MECA0010 – Reliability and stochastic modeling of engineered systems

Reliability: nonhomogeneous Poisson process, point estimation, and confidence-interval estimation

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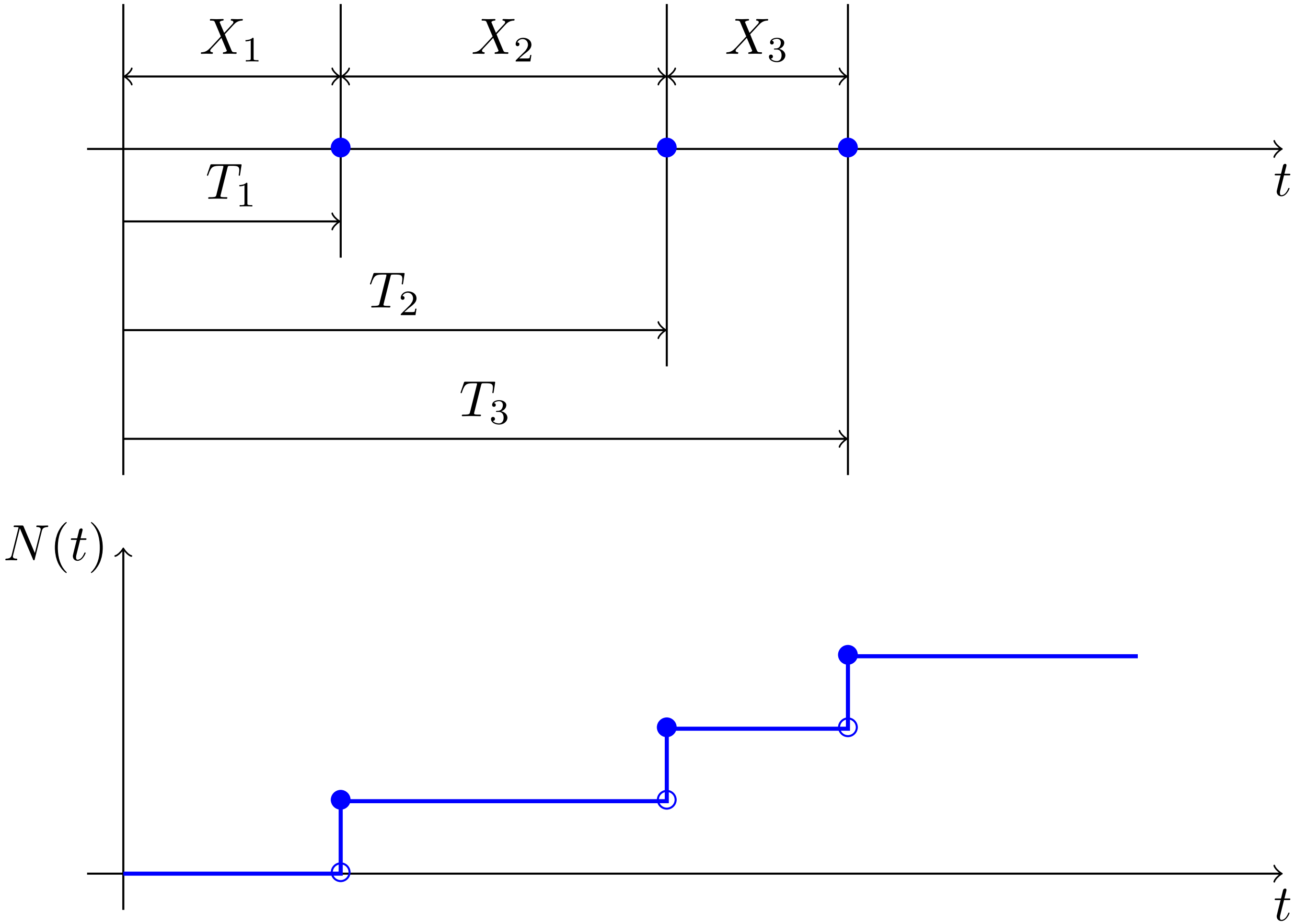
Outline

- Models for minimal repair:
  - Poisson process.
  - Homogeneous Poisson process.
  - Nonhomogeneous Poisson process.
- Statistical inference:
  - Role of statistical inference.
  - Point estimation.
  - Confidence-interval estimation.

Models for minimal repair

- **Poisson process** $\{N(t),\ t \geq 0\}$ with mean function m:



  - $N(0) = 0$.
  - $\{N(t),\ t \geq 0\}$ has independent increments.
  - for any $0 \leq s < t$, $N(t) - N(s)$ is a r.v. with Poisson distribution with mean $m(t) - m(s)$.

- The Poisson process $\{N(t),\ t \geq 0\}$ is **homogeneous** if the mean function m is of the form $m(t) = \lambda t$ with λ a positive constant, that is, if the average number of failures occurring increases linearly with the time interval under consideration.

- For a homogeneous Poisson process $\{N(t),\ t \geq 0\}$ with mean function m, we have:
  - the **first time to failure** obeys an exponential distribution with parameter λ.
  - more generally, the lengths of time between consecutive failures $\{X_n, n \geq 1\}$ are statistically independent and identically distributed with exponential distribution with parameter λ.
  - the conditional probability distribution of the **instants $(T_1, \ldots, T_n)$ at which the system suffers its first n failures given $\{N(t) = n\}$ admits as a density**

$$\rho_{(T_1,\ldots,T_n|N(t))}(t_1, \ldots, t_n|n) = \frac{n!}{t} 1\left(0 < t_1 < \ldots < t_n < t\right).$$

Nonhomogeneous Poisson process

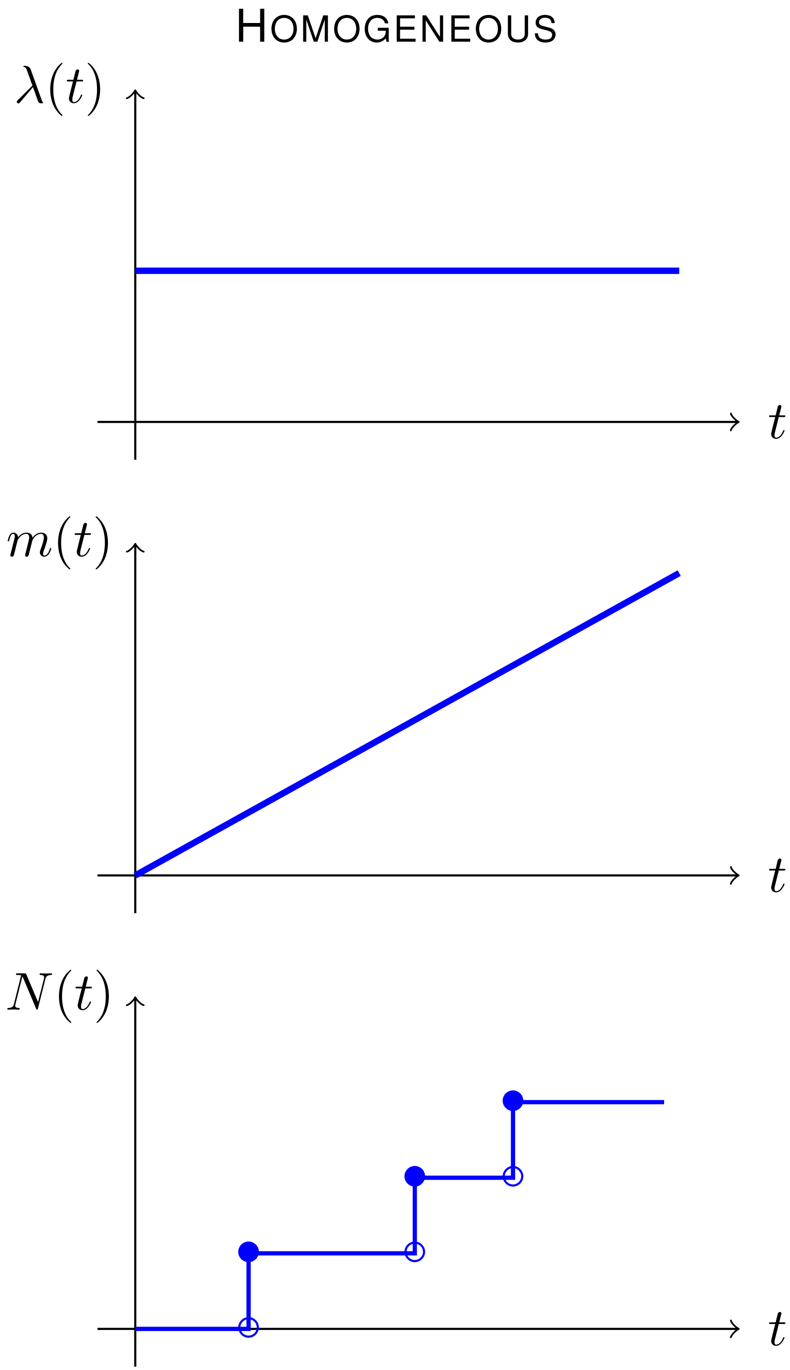
- Let $\{N(t),\ t \geq 0\}$ be a Poisson process with mean function m. If m is not of the form $m(t) = \lambda t$ with λ a positive constant, then $\{N(t),\ t \geq 0\}$ is a **nonhomogeneous** Poisson process.

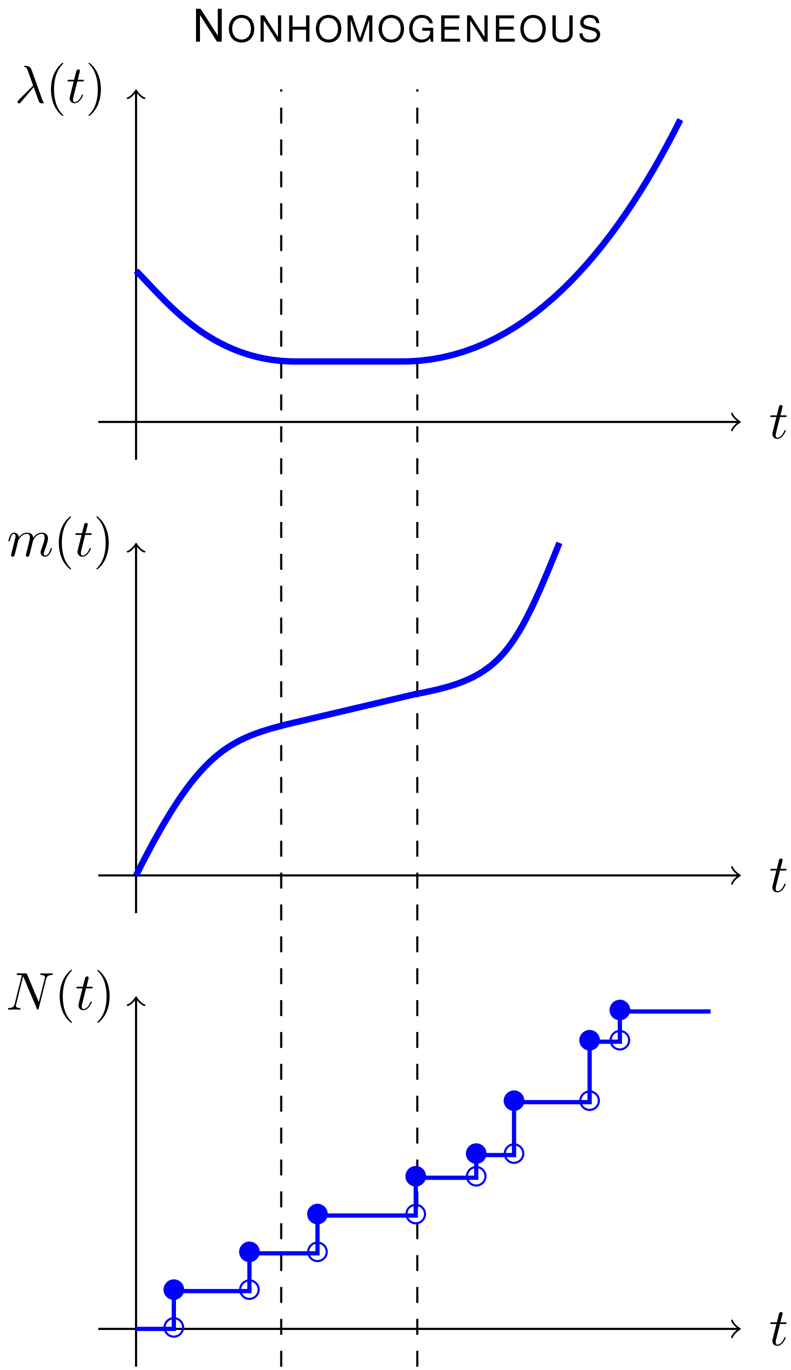
- Let $\{N(t),\ t \geq 0\}$ be a nonhomogeneous Poisson process with mean function m. If there exists a continuous function λ from $\mathbb{R}^+$ into $\mathbb{R}^+$ such that

$$m(t) = \int_0^t \lambda(s) ds,$$

then this function λ is called the **(instantaneous) intensity**.

Nonhomogeneous Poisson process





Nonhomogeneous Poisson process

- For a nonhomogeneous Poisson process $\{N(t),\ t \geq 0\}$ with (instantaneous) intensity λ, **the first time to failure** T_1 admits as a probability density function

$$\rho_{T_1}(t_1) = \text{exp}\left(-\int_0^{t_1} \lambda(s)ds\right)\lambda(t_1).$$

<u>Proof:</u>

$$P\left(T_1 > t_1\right) = P\left(N(t_1) = 0\right) = \text{exp}\left(-\int_0^{t_1} \lambda(s)ds\right)\frac{\left(\int_0^{t_1} \lambda(s)ds\right)^0}{0!}$$

$$P\left(T_1 \leq t_1\right) = 1 - \text{exp}\left(-\int_0^{t_1} \lambda(s)ds\right)$$

$$\rho\left(t_1\right) = \text{exp}\left(-\int_0^{t_1} \lambda(s)ds\right)\lambda(t_1).$$

Nonhomogeneous Poisson process

- It can be shown that for a nonhomogeneous Poisson process $\{N(t),\ t \geq 0\}$ with (instantaneous) intensity λ, the **joint probability distribution of the instants $(T_1, \ldots, T_n)$ at which the system suffers its first n failures** admits as a density

$$\rho_{(T_1,\ldots,T_n)}\left(t_1, \ldots, t_n\right) = \exp\left(-\int_0^{t_n} \lambda(s)ds\right) \prod_{i=1}^{n} \lambda\left(t_i\right) 1\left(0 < t_1 < \ldots < t_n\right).$$

- It can be shown that for a nonhomogeneous Poisson process $\{N(t),\ t \geq 0\}$ with (instantaneous) intensity λ, the **conditional probability distribution of the instants $(T_1, \ldots, T_n)$ at which the system suffers its first n failures given $\{N(t) = n\}$** admits as a density

$$\rho_{(T_1,\ldots,T_n|N(t))}\left(t_1, \ldots, t_n|n\right) = \frac{n!}{\left(-\int_0^t \lambda(s)ds\right)^n} \prod_{i=1}^{n} \lambda\left(t_i\right) 1\left(0 < t_1 < \ldots < t_n\right).$$

- **Example: Duane's power law model:**

$$\lambda(t) = \frac{\beta}{\alpha^\beta} t^{\beta-1},$$

where it should be noted that λ is continuous on $\mathbb{R}^+$ only if $\beta \geq 1$.

The first time to failure obeys the probability density

$$\begin{aligned}
\rho_{T_1}(t_1) &= \text{exp}\left(-\int_0^{t_1} \lambda(s)ds\right)\lambda(t_1) \\
&= \text{exp}\left(-\frac{t_1^\beta}{\alpha^\beta}\right)\frac{\beta}{\alpha^\beta} t_1^{\beta-1} \\
&= \frac{\beta}{\alpha}\left(\frac{t_1}{\alpha}\right)^{\beta-1} \text{exp}\left(-\left(\frac{t_1}{\alpha}\right)^\beta\right),
\end{aligned}$$

that is, a Weibull probability density function with parameters α and β.

Statistical inference

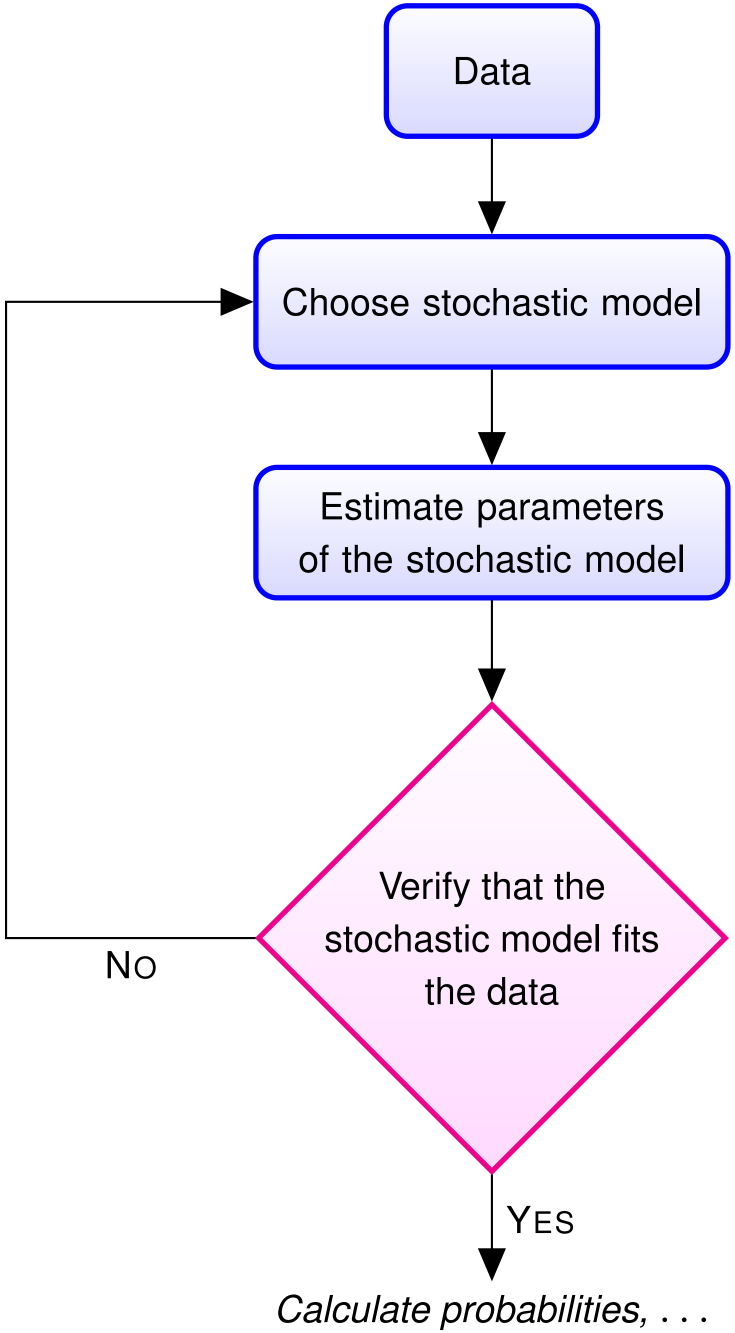
Role of statistical inference

- Previously, we looked at ways of stochastic modeling occurrences of failures, e.g., homogeneous and nonhomogeneous Poisson processes.

- What if we have **data** measured from the system failures and must **infer** a description of the occurrences of failures in terms of a stochastic model?

- This requires that we **choose a stochastic model** (e.g. homogeneous vs. nonhomogeneous Poisson process) and **determine the best choice of the parameters** (e.g. parameter λ of homogeneous Poisson process or parameters α and β of Duane's power law model for nonhomogeneous Poisson process).

Role of statistical inference



Role of statistical inference

- We will look at two **parameter-estimation methods**:
  - **point estimation** by using method of **maximum likelihood**,
  - **confidence-interval estimation**.
- We will look at two **model-selection methods**:
  - **goodness-of-fit testing**,
  - **trend testing**.

- Within the following **setting**:
  - **data**: samples $x_1, \ldots, x_\nu$,
  - **candidate stochastic model: probability density function** $\rho_X(x; \theta_1, \ldots, \theta_m)$, where $\theta_1, \ldots, \theta_m$ are the **unknown parameters** that must be estimated,

the **method of maximum likelihood** measures the plausability of the parameters given the data samples by the **likelihood**

$$\ell(\theta_1, \ldots, \theta_m) = \prod_{i=1}^{\nu} \rho_X(x_i; \theta_1, \ldots, \theta_m);$$

the **point estimate** $(\hat{\theta}_1, \ldots, \hat{\theta}_m)$ is then the value of the parameters that maximizes the likelihood:

$$(\hat{\theta}_1, \ldots, \hat{\theta}_m) = \text{arg} \max_{(\theta_1, \ldots, \theta_m)} \ell(\theta_1, \ldots, \theta_m)$$

- The maximum-likelihood point estimate $(\hat{\theta}_1, \ldots, \hat{\theta}_m)$ can be computed by solving

$$\frac{\partial \ell}{\partial \theta_i}(\hat{\theta}_1, \ldots, \hat{\theta}_m) = 0, \quad 1 \leq i \leq m;$$

sometimes, it is easier to maximize the "**loglikelihood**"

$$\frac{\partial \log \ell}{\partial \theta_i}(\hat{\theta}_1, \ldots, \hat{\theta}_m) = 0, \quad 1 \leq i \leq m.$$

- It can be shown that the method of maximum likelihood has good properties in terms of unbiasedness, consistency, efficiency, sufficiency, . . .

- For example, let us consider the following setting:
  - **data**: samples $x_1, \ldots, x_\nu$,
  - **candidate stochastic model: Gaussian probability density function** with **unknown mean** μ **and standard deviation** σ that must be estimated ;

in this case, the likelihood reads as

$$\ell(\mu, \sigma) = \prod_{i=1}^{\nu} \frac{1}{\sqrt{2\pi}\sigma} \text{exp}\left(-\frac{1}{2}\left(\frac{x_i - \mu}{\sigma}\right)^2\right),$$

so that the point estimate $(\widehat{\mu}, \widehat{\sigma})$ is obtained by solving

$$\frac{\partial \log \ell}{\partial \mu}(\widehat{\mu}, \widehat{\sigma}) = \sum_{i=1}^{\nu} \frac{(x_i - \hat{\mu})}{\hat{\sigma}^2} = 0 \quad \text{and} \quad \frac{\partial \log \ell}{\partial \sigma}(\hat{\mu}, \hat{\sigma}) = \frac{1}{\hat{\sigma}^2} \sum_{i=1}^{\nu} (x_i - \hat{\mu})^2 - \frac{\nu}{\hat{\sigma}} = 0,$$

thus leading to $\hat{\mu} = \frac{1}{\nu}\sum_{i=1}^{\nu} x_i$ and $\hat{\sigma} = \sqrt{\frac{1}{\nu}\sum_{i=1}^{\nu}(x_i - \hat{\mu})^2}$.

- Suppose that we observe up to time t a trajectory of a **homogeneous Poisson process**. Then, the setting is as follows:
  - **data**: n, the number of failures in the interval $[0, t]$, and $t_1, \ldots, t_n$, the time instants at which the system suffered these failures,
  - **candidate stochastic model: homogeneous Poisson process $\{N(t), t \geq 0\}$ with unknown parameter λ** to be estimated;

in this case, the likelihood reads as

$$\begin{aligned} \ell(\lambda) &= P(N(t) = n)\rho_{(T_1,\ldots,T_n|N(t))}(t_1, \ldots, t_n|n) \\ &= \exp(-\lambda t)\frac{(\lambda t)^n}{n!}\frac{n!}{t^n}1(0 < t_1 < \ldots < t_n < t), \end{aligned}$$

so that with $\log \ell(\lambda) = -\lambda t + n \log(\lambda t)$, the point estimate $\widehat{\lambda}$ is obtained by solving

$$\frac{\partial \log \ell}{\partial \lambda}\left(\hat{\lambda}\right) = -t + \frac{n}{\hat{\lambda}} = 0,$$

thus leading to $\hat{\lambda} = \frac{n}{t}$.

- Suppose that we observe up to time t a trajectory of a **nonhomogeneous Poisson process whose (instantaneous) intensity follows a Duane power law**. Then, the setting is as follows:
  - **data**: n, the number of failures in the interval $[0, t]$, and $t_1, \ldots, t_n$, the time instants at which the system suffered these failures,
  - **candidate stochastic model: nonhomogeneous Poisson process with unknown parameters α and β** to be estimated ;

in this case, the likelihood reads as

$$\begin{aligned} \ell(\lambda) &= P(N(t)=n)\rho_{(T_1,\ldots,T_n|N(t))}(t_1,\ldots,t_n|n) \\ &= \text{exp}\left(-\left(\frac{t}{\alpha}\right)^{\beta}\right)\frac{\left(\frac{t}{\alpha}\right)^{\beta n}}{n!}\frac{n!}{\left(\frac{t}{\alpha}\right)^{\beta n}}\prod_{i=1}^{n}\frac{\beta}{\alpha}\left(\frac{t_i}{\alpha}\right)^{\beta-1} \end{aligned}$$

so that the point estimate is obtained by solving $\frac{\partial \log \ell}{\partial \alpha}\left(\hat{\alpha}, \hat{\beta}\right) = 0$ and $\frac{\partial \log \ell}{\partial \beta}\left(\hat{\alpha}, \hat{\beta}\right) = 0$,

which ultimately leads to $\frac{1}{\hat{\beta}} = \log t - \frac{1}{n}\sum_{i=1}^{n}\log t_i$ and $\log \hat{\alpha} = \log t - \frac{1}{\hat{\beta}}\log n$.

- Within the following setting:
  - **data**: samples $x_1, \ldots, x_\nu$,
  - **candidate stochastic model**: probability density function $\rho_X\left(x; \theta_1, \ldots, \theta_m\right)$, where $\theta_1, \ldots, \theta_m$ are the unknown parameters to be estimated,

the **method of confidence interval estimation** consists in setting a confidence level α and then inferring from the data **intervals**

$$\left[\hat{\theta}_1^-, \hat{\theta}_1^+\right], \ldots, \left[\hat{\theta}_m^-, \hat{\theta}_m^+\right],$$

which are such that if the data samples were independent and identically distributed samples from $\rho_X\left(x; \theta_1, \ldots, \theta_m\right)$ with "true values" $\theta_1, \ldots, \theta_m$ of the parameters, then these intervals would be more than γ-likely to contain $\theta_1, \ldots, \theta_m$, that is,

$$P\left(\theta_1 \in \left[\widehat{\Theta}_1^-, \widehat{\Theta}_1^+\right], \ldots, \theta_m \in \left[\widehat{\Theta}_m^-, \widehat{\Theta}_m^+\right]\right) \geq \gamma.$$

- For example, let us consider the following setting:
  - **data**: samples $x_1, \dots, x_\nu$,
  - **candidate stochastic model**: **Gaussian probability density function with unknown mean μ and known standard deviation σ**;

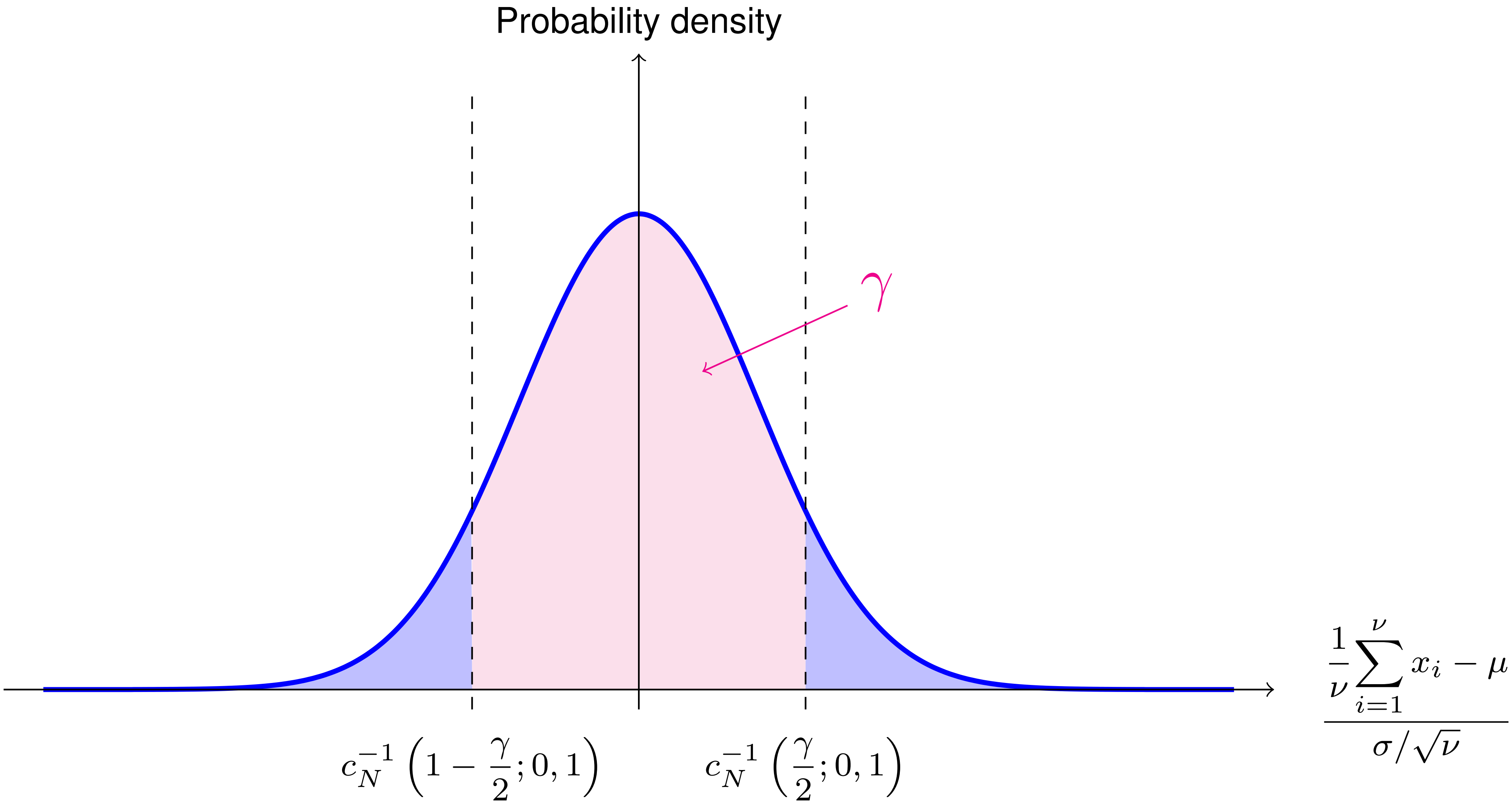
  in this case, the following interval is a γ-confidence interval for the unknown mean:

$$\left[\frac{1}{\nu}\sum_{i=1}^{\nu} x_i - c_N^{-1}\left(1-\frac{\gamma}{2};0,1\right)\frac{\sigma}{\sqrt{\nu}}, \frac{1}{\nu}\sum_{i=1}^{\nu} x_i + c_N^{-1}\left(\frac{\gamma}{2};0,1\right)\frac{\sigma}{\sqrt{\nu}}\right]$$

  where $c_N^{-1}(\,\cdot\,;0,1)$ is the inverse of the cumulative distribution function of a Gaussian random variable with mean 0 and standard deviation 1. Indeed, denoting ν statistically independent copies of a Gaussian random variable with mean μ and standard deviation σ by $X_1, \dots, X_\nu$, we have that $\frac{1}{\nu}\sum_{i=1}^{\nu} X_i$ is a Gaussian random variable with mean μ and standard deviation $\frac{\sigma}{\sqrt{\nu}}$, so that

$$P\left(\frac{1}{\nu}\sum_{i=1}^{\nu} X_i - c_N^{-1}\left(1-\frac{\gamma}{2};0,1\right)\frac{\sigma}{\sqrt{\nu}} \le \mu \le \frac{1}{\nu}\sum_{i=1}^{\nu} X_i + c_N^{-1}\left(\frac{\gamma}{2};0,1\right)\frac{\sigma}{\sqrt{\nu}}\right) \ge \gamma.$$

Confidence interval estimation



$$P\left(\frac{1}{\nu}\sum_{i=1}^{\nu} X_i - c_N^{-1}\left(1-\frac{\gamma}{2};0,1\right)\frac{\sigma}{\sqrt{\nu}} \leq \mu \leq \frac{1}{\nu}\sum_{i=1}^{\nu} X_i + c_N^{-1}\left(\frac{\gamma}{2};0,1\right)\frac{\sigma}{\sqrt{\nu}}\right) \geq \gamma.$$

- Suppose that we observe up to time t a trajectory of a **homogeneous Poisson process**. Then, the setting is as follows:
  - **data**: n, the number of failures in the interval $[0, t]$, and $t_1, \ldots, t_n$, the time instants at which the system suffered these failures,
  - **candidate stochastic model**: **homogeneous Poisson process** $\{N(t), t \geq 0\}$ **with unknown parameter** λ to be estimated ;

in this case, the following intervals are γ-confidence intervals:

$$\left[0, \frac{1}{2t} c_{\chi^2}^{-1}\left(\gamma; 2(n+1)\right)\right],$$

$$\left[\frac{1}{2t} c_{\chi^2}^{-1}\left(1-\gamma; 2n\right), +\infty\right],$$

$$\left[\frac{1}{2t} c_{\chi^2}^{-1}\left((1-\gamma)/2; 2n\right), \frac{1}{2t} c_{\chi^2}^{-1}\left((1+\gamma)/2; 2(n+1)\right)\right],$$

where $c_{\chi^2}^{-1}\left(\,\cdot\,; n\right)$ is the inverse of the cumulative distribution function of a χ^2 random variable with n degrees of freedom.

Confidence interval estimation

Proof:

$$\gamma \overset{?}{\leq} P\left(0 \leq \lambda t \leq \frac{t}{2t} c_{\chi^2}^{-1}(\gamma; 2(N(t)+1))\right)$$

$$\gamma \overset{?}{\leq} P\left(c(N(t);\lambda t) \geq c\left(N(t); \frac{t}{2t} c_{\chi^2}^{-1}(\gamma; 2(N(t)+1))\right)\right)$$

$$\gamma \overset{?}{\leq} P\left(c(N(t);\lambda t) \geq 1 - c_{\chi^2}\big(c_{\chi^2}^{-1}\left(\gamma; 2(N(t)+1)\right); 2(N(t)+1)\big)\right)$$

$$\gamma \overset{!}{\leq} P\big(c(N(t);\lambda t) \geq 1-\gamma\big),$$

where $c(\,\cdot\,; m)$ is the cumulative distribution function of a Poisson random variable with parameter m ; please note that the passage from the first to the second inequality holds because $c(n;\,\cdot\,)$ is monotically decreasing.

Proof (continued):

$$
\begin{aligned}
c(n;m) &= P(X \leq n) && \text{where } X \text{ is a Poisson random variable with parameter } m, \\
&= 1 - P(X \geq n+1) \\
&= 1 - P(T_{n+1} \leq m) && \text{where } T_{n+1} \text{ is the time of the } (n+1)\text{-th failure in a homogeneous Poisson process with parameter 1,} \\
&= 1 - \int_0^m \underbrace{\frac{1}{n!} \exp\left(-t_{n+1}\right) \left(t_{n+1}\right)^n}_{\text{gamma pdf with parameter } n+1 \text{ and } 1} \; dt_{n+1} \\
&= 1 - \int_0^{2m} \frac{1}{n!} \frac{1}{2^{n+1}} \exp\left(-\frac{y}{2}\right) y^n dy \\
&= 1 - c_{\chi^2}\left(2m; 2(n+1)\right);
\end{aligned}
$$

here, the fourth equality follows from the fact that the sum of $n+1$ statistically independent exponential random variables with parameter 1 is a gamma random variable with parameters $n+1$ and 1.

Confidence interval estimation

- Suppose that we observe up to time t a trajectory of a **nonhomogeneous Poisson process whose (instantaneous) intensity follows a Duane power law**. Then, the setting is as:
  - **data**: n, the number of failures in the interval $[0, t]$, and $t_1, \ldots, t_n$, the time instants at which the system suffered these failures,
  - **candidate stochastic model**: **nonhomogeneous Poisson process with unknown parameters α and β** to be estimated ;

  in this case, the following intervals are γ-confidence intervals for β:

$$\left[0, \frac{\widehat{\beta}}{2n} c_{\chi^2}^{-1}(\gamma; 2n)\right],$$

$$\left[\frac{\widehat{\beta}}{2n} c_{\chi^2}^{-1}(1-\gamma; 2n), +\infty\right],$$

$$\left[\frac{\widehat{\beta}}{2n} c_{\chi^2}^{-1}((1-\gamma)/2; 2n), \frac{\widehat{\beta}}{2n} c_{\chi^2}^{-1}((1+\gamma)/2; 2n)\right];$$

  it is more difficult to establish similar intervals for α.

References

Suggested reading material

- L. Wehenkel. Eléments de statistiques. Université de Liège. Lecture notes.

Additional references also consulted to prepare this lecture

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