t')\big) s_F(\omega) d\omega$ in the expression for $r_{U_F}$, we obtain:

$$r_{U_F}(t-t') = \frac{1}{2\pi}\int_{\mathbb{R}} \exp\big(i\omega(t-t')\big)\left(\int_{\mathbb{R}} \exp(-i\omega s)h(s)ds\right) s_F(\omega)\left(\int_{\mathbb{R}} \exp(i\omega s')h(s')ds'\right) d\omega.$$

  Hence, we obtain the **power spectral density function** of $\{U_F(t),\ t \in \mathbb{R}\}$ as follows:

$$\boxed{s_{U_F}(\omega) = \left|\hat{h}(\omega)\right|^2 s_F(\omega), \quad \forall \omega \in \mathbb{R}.}$$

- **Stochastic Initial-Value Problem (Stochastic IVP)**:

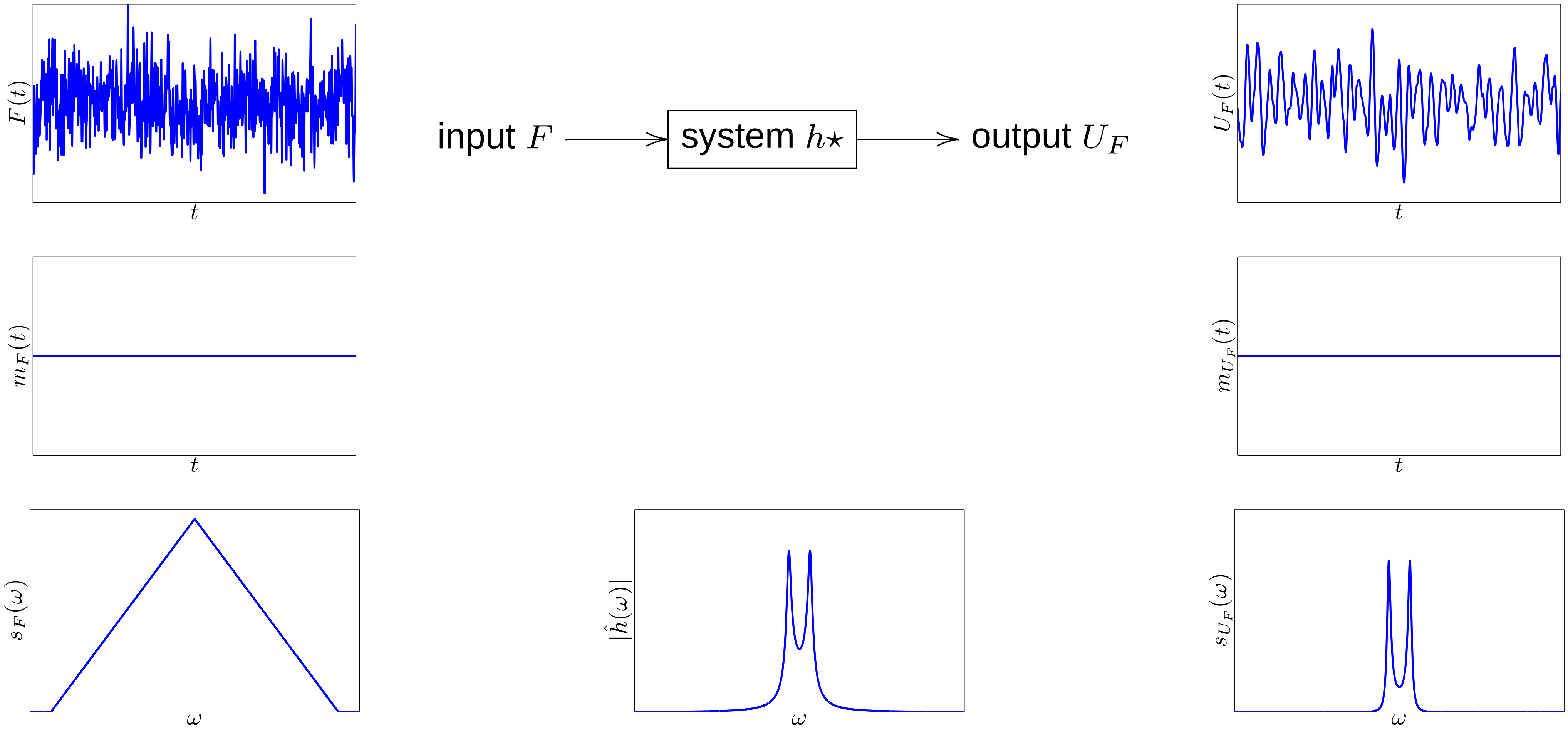
$$\left\{\begin{array}{l} \displaystyle\sum_{k=0}^{q} b_k \frac{d^k U}{dt^k}(t) = F(t), \quad t > 0, \quad b_q \neq 0, \quad q \geq 0, \\ \displaystyle U(0) = U_0, \ldots, \frac{d^{q-1}U}{dt^{q-1}}(0) = U_{q-1}, \end{array}\right. .$$

  If poles of $1/p$ are located left of imaginary axis, we have

$$U(t) = \underbrace{U_h(t)}_{\text{transient response}} + \underbrace{U_F(t)}_{\text{forced response}} \quad \text{with} \quad \lim_{t\to+\infty}\sqrt{\int_{\Theta} |U(t) - U_F(t)|^2 dP} = 0,$$

  that is, the response $U$ converges to the forced response $U_F$ as time advances.

# Stochastic processes

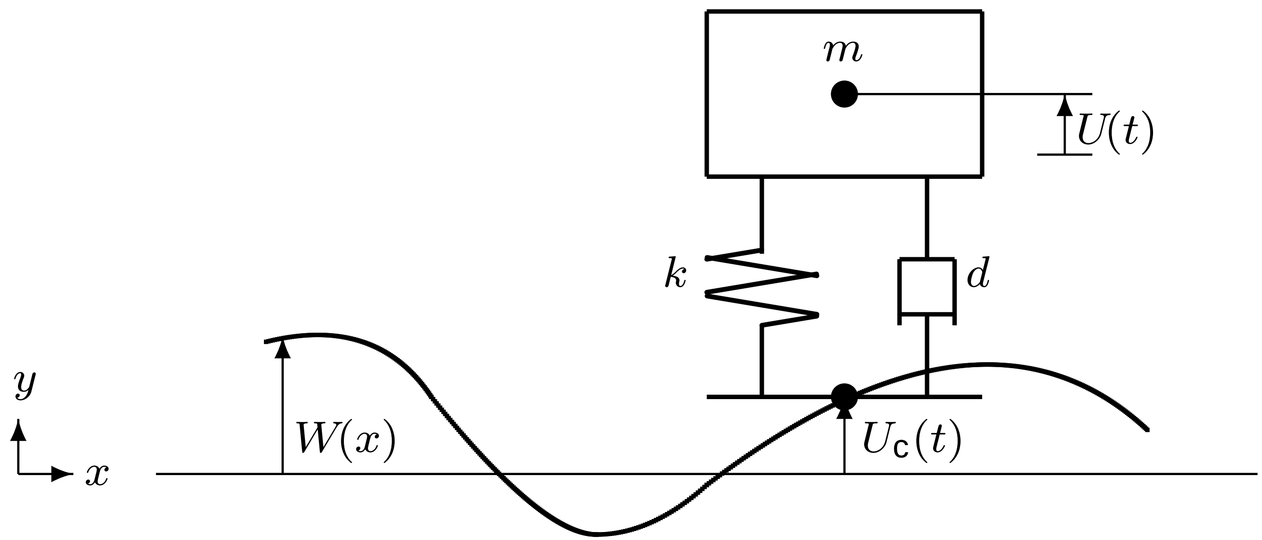



Linear filtering of a mean-square stationary zero-mean second-order stochastic process.

# Vocabulary

| time | space |
| --- | --- |
| frequency $f$ | |
| circular frequency $\omega = 2\pi f$ | wavenumber $\xi$ |
| period $\frac{1}{f} = \frac{2\pi}{\omega}$ | wavelength $\frac{2\pi}{\xi}$ |

# Exercise




- We consider a highly simplified model of a vehicle that moves over an uneven support at a constant horizontal velocity of $v$.
- Let the vehicle body be represented by a mass, and let the vehicle suspension be represented by a spring and dashpot that link the vehicle body to the contact point with the support.
- Let the support unevenness be modeled by a mean-square stationary zero-mean second-order stochastic process $\{W(x),\ x \in \mathbb{R}\}$ with power spectral density function $s_W$.
- Let $\{U_{\mathsf{c}}(t),\ t \in \mathbb{R}\}$ be the vertical displacement of the contact point caused by the support unevenness.
- Let $\{U(t),\ t \in \mathbb{R}\}$ be the resulting vertical displacement of the vehicle body with respect to the static equilibrium configuration on a flat support.

# Exercise

1. Using the fact that the vehicle rolls at a horizontal velocity of $v$, write the equation that relates the vertical displacement of the contact point $\{U_{\mathsf{c}}(t),\ t \in \mathbb{R}\}$ to the support unevenness $\{W(x),\ x \in \mathbb{R}\}$.
2. Write the equations that relate the mean function, the autocorrelation function, and the power spectral density function of $\{U_{\mathsf{c}}(t),\ t \in \mathbb{R}\}$ to those of $\{W(x),\ x \in \mathbb{R}\}$.
3. Express dynamical equilibrium and deduce the stochastic differential equation that governs the displacement $\{U(t),\ t \in \mathbb{R}\}$ of the vehicle body.
4. Consider the stochastic differential equation that you obtained under 3 as a convolution transformation whose input is the stochastic process $\{U_{\mathsf{c}}(t),\ t \in \mathbb{R}\}$ and whose output is the stochastic process $\{U(t),\ t \in \mathbb{R}\}$. Write the expression for the FRF $\hat{h}$ associated with this convolution filter. Interpret your result: indicate frequency regions wherein the vehicle body follows the motion of the contact point (vibration transmission) and wherein the vehicle body is isolated from the motion of the contact point (vibration isolation).
5. Write the equations that relate the mean function and the power spectral density function of $\{U(t),\ t \in \mathbb{R}\}$ to those of $\{W(x),\ x \in \mathbb{R}\}$.
6. Consider $s_W(\xi) = s_0/\big(1 + \frac{|\xi|}{\xi_0}\big)^{\alpha}$ and the numerical values of $s_0 = 5 \times 10^{-5}\,\mathrm{m}^3$, $\alpha = 2$, $\xi_0 = 0.5\,\mathrm{m}^{-1}$, $m = 470\,\mathrm{kg}$, $k = 36 \times 10^{+3}\,\mathrm{N/m}$, and $d = 0.10 \times 2 \times \sqrt{mk}$. Plot $s_W$ and $|\hat{h}|$, first on a linear scale and then on a loglog scale. Subsequently, plot $s_U$ for $v = 20\,\mathrm{m/s}$ and $v = 40\,\mathrm{m/s}$, first on a linear scale and then on a loglog scale. Interpret your results.

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