MECA0010 – Reliability and stochastic modeling of engineered systems

Reliability: Hypothesis testing

Maarten Arnst and Marco Lucio Cerquaglia

December 6, 2017

Statistical inference

Hypothesis testing

- Stochastic model (random variable, stochastic process, . . .)

$\Downarrow$ Theory

Probabilistic characterization (PDF, quantiles, . . .) of "test statistics" (sample mean, sample variance, . . .).

Hypothesis tests: can we go in the opposite direction ? Can we use "test statistics" to infer conclusions about the stochastic model ?

Hypothesis testing

- Suppose that we observe ν statistically independent trajectories up to time t of a failure counting process. Then, the setting is as
  - **data:** the numbers of failures in each interval $[0, t]$ and the time instants at which the failures occurred,
  - **candidate stochastic model: Poisson process**.

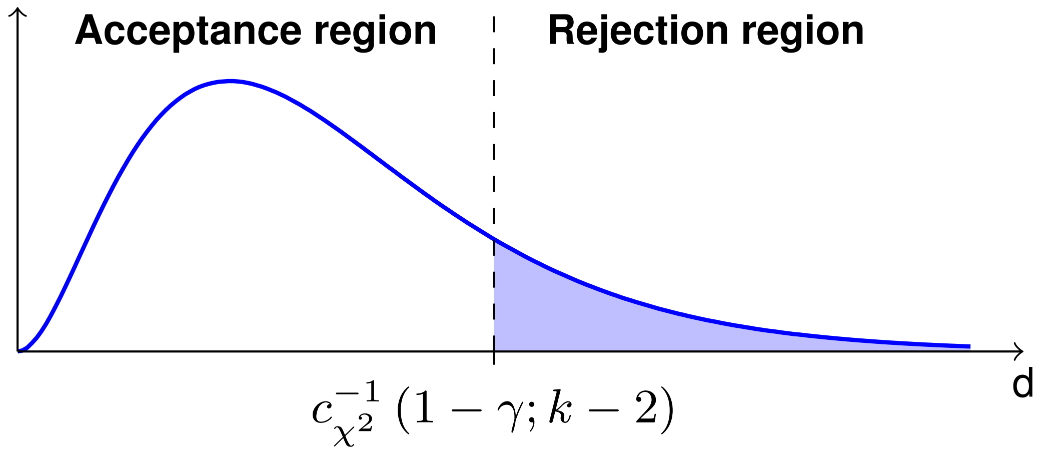
  In this case, the **goodness of fit of the Poisson distribution** can be tested:
  - **Null hypothesis:** Poisson distribution is suitable,
  - **Alternative hypothesis:** Poisson distribution is not suitable.

  The **chi-squared test** is a particular goodness-of-fit test in which under the null hypothesis, the test statistic is a sample of a chi-squared distribution.

- Chi-squared hypothesis test:
  - **The chi-squared test statistic** measures the goodness of fit in terms of the sum of the squares of the differences between the observed and calculated outcome frequencies, divided by the calculated outcome frequencies:

$$d = \sum_{i=1}^{k} \frac{(f_i - e_i)^2}{e_i} \quad \text{with} \quad \begin{cases} f_i : \text{observed frequency for i-th value/bin,} \\ e_i : \text{calculated frequency for i-th value/bin.} \end{cases}$$

  - Under the null hypothesis, the chi-squared test statistic is a sample from, approximately, the **chi-squared distribution** with $k-2$ degrees of freedom.



  - **Accept** the null hypothesis if the test statistic lies within the acceptance region and **reject** the null hypothesis otherwise.

Hypothesis testing

- Let us consider an example involving failures of 66 machines over one day:
  - 20 with zero failure, 23 with 1 failure, 15 with 2 failures, 6 with 3 failures, and 2 with 4 failures.
  - parameter estimation:

$$\widehat{m} = 0 \times \frac{20}{66} + 1 \times \frac{23}{66} + 2 \times \frac{15}{66} + 3 \times \frac{6}{66} + 4 \times \frac{2}{66} = 1.197 \text{ failure/day.}$$

  - chi-squared test statistic:

Number of failures	f_i	e_i	$(f_i - e_i)^2 / e_i$
0	20	19.94	1.18e^{-4}
1	23	23.86	0.0310
2	15	14.29	0.0353
3	6	5.69	0.0169
4	2	1.70	0.0528
>4	0	0.52	0.52
			$d = 0.65$

  - Since $d = 0.65 < c_{\chi^2}^{-1}(0.95; 4) = 9.48$, we accept null hypothesis at $\gamma = 95\%$ significance.

Hypothesis testing

- Suppose that we observe a trajectory up to time t of a Poisson process. Then, the setting is as:
  - **data:** n, the number of failures in the interval $[0, t]$, and $t_1, \ldots, t_n$, the time instants at which the failures occurred,
  - **candidate stochastic model: Poisson process.**

  In this case, the homogeneity of the Poisson process can be tested:
  - **Null hypothesis:** the Poisson process is homogeneous.
  - **Alternative hypothesis:** the rate of occurrence of failures decreases (increases).

  The **logarithm test** is a particular **trend test** in which under the null hypothesis, the test statistic is a sample of a chi-squared distribution.

- Logarithm hypothesis test:
  - The **logarithm test statistic** measures homogeneity by

$$\nu = -2\sum_{i=1}^{n} \log \frac{t_i}{t}.$$

  - Under the null hypothesis, the logarithm test statistic is a sample from the **chi-squared distribution** with n degrees of freedom.

Indeed, under the null hypothesis, the time instants at which the failure occurred are statistically independent and uniformly distributed in the interval $[0, t]$. It can be shown that the sign-reversed double of the sum of the logarithms of n statistically independent uniform random variables with values in $[0, 1]$ is a chi-squared random variable with n degrees of freedom.

- **Accept** the null hypothesis if the test statistic lies in the acceptance region $\left[0, c_{\chi^2}^{-1}(\gamma, n)\right]$ $\left(\left[c_{\chi^2}^{-1}(1-\gamma, n), +\infty\right]\right)$ and **reject** otherwise.

References

Suggested reading material

- L. Wehenkel. Eléments de statistiques. Université de Liège. Lecture notes.

Additional references also consulted to prepare this lecture

- A. Ang and W. Tang. Probability concepts in engineering. John Wiley & Sons, 2007.
- C. Cocozza-Thivent. Processus stochastiques et fiabilité des systèmes. Springer, 1997.
- D. Foata and A. Fuchs. Processus stochastiques: processus de Poisson, chaînes de Markov et martingales. Dunod, 2004.
- M. Rausland and A. Hoyland. System reliability theory: models, statistical methods, and applications. Wiley, 2014.
- H. Procaccia, E. Ferton, and M. Procaccia. Fiabilité et maintenance des matériels industriels réparables et non réparables. Lavoisier, 2011.
- C. Soize. The Fokker–Planck equation for stochastic dynamical systems and its explicit steady state solutions. World Scientific Publishing, 1994.