MECA0010 – Reliability and stochastic modeling of engineered systems

Reliability: introduction and homogeneous Poisson process

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Motivation

- **Reliability of engineered systems** and its evolution through time.

- Significance to engineering:
  - performance,
  - maintenance,
  - qualification and certification,
  - warranty,
  - . . .

- Focus here on **statistical inferences of reliability characteristics from data** (as opposed to physical modeling of failure mechanisms and degradation).

Indicators of reliability

Introduction

- We distinguish between non-repairable and repairable systems:
  - A **non-repairable system** is a system for which after failure, the only option is to replace the entire system with a new one.
  - A **repairable system** is a system which, after failure, can be restored to an operative condition by a maintenance action other than replacement of the entire system. Replacing the entire system by a new one may be an option, but it is not the only one.
- We will assume that the description of the system state at any time is reduced to two categories: **operative** and **failed**.
- As a model of the evolving system state, we consider a stochastic process $\{X(t), t \in \mathbb{R}^+\}$ that is
  - indexed by time t,
  - with values in a finite set of states $\mathcal{E}$,
  - with $\mathcal{E} = \mathcal{M} \cup \mathcal{P}$ with $\mathcal{M} \cap \mathcal{P} = \emptyset$,
  - where $\mathcal{M}$ is the finite set of operative states,
  - and $\mathcal{P}$ is the finite set of failed states.

- **(Instantaneous) availability:**

$$d(t) = P\big(X(t) \in \mathcal{M}\big),$$

the probability that the system is operative at a given time.

- **Reliability:**

$$\tau(t) = P\big(\forall s \in [0, t] : X(s) \in \mathcal{M}\big),$$

the probability that the system is operative over the entire interval $[0, t]$.

- **Lifetime = time to failure:**

$$T = \inf\left\{s \geq 0 : X(s) \in \mathcal{P}\right\},$$

where we have the property $\tau(t) = P\left(T > t\right)$.

- **Mean time to failure:**

$$\text{mttf} = E\{T\}.$$

- Relationship between mean time to failure and reliability:

$$\begin{aligned}\text{mttf} = E\{T\} &= \int_{\mathbb{R}_0^+} t\rho_T(t)dt = \big[t\big(c_T(t) - 1\big)\big]_0^{+\infty} - \int_{\mathbb{R}_0^+} \big(c_T(t) - 1\big)dt \\ &= \int_{\mathbb{R}_0^+} \big(1 - c_T(t)\big)dt = \int_{\mathbb{R}_0^+} P\left(T > t\right)dt = \int_{\mathbb{R}_0^+} \tau(t)dt.\end{aligned}$$

Repairable systems

- (Instantaneous) availability.

- Reliability.

- First time to failure.

- **Maintainability:**

$$m(t) = P\Big(\exists s \in [0,t], X(s) \in \mathcal{M} \Big| X(0) \in \mathcal{P}\Big),$$

the probability that the repair of the system is completed before the time t given that the system was in a failed state at the initial time instant.

- **Mean time to repair:**

$$\text{mttr} = E\Big\{ \inf \left\{ s \geq 0 : X(s) \in \mathcal{M} \middle| X(0) \in \mathcal{P} \right\} \Big\}.$$

- **Rate of occurence of failures:**

$$\lambda(t) = \begin{cases} \dfrac{\rho_T(t)}{1 - c_T(t)} & \text{if } 1 - c_T(t) \neq 0, \\ 0 & \text{if } 1 - c_T(t) = 0. \end{cases}$$

- Interpretation as a rate:

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} P\big(t \leq T \leq t + \Delta t \big| T > t\big)$$

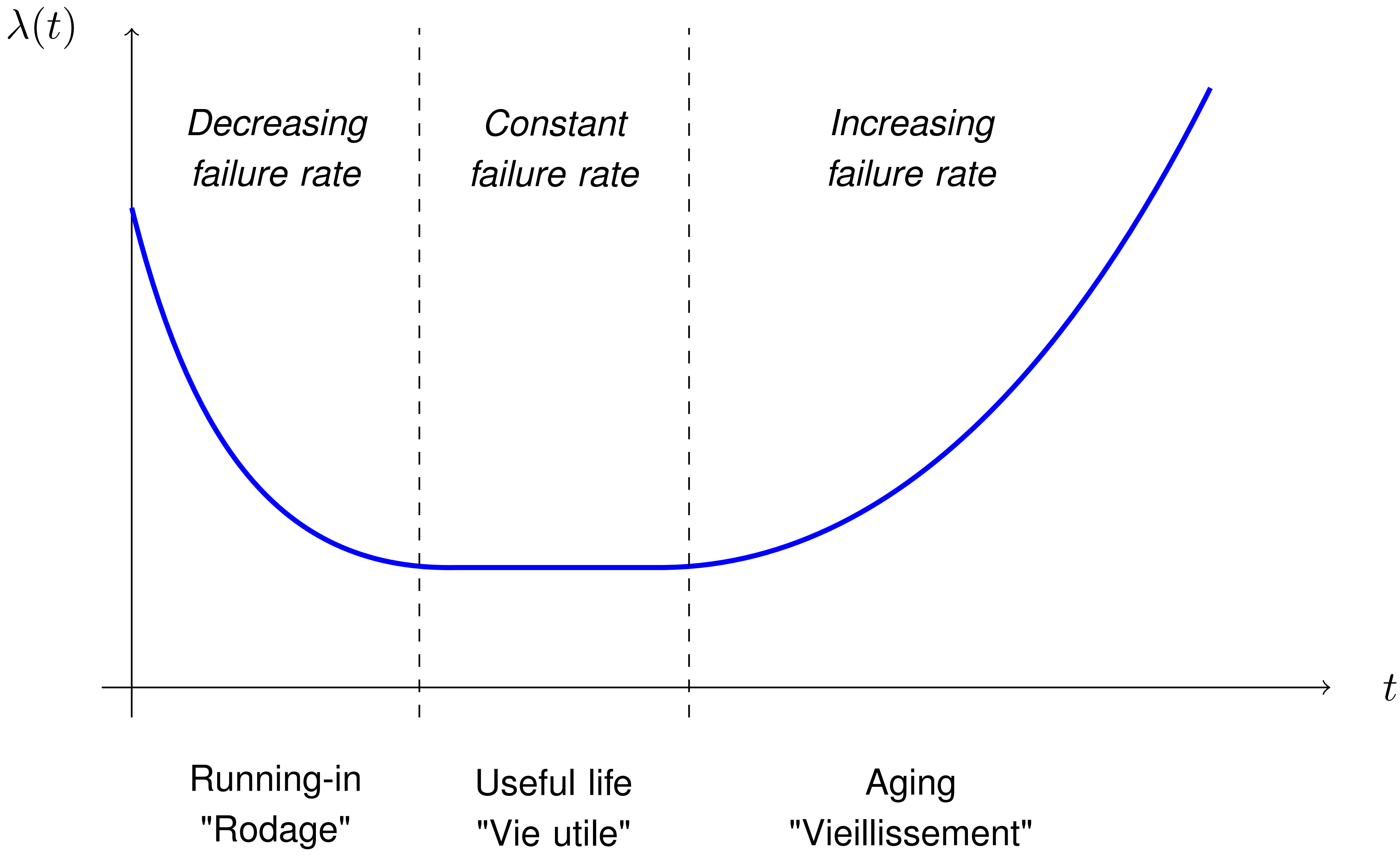
$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \frac{P\left(t < T \leq t + \Delta t\right)}{P\left(T > t\right)}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \frac{c_T(t + \Delta t) - c_T(t)}{1 - c_T(t)}$$

$$= \frac{\rho_T(t)}{1 - c_T(t)}.$$

Rate of occurence of failures

- **Bathtub curve ("courbe en baignoire"):**



Models for minimal repair

Introduction

- To model the system evolution, it is essential to describe the characteristics of the system after maintenance.

- Many repairs in industry do not return a system to a state "as good as new" but rather result in reliability characteristics that are basically the same as they were juste before the failure occurred, that is, the repaired system is **"as good as old"**. In other words, the **"age" of the system after repair is the same as it was before failure**.

- This behavior frequently arises in the maintenance of complex systems for which after failure, only small parts of the system are replaced by new ones.

- We assume that the **time to repair is negligible** (compared to the time of failure).

Counting and point processes

- The reliability of a repairable system that undergoes **"minimal" repairs** that restore the system to a state "as good as old" after failure and for which the **time to repair is negligible** can be described in different ways, such as by describing the **number of failures suffered by the system up to a given time t**, by means of the **lengths of time between consecutive failures**, and so forth.

- **Counting process:**

$$\{N(t), t \geq 0\},$$

where $N(t)$ is the number of failures suffered by the system in the interval $[0, t]$, with $N(0) = 0$.

- We assume that the **time to repair is negligible** (compared to the time of failure).

- **Point process:**

$$\{T_n, n \geq 1\},$$

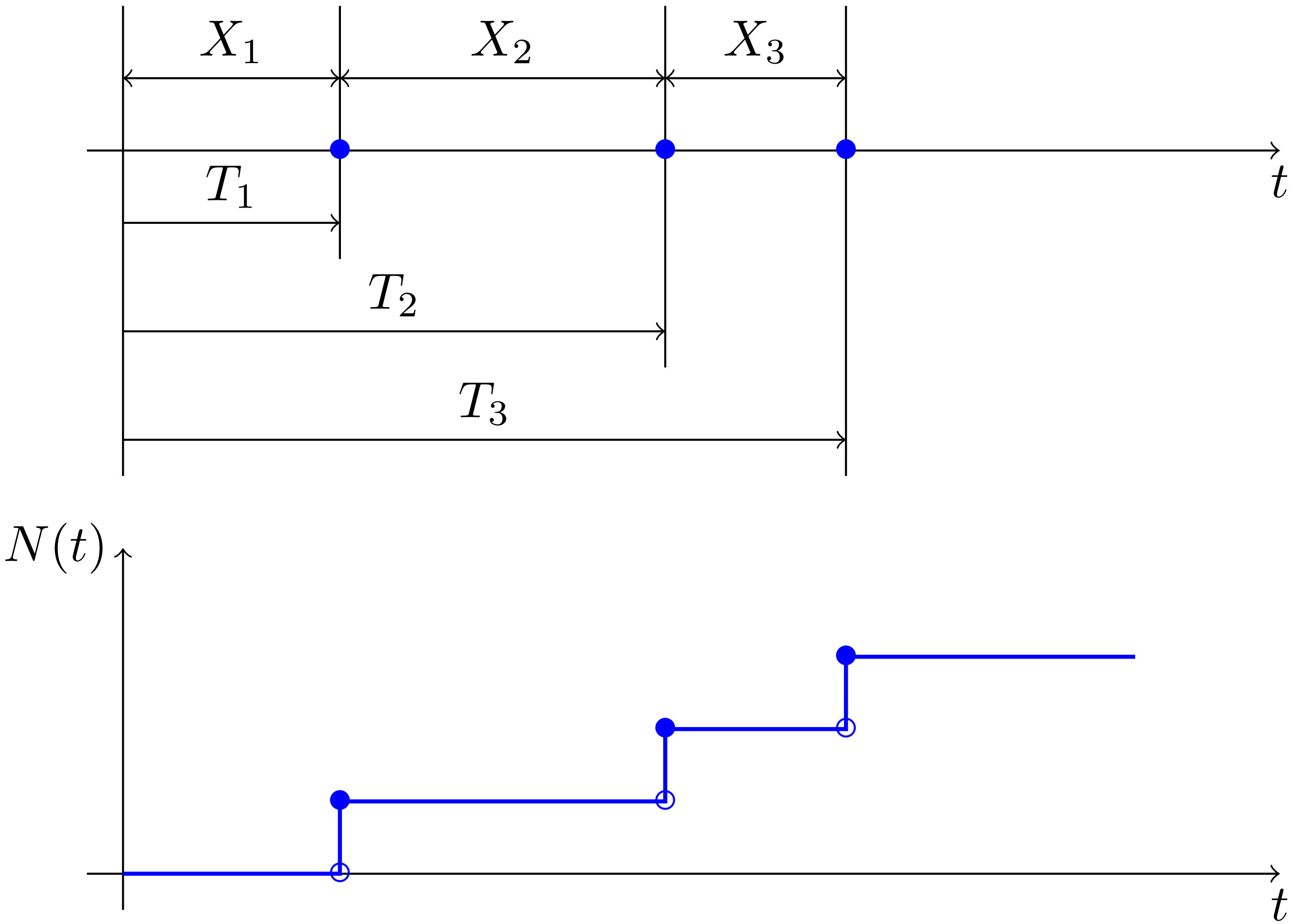
which collects the time instants at which the system suffers failures.

- **Lengths of time between consecutive failures:**

$$\{X_n, n \geq 1\},$$

with $X_1 = T_1$ and $X_n = T_n - T_{n-1}$ for $n \geq 2$.

- **Counting process** $\{N(t),\ t \geq 0\}$:



Poisson process

- A stochastic process $\{N(t), t \geq 0\}$ indexed by $\mathbb{R}^+$ with values in $\mathbb{N}$ is a **Poisson process** with mean function m, where $t \mapsto m(t)$ is an increasing and positive function, if and only if
  - $\{N(t), t \geq 0\}$ has **independent increments**, that is, for any $0 \leq s < t \leq u < v < +\infty$, the random variables $N(v) - N(u)$ and $N(t) - N(s)$ are statistically independent.
  - **N(0) = 0.**
  - for any $0 \leq s < t$, $N(t) - N(s)$ is a r.v. with **Poisson distribution** with mean $m(t) - m(s)$:

$$P\Big\{N(t) - N(s) = n\Big\} = \frac{\big(m(t) - m(s)\big)^n}{n!} \text{exp}\Big(- \big(m(t) - m(s)\big)\Big).$$

Homogeneous Poisson process

- The Poisson process $\{N(t), t \geq 0\}$ is **homogeneous** if the mean function m is of the form $m(t) = \lambda t$ with λ a positive constant, that is, if the average of the number of failures occuring increases linearly with the time interval under consideration.

- Let $\{N(t), t \geq 0\}$ be a homogeneous Poisson process with parameter λ. Then, the **first time to failure obeys an exponential distribution with parameter λ**.

Proof:

$$\lim_{h \to 0} \frac{P\left(0 < t_1 \leq T_1 < t_1 + h\right)}{h} = \lim_{h \to 0} \frac{\frac{(\lambda t_1)^0}{0!} \exp\left(-\lambda t_1\right) \frac{(\lambda h)^1}{1!} \exp\left(-\lambda h\right)}{h} = \lambda \exp\left(-\lambda t_1\right).$$

Homogeneous Poisson process

- More generally, let $\{N(t), t \geq 0\}$ be a homogeneous Poisson process with parameter λ, then the **lengths of time between consecutive failures $\{X_n, n \geq 1\}$ are statistically independent and identically distributed with exponential distribution with parameter λ.**

- **Link with rate of occurrence of failures:**

$$\frac{\rho_T(t)}{1 - c_T(t)} = \frac{\lambda \text{exp}\,(-\lambda t)}{\text{exp}\,(-\lambda t)} = \lambda.$$

Homogeneous Poisson process $\equiv$ no aging.

Homogeneous Poisson process

- Let $\{N(t), t \geq 0\}$ be a homogeneous Poisson process with parameter λ. Then, the probability distribution of the instants at which the system suffers its first n failures admits as a density

$$\rho_{(T_1,\ldots,T_n)}\left(t_1,\ldots,t_n\right) = \lambda^n \text{exp}\left(-\lambda t_n\right) 1\left(0 < t_1 < \ldots < t_n\right),$$

and the probability distribution of these instants given that $\{N(t) = n\}$ admits as a density

$$\rho_{(T_1,\ldots,T_n|N(t))}\left(t_1,\ldots,t_n|n\right) = \frac{n!}{t^n} 1\left(0 < t_1 < \ldots < t_n < t\right).$$

Proof:

$$\begin{aligned}
\rho_{(T_1,\ldots,T_n)}\left(t_1,\ldots,t_n\right) &= \lambda\text{exp}\left(-\lambda t_1\right) \times \ldots \times \lambda\text{exp}\left(-\lambda(t_n - t_{n-1})\right) \times 1\left(0 < t_1 < \ldots < t_n\right) \\
&= \lambda^n \text{exp}\left(-\lambda t_n\right) 1\left(0 < t_1 < \ldots < t_n\right).
\end{aligned}$$

Proof (continued):

$$
\begin{aligned}
&\rho_{(T_1,\ldots,T_n|N(t))}\,(t_1,\ldots,t_n|n)\\
&= c\lambda^{n+1}\text{exp}\left(-\lambda t_{n+1}\right)1\left(0<t_1<\ldots<t_n<t<t_{n+1}\right)\\
&= \int_t^{+\infty} c\lambda^{n+1}\text{exp}\left(-\lambda t_{n+1}\right)1\left(0<t_1<\ldots<t_n<t\right)dt_{n+1}\\
&= c\lambda^{n}\text{exp}\left(-\lambda t\right)1\left(0<t_1<\ldots<t_n<t\right).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\int_{\mathbb{R}^n} c\lambda^{n}\text{exp}\left(-\lambda t\right)1\left(0<t_1<\ldots<t_n<t\right)\mathrm{d}t_1\ldots\mathrm{d}t_n = 1\\
&\Rightarrow \quad c = \frac{1}{\lambda^n\text{exp}\left(-\lambda t\right)}\frac{n!}{t^n}
\end{aligned}
$$

since

$$
\int_{\mathbb{R}^n} 1\left(0<t_1<\ldots<t_n<t\right)\mathrm{d}t_1\ldots dt_n = \frac{n!}{t^n}\cdot
$$

References

Suggested reading material

- L. Wehenkel. Eléments de statistiques. Université de Liège. Lecture notes.

Additional references also consulted to prepare this lecture

- A. Ang and W. Tang. Probability concepts in engineering. John Wiley & Sons, 2007.
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- C. Soize. The Fokker–Planck equation for stochastic dynamical systems and its explicit steady state solutions. World Scientific Publishing, 1994.