Consistency of stress smoothing by convolution

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1. Introduction

A very classical stress smoothing procedure consists to generate from the finite elements stress field $\widetilde{\sigma}_h$ a new stress field of the same form as the displacements,

$$\widetilde{\sigma}_h = \sum_{i=1}^{Node} N_i \widetilde{\sigma}_{h_i}$$

The main difficulty of such a scheme is to obtain sufficiently accurate nodal values of the stresses. In the so-called local methods, these nodal values are obtained as a weighted mean of stresses at selected points.

The main idea of the present work was to replace these discrete averages by an integral one, which is more systematic. Here,

$$\widetilde{\sigma}_{hi} = \int_{R^n} \sigma_h(x_i + y)\varphi(y)dy\,,$$

with a radial convolution kernel φ whose support is a ball of radius R, and which verifies the normalization condition $\int_{R^n} \varphi(y)dy = 1$. For internal nodes, R may be chosen so the ball lies inside the patch of elements containing node i (fig. 1).

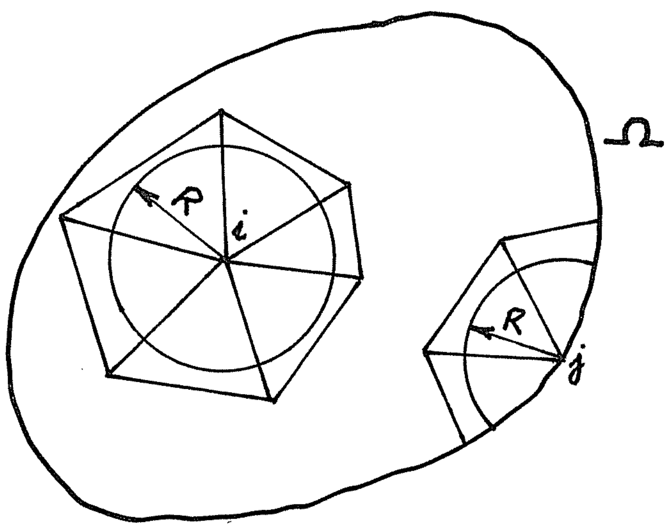

FIG. 1

The ball is then inside the body Ω. On the boundary of Ω (point j) the integration may be restricted to the part of the ball that is contained in Ω. The result is then divided by the sum of weights, that is,

$$\sigma_{hj} = \frac{\int\limits_{\Omega - x_j} \sigma_h(x_j + y)\varphi(y)dy}{\int\limits_{\Omega - x_j} \varphi(y)dy}$$

2. Kernel families

In order to obtain a kernel family φ_R with support in B_R and unit integral, the following scheme is available

- Select some radial function $\overline{\varphi}(x)$ whose support is B_1 and such that
$$\int\limits_{B_1} \overline{\varphi}(x)dx \neq 0$$
- Set
$$\varphi_1(x) = \frac{\overline{\varphi}(x)}{\int\limits_{B_1} \overline{\varphi}(x)dx}$$
- On a ball B_R, the kernel will be
$$\varphi_R(x) = \frac{1}{R^n}\varphi_1(x)$$

The best known kernel is the *canonical mollifier* defined by

$$\overline{\varphi}(x) = \begin{cases} \exp\left(\dfrac{1}{|x|^2 - 1}\right) & \text{for } |x| < 1 \\ 0 & \text{elsewhere} \end{cases}$$

But more simple choices are possible.

3. First experimentations

This method, with the canonical mollifier, has been tested by Do Viet Tuyen [5] on 2-D stress fields. The conclusions are as follows

- With *1st degree* elements (σ of degree 0), the convolution method leads to good effectivity indices.
- With *2nd degree* elements, it is no more the case, and strangely, the lowest the kernel support radius R, the best the effectivity index.

If it is noted that for 0-degree stresses, *any* radial convolution leads to the same result, a fact that is not true for higher degrees, this all seems to ask the following question : *are there better suited kernels ?*

4. A consistency condition

The convolution process may be considered as an operator T which from a given function f leads to a new function $\tilde{f} = Tf$. If f_h is an approximation of f, let us consider the difference between the true function f and $\tilde{f}_h = Tf_h$. One has

$$f - Tf_h = f - Tf + Tf - Tf_h$$

So, when $f_h \to f$, one has

$$\|Tf - Tf_h\| \leq \|T\| \|f - f_h\|$$

But it remains the term $f - Tf$ which is independent of f_h and has to be zero in order to ensure that $Tf_h \to f$.
This is to say that *the adopted transformation has to be neutral when applied to the true stress field*: it is a consistency condition.

5. Internal consistency with harmonic functions

A neutral kernel for any function is of course not attainable (it would be a representation of Dirac's measure). But restricting this aim to some classes of functions, this property may be true. Let us first consider *harmonic* functions. Noting

$$\omega_i = x_i / r, \quad r = |x|,$$

let us compute the following integral :

$$\boxed{I = \int_{|\omega|=1} [f(R\omega) - f(o)] d\omega} \qquad (1)$$

It is easy to see that

$$I = \int_{|\omega|=1} d\omega \int_o^R D_r f(r\omega) dr$$

Now, defining the function e(r) by

$$\frac{de}{dr} = \frac{1}{r^{n-1}}, \quad e(\infty) = o,$$

it is clear that

$$I = \int_{|\omega|=1} d\omega \int_o^R D_r f(r\omega) D_r e(r) . r^{n-1} dr$$

$$= \int_{B_R} D_i f D_i e dx$$

and after an integration by parts,

$$\boxed{I = \int_{B_R} [e(R) - e(r)] \Delta f dx} \tag{2}$$

This result contains the well-known *mean value theorem for harmonic functions* [1] : For any harmonic function f,

$$f(o) = \frac{1}{\omega_n} \int_{|\omega|=1} f(R\omega) d\omega \qquad \forall R$$

where ω_n is the superficial measure of the n-D unit sphere.

Multiplying this result by any radial function $\varphi_R(r)$ and by r^{n-1}, one obtains

$$f(o)\varphi_R(r) r^{n-1} = \frac{1}{\omega_n} \int_{|\omega|=1} f(r\omega)\varphi_R(r) r^{n-1} d\omega$$

~~An integration from 0 to R leads to~~

$$f(o) \int_{B_R} \varphi_R(r) dy = \int_{B_R} f(r\omega)\varphi_R(r) dy \tag{3}$$

This, translated of x, gives

$$f(x) \int_{B_R(x)} f(x+y)\varphi_R(y) dy \tag{4}$$

Where is supposed to be of unit integral.

The result is thus that *any normalized radial kernel is neutral for harmonic functions.* Although of simple nature, this result is seldom citated [3].

6. The case of biharmonic functions

Unfortunately, elastic stress fields are not generally harmonic, so that from (2), a convolution by any radial kernel is *not consistent*. However, in the case of an isotropic homogeneous solid, submitted to body forces of degree 2 at most, it may be proved that the stress field is *biharmonic*.

The Laplacian of a biharmonic function being harmonic, one has

$$I = \int_{B_R} [e(R) - e(r)] \Delta f dx = \Delta f(o). \int_{B_R} [e(R) - e(r)] dx$$

since $[e(R) - e(r)]$ is a radial kernel. After elementary transformations, this leads to

$$I = \Delta f(o). \frac{\omega_n R^2}{2n} \tag{5}$$

Returning to the definition of I, multiplying both sides of this equality by $r^{n-1}\varphi_R(r)$ where φ_R is a normalized radial kernel, and integrating from O to R leads to the following result

$$f(o) = (f * \varphi_R)_o + \frac{AW_n}{2n} \Delta f(o).R^2 \tag{6}$$

where

$$A = \int_o^1 \rho^{n+1} \varphi_1(\rho) d\rho \tag{7}$$

is independent of R. So, *the consistency error on a biharmonic function is proportional to* R^2.

The consistency condition for biharmonic functions is that the kernel verifies A = 0, that is

$$\boxed{A = \int_{B_1} \rho^2 \varphi_1 dx = 0}$$

Two comments about this condition

- A biharmonic-consistent kernel *cannot be positive everywhere* since $\varphi_1 \geq 0$ would imply A > 0. Note that the existence of positive consistent kernels would imply a maximum theorem such as for harmonic functions, and this is not true.

- In the 2-D case, it follows from Goursat's theorem [6] that any biharmonic function f is of the form

  $$f = \alpha + r^2 \beta$$

  where α and β are harmonic. Therefore, with a radial kernel φ,

  $$\int_{B_R} f\varphi dx = \int_{B_R} \alpha\varphi dx + \int_{B_R} \beta r^2 \varphi dx = \alpha(o) \int_{B_R} \varphi dx + \beta(o) \int_{B_R} r^2 \varphi dx$$

  and this reduces to

$$f(o)\int_{B_R}\varphi dx = \alpha(o)\int_{B_R}\varphi dx$$

only if

$$\int_{B_R} r^2\varphi dx = 0,$$

that is, our consistency condition.

7. Biharmonic-consistent kernels

A direct derivation of biharmonic-consistent kernels may be performed starting from the equation

$$\int_{|\omega|=1} f(r\omega)d\omega = \omega_n f(o) + \frac{\omega_n r^2}{2n}\Delta f(o)$$

by somewhat long developments inspired from ideas of Parton and Perline [2]. The result is as follows :

Let φ_R be a normalized radial kernel such that

- $\lim_{r\to R}\varphi_R(r) = 0$
- $\lim_{r\to o} r^n\varphi_R(r) = 0$

Then, a biharmonic-consistent kernel is given by

$$\boxed{\Psi_R = \frac{1}{2}\frac{1}{r^{n+1}}\frac{d}{dr}\left(r^{n+2}\varphi_R\right) = \frac{1}{2}\left[(n+2)\varphi_R + r\frac{d\varphi_R}{dr}\right]}$$

In fact, it is normalized, as

$$\int_o^R \Psi_R r^{n-1}dr = \frac{1}{2}\int_o^R \frac{1}{r^2}\frac{d}{dr}\left(r^{n+2}\varphi_R\right)dr$$

$$= \frac{1}{2}\left[r^n\varphi_R\right]_o^R + \int_o^R r^{n-1}\varphi_R dr = \int_o^R r^{n-1}\varphi_R dr$$

It also verifies the consistency condition, since

$$\int_o^R \Psi_R r^{n+1}dr = \int_o^R \frac{d}{dr}\left(r^{n+2}\varphi_R\right)dr = \left[r^{n+2}\varphi_R\right]_o^R = 0$$

8. Example

It is generally admitted that a suitable kernel has to fastly decay with r. Let us consider, for the 2-D case

$$\varphi_R = \frac{1}{\pi R}\left(\frac{1}{r} - \frac{1}{R}\right)$$

The corresponding Ψ_R is

$$\Psi_R = \frac{1}{2\pi R}\left[4\left(\frac{1}{r} - \frac{1}{R}\right) - \frac{1}{r}\right] = \frac{1}{2\pi R}\left(\frac{3}{r} - \frac{4}{R}\right)$$

9. Conclusions

Up to now, it has not been possible to test the presented kernels. However, the results obtained by arbitrary kernels are explained, since

- for zero degree element stresses, *any* radial kernel leads to a weighted average where the weights are the angular extensions of the adjacent elements.
- for higher degree elements, the result depends on the kernel, and the best result was obtained for $R \to 0$, in conformity with our consistency analysis.

The question of boundary nodes remains somewhat open since our truncated convolution is questionable. Other ways are possible, such as equilibrium verification.

It should also be mentioned that the convolution procedure may be used for 3-D problems without requiring any superconvergence result.

Références

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