MECA0010 – Reliability and stochastic modeling of engineered systems

Uncertainty quantification

Part 3 of 4 — Maximum entropy principle. Stochastic sensitivity analysis.

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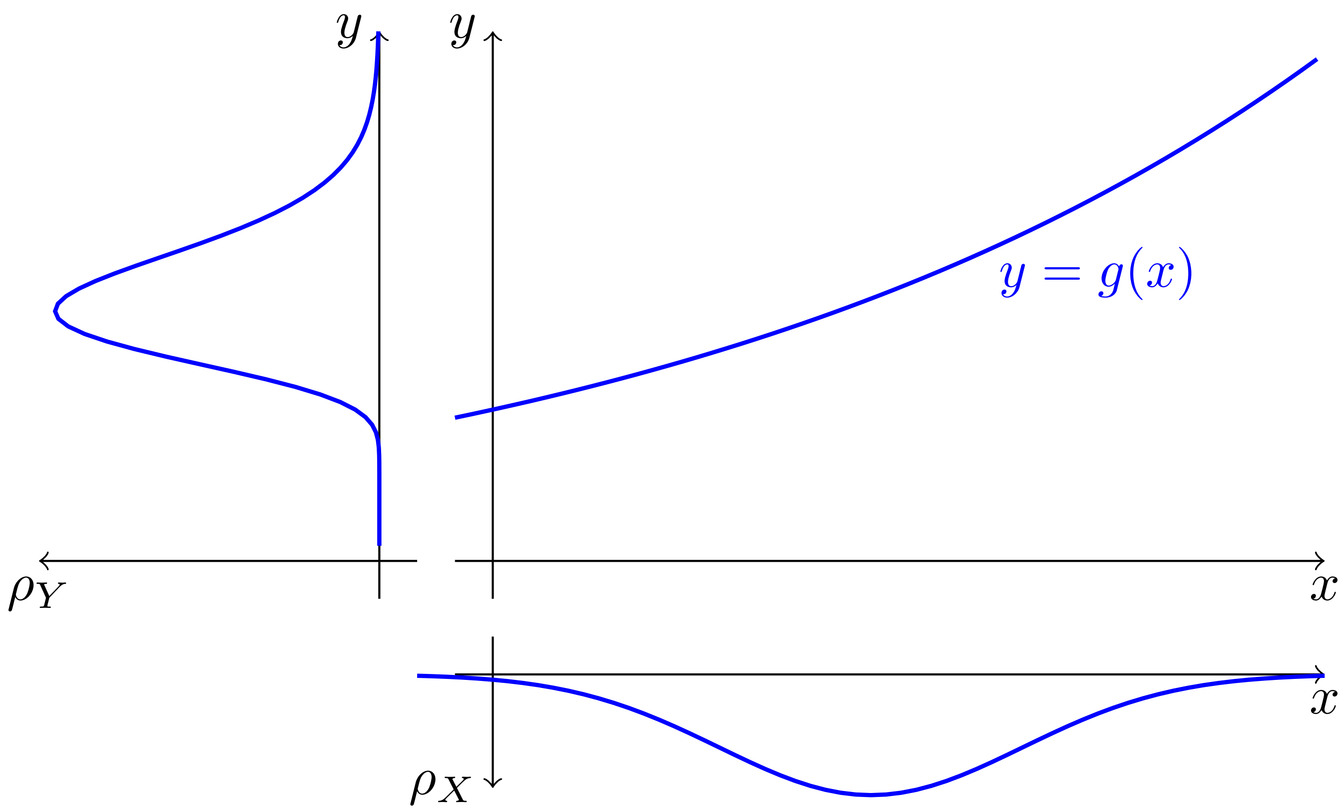
Maximum entropy principle

Motivation



- Recall that if the transformation g is nonlinear, then the probability density function of the uncertain input variables must be known in order to be able to determine statistical descriptors of the uncertainty induced in the output variable.
- The maximum entropy principle is a method for constructing a probability density function for the input variables on the basis of the information that is available about their uncertainty.

Outline

- Motivation.
- Outline.
- Entropy.
- Available information.
- Maximum entropy principle.
- Examples.
- Conclusion and references.

Entropy

Entropy of a discrete random variable

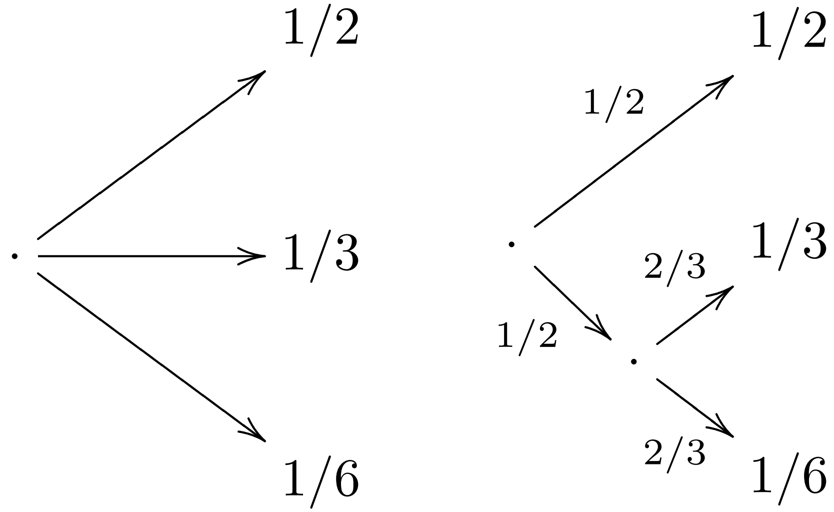
- **Context**: Let $\boldsymbol{X}$ be a discrete random variable that can take the values $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$. Let these values $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ occur with probabilities $p_1, \ldots, p_n$, that is,

$$P_{\boldsymbol{X}}(\boldsymbol{x}_j) = p_j, \quad 1 \leq j \leq n, \quad 0 \leq p_1, \ldots, p_n \leq 1, \quad p_1 + \ldots + p_n = 1.$$

- **Intuition**: If $p_i = 1$ and $p_j = 0$ for $i \neq j$, there is "no uncertainty." If $p_1 = p_2 = \ldots = p_n = 1/n$, there is "maximal uncertainty."

- **Shannon's axioms for gauging the "amount of uncertainty:"**

(i) $(p_1, \ldots, p_n) \mapsto s_n(p_1, \ldots, p_n)$ is continuous from $[0, 1]^n$ into $\mathbb{R}$.
(ii) If $p_1 = p_2 = \ldots = p_n = \frac{1}{n}$, then $n \mapsto s_n\left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$ is monotonically increasing.
(iii) $s_n(p_1, \ldots, p_n)$ is symmetric in $p_1, \ldots, p_n$.
(iv) $s_n(p_1, \ldots, p_n) = s_{n-1}(p_1 + p_2, p_3, \ldots, p_n) + (p_1 + p_2)s_2\left(\frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2}\right)$.



$$s_3\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right) = s_2\left(\frac{1}{2} + \frac{1}{2}\right) + \frac{1}{2}s_2\left(\frac{2}{3}, \frac{1}{3}\right).$$

Entropy of a discrete random variable

- There exists one and only one function that satisfies the requirements (i)-(iv), namely,

$$s_n(p_1, \ldots, p_n) = -\sum_{j=1}^{n} p_j \log(p_j), \qquad 0 \log(0) = 0.$$

Here, s_n is called the **Shannon entropy** associated with $p_1, \ldots, p_n$.

- $s_n(p_1, \ldots, p_n) \geq 0$.
- $\max_{0 \leq p_1, \ldots, p_n \leq 1} s_n(p_1, \ldots, p_n) = \log(n)$.
- $s_{nm}(P_{(\boldsymbol{X},\boldsymbol{Y})}) \leq s_n(P_{\boldsymbol{X}}) + s_m(P_{\boldsymbol{Y}})$.
- $s_{nm}(P_{\boldsymbol{X}} \times P_{\boldsymbol{Y}}) = s_n(P_{\boldsymbol{X}}) + s_m(P_{\boldsymbol{Y}})$.

C. Shannon. A mathematical theory of communication. *Bell System Technical Journal*, 27:379-423, 1948.

Entropy of a continuous random variable

- Let $\boldsymbol{X}$ be a random variable with values in $\mathbb{R}^n$. Let $\boldsymbol{X}$ admit a probability density function $\rho_{\boldsymbol{X}}$, that is, for any meaningful subset $\mathcal{B}$ of $\mathbb{R}^n$, we have $P_{\boldsymbol{X}}(\mathcal{B}) = \int_{\mathcal{B}} \rho_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x}$. Then, the **Shannon entropy** associated with $\rho_{\boldsymbol{X}}$, denoted by $s(\rho_{\boldsymbol{X}})$, is defined as follows:

$$s(\rho_{\boldsymbol{X}}) = -\int_{\mathbb{R}^n} \rho_{\boldsymbol{X}}(\boldsymbol{x}) \log\big(\rho_{\boldsymbol{X}}(\boldsymbol{x})\big) d\boldsymbol{x}.$$

  - $-\infty < s(\rho_{\boldsymbol{X}}) < +\infty$.
  - $s_{nm}(P_{(\boldsymbol{X},\boldsymbol{Y})}) \leq s_n(P_{\boldsymbol{X}}) + s_m(P_{\boldsymbol{Y}})$.
  - $s_{nm}(P_{\boldsymbol{X}} \times P_{\boldsymbol{Y}}) = s_n(P_{\boldsymbol{X}}) + s_m(P_{\boldsymbol{Y}})$.

  The entropy of a continuous random variable is not necessarily positive, and it depends on the coordinate system.

Available information

Sources of available information

- **Data** in the form of either new experimental results or higher fidelity models.

- **Mechanical and physical constraints** imposed by applicable mechanical and physical laws:
  - positiveness and symmetry of mechanical properties involved in constitutive equations,
  - positiveness and symmetry of reduced matrices involved in reduced-order models,
  - causality and stability properties of dynamical systems,
  - ...

  Consistency with mechanics and physics requires the assignment of a vanishing probability to those values of the uncertain features that do not satisfy the mechanical and physical constraints.

- **Other sources of information** can also contribute to the available information. The combined information provided by the mechanical and physical constraints and these other sources of information is often referred to as the prior information.

Maximum entropy principle

General formulation

- The maximum entropy principle allows probability distributions to be constructed in a manner that is consistent with the available information.

- The **maximum entropy principle** consists in choosing, out of all probability distributions consistent with a given set of constraints, the one that has maximum entropy.

- E. Jaynes. Information theory and statistical mechanics. *The Physical Review*, 106:620-630, 1957.

Formulation involving moment constraints

- We consider cases wherein
  - the **support** of $\rho_{\boldsymbol{X}}$ is given, that is, a subset $\mathcal{K}$ of $\mathbb{R}^n$ is given such that

$$\rho_{\boldsymbol{X}}(\boldsymbol{x}) = 0 \text{ for } \boldsymbol{x} \notin \mathcal{K},$$

  - m **generalized moments** are given, written as

$$\int_{\mathbb{R}^n} \boldsymbol{g}_j(\boldsymbol{x})\rho_{\boldsymbol{X}}(\boldsymbol{x})d\boldsymbol{x} = \overline{\boldsymbol{g}}_j, \quad j = 1, \ldots, m.$$

  Thus, here, the support $\mathcal{K}$ and the m generalized moments described by $\boldsymbol{g}_1, \ldots, \boldsymbol{g}_m$ and $\overline{\boldsymbol{g}}_1, \ldots, \overline{\boldsymbol{g}}_m$ constitute the available information.

- The probability density function $\rho_{\boldsymbol{X}}$ is obtained by solving the **optimization problem**

$$\max_{\rho_{\boldsymbol{X}} \in \mathcal{C}_{\text{ad}}} s(\rho_{\boldsymbol{X}}),$$

  subject to:

$$\int_{\mathbb{R}^n} \rho_{\boldsymbol{X}}(\boldsymbol{x})d\boldsymbol{x} = 1,$$

$$\int_{\mathbb{R}^n} \boldsymbol{g}_j(\boldsymbol{x})\rho_{\boldsymbol{X}}(\boldsymbol{x})d\boldsymbol{x} = \overline{\boldsymbol{g}}_j, \quad j = 1, \ldots, m,$$

  in which $\mathcal{C}_{\text{ad}} = \{\rho_{\boldsymbol{X}} \ : \ \rho_{\boldsymbol{X}}(\boldsymbol{x}) = 0 \text{ for } \boldsymbol{x} \notin \mathcal{K}\}$.

Formulation involving moment constraints

- This optimization problem can be readily solved using the method of the **Lagrange multipliers**:
  - First, the Lagrangian is formed as follows:

$$\mathcal{L}(\rho_{\boldsymbol{X}},\lambda_0,\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_m)=s(\rho_{\boldsymbol{X}})-(\lambda_0-1)\left(\int_{\mathbb{R}^n}\rho_{\boldsymbol{X}}(\boldsymbol{x})d\boldsymbol{x}-1\right)-\sum_{j=1}^{m}\boldsymbol{\lambda}_j^{\mathrm{T}}\left(\int_{\mathbb{R}^n}\boldsymbol{g}_j(\boldsymbol{x})\rho_{\boldsymbol{X}}(\boldsymbol{x})d\boldsymbol{x}-\overline{\boldsymbol{g}}_j\right).$$

  - Subsequently, stationarity is expressed as follows:

$$\delta\mathcal{L}=\int_{\mathbb{R}^n}\left(-\log\left(\rho_{\boldsymbol{X}}(\boldsymbol{x})\right)-1-(\lambda_0-1)-\sum_{j=1}^{m}\boldsymbol{\lambda}_j^{\mathrm{T}}\boldsymbol{g}_j(\boldsymbol{x})\right)\delta\rho_{\boldsymbol{X}}(\boldsymbol{x})d\boldsymbol{x}$$

$$-\left(\int_{\mathbb{R}^n}\rho_{\boldsymbol{X}}(\boldsymbol{x})d\boldsymbol{x}-1\right)\delta\lambda_0-\left(\int_{\mathbb{R}^n}\boldsymbol{g}_j(\boldsymbol{x})\rho_{\boldsymbol{X}}(\boldsymbol{x})d\boldsymbol{x}-\overline{\boldsymbol{g}}_j\right)^{\mathrm{T}}\delta\boldsymbol{\lambda}_j=0,$$

  thus leading to

  (i) $-\log\left(\rho_{\boldsymbol{X}}(\boldsymbol{x})\right)-1-(\lambda_0-1)-\sum_{j=1}^{m}\boldsymbol{\lambda}_j^{\mathrm{T}}\boldsymbol{g}_j(\boldsymbol{x})=0,$

  (ii) $\int_{\mathbb{R}^n}\rho_{\boldsymbol{X}}(\boldsymbol{x})d\boldsymbol{x}-1=0,$

  (iii) $\int_{\mathbb{R}^n}\boldsymbol{g}_j(\boldsymbol{x})\rho_{\boldsymbol{X}}(\boldsymbol{x})d\boldsymbol{x}-\overline{\boldsymbol{g}}_j=0.$

  - Finally, the solution to the constrained optimization problem is obtained from (i) as follows:

$$\rho_{\boldsymbol{X}}(\boldsymbol{x})=1_{\mathcal{K}}(\boldsymbol{x})\exp\left(-\lambda_0-\sum_{j=1}^{m}\boldsymbol{\lambda}_j^{\mathrm{T}}\boldsymbol{g}_j(\boldsymbol{x})\right),$$

  in which λ_0 and $\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_m$ have to be determined in such a way that (ii) and (iii) are fulfilled.

Examples

- **Support $]-\infty,+\infty[$, mean given, variance given:**
  - We have to solve the following optimization problem:

$$\max s(\rho_X),$$

subject to

$$\int_{\mathbb{R}} \rho_X(x)dx = 1,$$

$$\int_{\mathbb{R}} x\rho_X(x)dx = m_X,$$

$$\int_{\mathbb{R}} (x-m_X)^2\rho_X(x)dx = \sigma_X^2.$$

  - Applying the method of the Lagrange multipliers, we obtain

$$\rho_X(x) = \exp\big(-\lambda_0 - \lambda_1 x - \lambda_2(x - m_X^2)\big).$$

  - After using the constraints to determine the values of λ_0, λ_1, and λ_2, we obtain

$$\rho_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X}\exp\left(-\frac{(x-m_X)^2}{2\sigma_X^2}\right),$$

that is, we obtain the **Gaussian probability density function** with mean m_X and variance σ_X^2.

- **Support $]0, +\infty[$, mean given, log-mean given:**
  - We have to solve the following optimization problem:

$$\max_{\rho_X \in \mathcal{C}_{\text{ad}}} s(\rho_X),$$

subject to

$$\int_{\mathbb{R}} \rho_X(x)dx = 1,$$

$$\int_{\mathbb{R}} x\rho_X(x)dx = m_X,$$

$$\int_{\mathbb{R}} \log(x)\rho_X(x)dx = c_X,$$

with $\mathcal{C}_{\text{ad}} = \{\rho_X \ : \ \rho_X(x) = 0 \text{ if } x \notin \mathbb{R}_0^+\}$.

  - Applying the method of the Lagrange multipliers, we obtain

$$\rho_X(x) = 1_{\mathbb{R}_0^+}(x) \exp\big(-\lambda_0 - \lambda_1 x - \lambda_2 \log(x)\big).$$

  - After using the constraints to determine the values of λ_0, λ_1, and λ_2, we obtain

$$\rho_X(x) = 1_{\mathbb{R}_0^+}(x)\frac{1}{m_X}\left(\frac{1}{\delta_X^2}\right)^{\frac{1}{\delta_X^2}} \frac{1}{\Gamma(\frac{1}{\delta_X^2})}\left(\frac{x}{m_X}\right)^{\frac{1}{\delta_X^2}-1} \exp\left(-\frac{x}{\delta_X^2 m_X}\right);$$

we obtain the **gamma probability density function** with mean m_X and dispersion level δ_X.

- **Support $]a, b[$, mean given**:
  - We have to solve the following optimization problem:

$$\max_{\rho_X \in \mathcal{C}_{\text{ad}}} s(\rho_X),$$

subject to

$$\int_{\mathbb{R}} \rho_X(x)dx = 1,$$

$$\int_{\mathbb{R}} x\rho_X(x)dx = m_X,$$

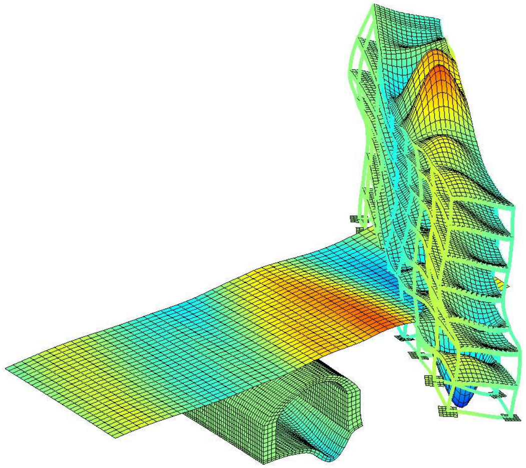
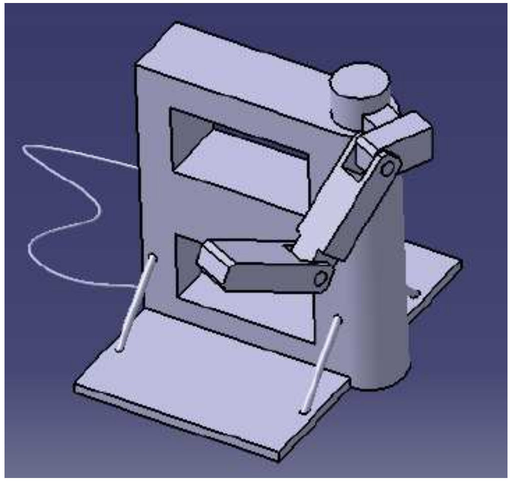
with $\mathcal{C}_{\text{ad}} = \{\rho_X \ : \ \rho_X(x) = 0 \text{ if } x \notin]a, b[\}$.

  - Applying the method of the Lagrange multipliers, we obtain

$$\rho_X(x) = 1_{]a,b[}(x) \exp\big(-\lambda_0 - \lambda_1 x\big),$$

where λ_0 and λ_1 have to be determined numerically using the constraints.

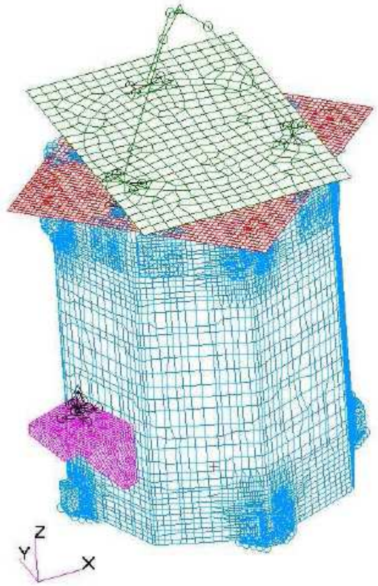
Conclusion and references

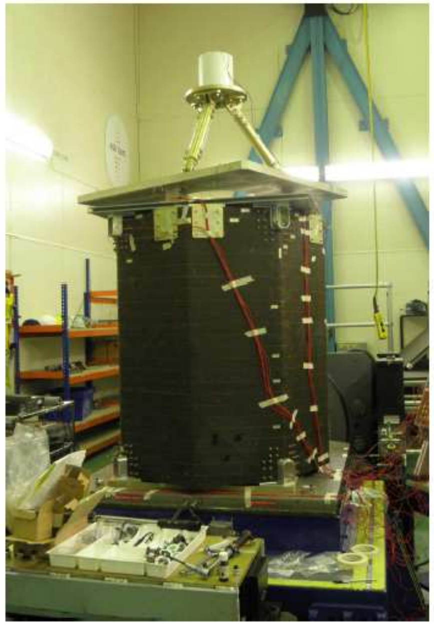
-   

- The maximum entropy principle allows probabilistic models consistent with mechanical and physical constraints to be constructed in a wide variety of engineering applications, including reduced-order models for the dynamical behavior of structures, mistuned bladed disks, rigid body mechanics and robotics, soil structure interaction, fluid dynamics, among others.

- E. Capiez-Lernout and C. Soize. Nonparametric modeling of random uncertainties for dynamic response of mistuned bladed disks. Journal of Engineering for Gas Turbines and Power, 2004.
- M. Arnst, D. Clouteau, H. Chebli, R. Othman, and G. Degrande. A non-parametric probabilistic model for ground-borne vibrations in buildings. Probabilistic Engineering Mechanics, 2005.
- A. Batou and C. Soize. Rigid multibody system dynamics with uncertain rigid bodies. Multibody System Dynamics, 2012.
- J. Guilleminot and C. Soize. On the Statistical Dependence for the Components of Random Elasticity Tensors Exhibiting Material Symmetry Properties. Journal of Elasticity, 2012.

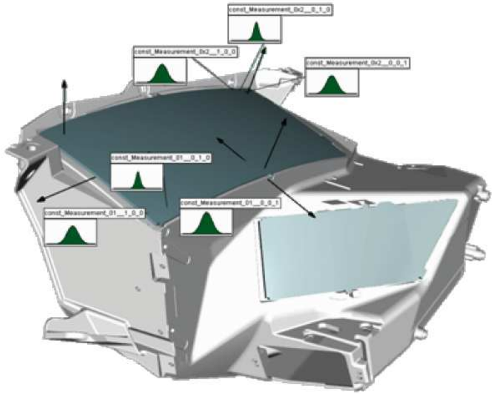
Stochastic sensitivity analysis





What are the most significant sources of uncertainties and errors?

Where to direct efforts aimed at improving predictions?



Engineering design.

Sensitivity of performance with respect to manufacturing tolerances?

Outline

- Motivation.
- Outline.
- Context.
- Variance-based sensitivity analysis.
- Example: Metal forming.
- References.

Context

Model problem

$$\underbrace{z}_{\text{output variable}} = \underbrace{g}_{\text{function}}(\underbrace{x\ ,\ y}_{\text{input variables}}).$$

Let x and y be uncertain (e.g., imperfect knowledge at design time, imperfect manufacturing when compared to the design,...).

Let us note that we consider only two input variables only for the sake of the simplicity of the system of notation. All the methods described later can be extended without difficulty to problems involving an arbitrary number of input variables.

Introduction to sensitivity analysis

- There exist many types of sensitivity analysis:
  - elementary effects:

$$\frac{g(x+\Delta x, y) - g(x,y)}{\Delta x} \quad \text{and} \quad \frac{g(x, y+\Delta y) - g(x,y)}{\Delta y},$$

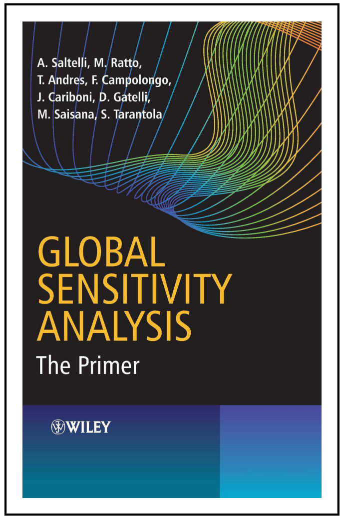
  - differentiation-based sensitivity analysis,

$$\frac{\partial g}{\partial x}(x,y) \quad \text{and} \quad \frac{\partial g}{\partial y}(x,y),$$

  - regression analysis,

$$z = g(x,y) \approx c_{00} + c_{10}x + c_{01}y + c_{20}x^2 + c_{11}xy + c_{02}y^2 + \ldots,$$

  - correlation analysis,
  - methods involving scatter plots,
  - ...



Sensitivity analysis of uncertainties

- In a context wherein the input variables are uncertain, new types of question can be asked.
- For example, one can ask which one of the uncertain input variables is most significant in inducing uncertainty in the output variable?
- Such new questions lead to new types of sensitivity analysis, such as variance-based sensitivity analysis. . .

Variance-based sensitivity analysis

Problem setting

- **Characterization of uncertainty**:
  - Two statistically independent sources of uncertainty modeled as two statistically independent random variables X and Y with probability distributions P_X and P_Y:
$$(X, Y) \sim P_X \times P_Y.$$

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- **Propagation of uncertainty**:
  - We assume that the relationship between the sources of uncertainty and the predictions is represented by a nonlinear function g:

Sources of uncertainty	Problem	Prediction
(X, Y)	$Z = g(X, Y)$	Z

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  - The probability distribution P_Z of the prediction is obtained as the image of the probability distribution $P_X \times P_Y$ of the sources of uncertainty under the function g:
$$Z \sim P_Z = (P_X \times P_Y) \circ g^{-1}.$$

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$$Z \sim P_Z = (P_X \times P_Y) \circ g^{-1}.$$
- **Sensitivity analysis**:
  - Is either X or Y most significant in inducing uncertainty in Z?

Geometrical point of view

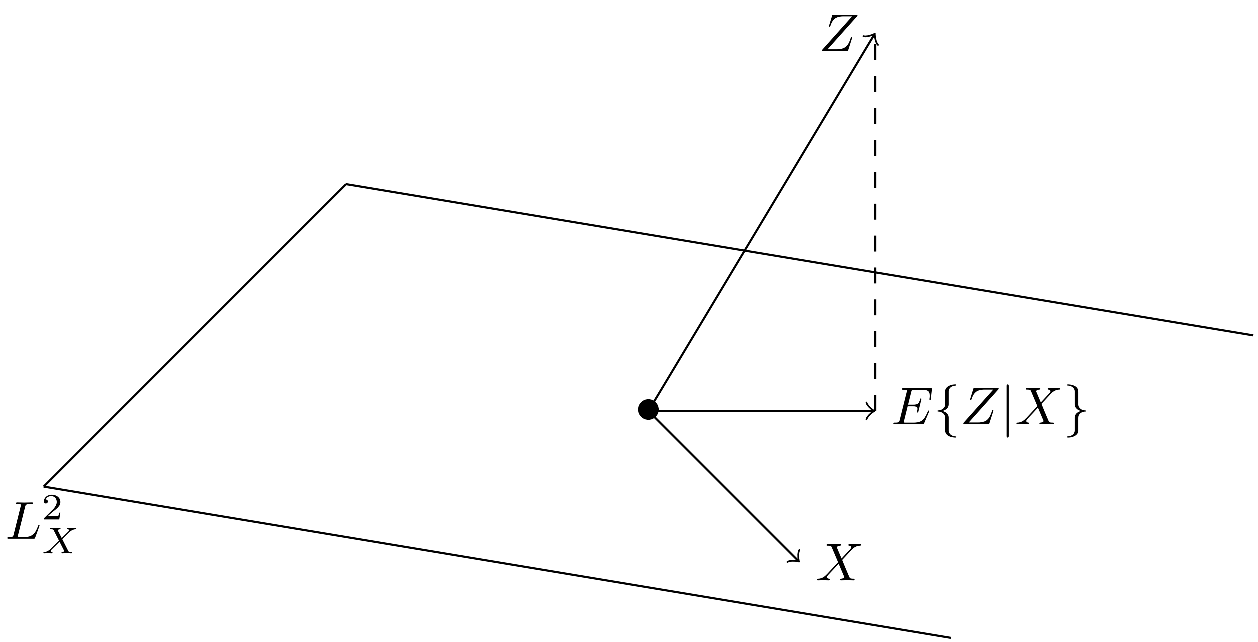
- **Least-squares-best approximation of function g with function of only one input**:
  - Assessment of the significance of the source of uncertainty X:

$$g_X^* = \arg\min_{f_X^*} \iint \left| g(x,y) - f_X^*(x) \right|^2 P_X(dx) P_Y(dy).$$

  - By means of the calculus of variations, it can be readily shown that the solution is given by

$$g_X^* = \int g(\cdot, y) P_Y(dy).$$

  - In the geometry of the space of $P_X \times P_Y$-square-integrable functions, g_X^* is the orthogonal projection of function g of x and y onto the subspace of functions of only x:



Geometrical point of view

- **Expansion of function g in terms of main effects and interaction effects**:
  - Extension to assessment of significance of both sources of uncertainty X and Y:
$$g(x,y) = g_0 + \underbrace{g_X(x)}_{\text{main effect of } X} + \underbrace{g_Y(y)}_{\text{main effect of } Y} + \underbrace{g_{(X,Y)}(x,y)}_{\text{interaction effect of } X \text{ and } Y},$$
where
$$g_0 = \iint g(x,y) P_X(dx) P_Y(dy),$$
$$g_X(x) = g_X^*(x) - g_0 = \int g(x,y) P_Y(dy) - g_0,$$
$$g_Y(y) = g_Y^*(y) - g_0 = \int g(x,y) P_X(dx) - g_0.$$
  - Because they are obtained via orthogonal projection, the functions g_0, g_X, g_Y, and $g_{(X,Y)}$ are orthogonal functions.
  - The property that g_0, g_X, g_Y, and $g_{(X,Y)}$ are orthogonal provides a link with other expansions, such as the polynomial chaos expansion.

Statistical point of view

- **Sensitivity indices = mean-square values of main effects and interaction effects**:
  - Quantitative insight into the significance of X and Y in inducing uncertainty in Z:

$$\underbrace{\iint \left|g(x,y)-g_0\right|^2 P_X(dx)P_Y(dy)}_{=\sigma_Z^2}$$

$$= \underbrace{\int \left|g_X(x)\right|^2 P_X(dx)}_{=s_X} + \underbrace{\int \left|g_Y(y)\right|^2 P_Y(dy)}_{=s_Y} + \underbrace{\iint \left|g_{(X,Y)}(x,y)\right|^2 P_X(dx)P_Y(dy)}_{=s_{(X,Y)}}.$$

  - Because g_X, g_Y, and $g_{(X,Y)}$ are orthogonal, there are no double product terms.
  - Thus, the expansion of g (geometry) reflects a **partitioning of the variance** of Z into terms that are the variances of the main and interaction effects of X and Y (statistics), where:

    s_X = portion of the variance of Z that is explained as stemming from X,

    s_Y = portion of the variance of Z that is explained as stemming from Y.

Statistical point of view

- By the conditional variance identity, we have

$$s_X = V\{E\{Z|X\}\} = V\{Z\} - E\{V\{Z|X\}\},$$
$$s_Y = V\{E\{Z|Y\}\} = V\{Z\} - E\{V\{Z|Y\}\},$$

so that s_X and s_Y may also be interpreted as **expected reductions of amount of uncertainty**:

s_X = expected reduction of variance of Z if there were no longer uncertainty in X,

s_Y = expected reduction of variance of Z if there were no longer uncertainty in Y.

In contrast to the expansion of g and the variance partitioning of Z, these expressions and these interpretations of s_X and s_Y remain valid even if X and Y are statistically dependent.

Example

- Let us consider a simple problem wherein X and Y are uniform r.v. with values in $[-1, 1]$,

$$X \sim \mathcal{U}([-1, 1]),$$
$$Y \sim \mathcal{U}([-1, 1]),$$

and the function g is given by

$$z = g(x, y) = x + y^2 + xy.$$

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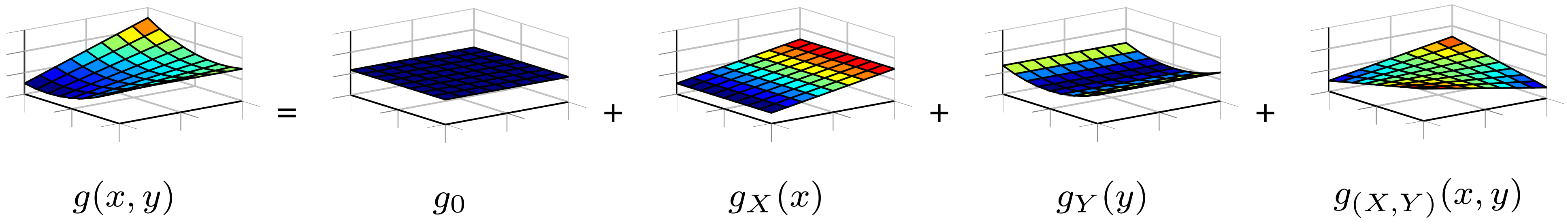
- This problem has the expansion
$$g(x,y) = g_0 + g_X(x) + g_Y(y) + g_{(X,Y)}(x,y),$$



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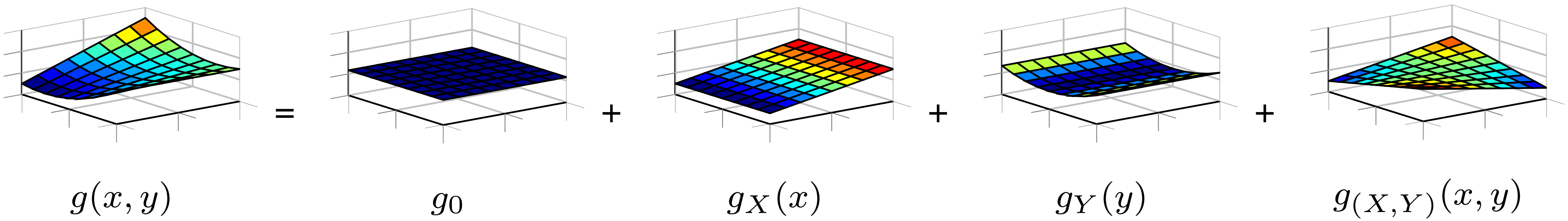


- To this expansion corresponds the variance partitioning

$$\sigma_Z^2 = s_X + s_Y + s_{(X,Y)},$$

$$\sigma_Z^2 = \frac{28}{45}, \quad s_X = \frac{1}{3} = 53.57\%\sigma_Z^2, \quad s_Y = \frac{8}{45} = 28.57\%\sigma_Z^2, \quad s_{(X,Y)} = \frac{1}{9} = 17.86\%\sigma_Z^2.$$

Computation

- Computation by means of a **stochastic expansion method**:

$$s_X \approx \sum_{\alpha \neq 0} c_{(\alpha,0)}^2, \qquad s_Y \approx \sum_{\beta \neq 0} c_{(0,\beta)}^2, \quad \text{with} \quad g(x,y) = \sum_{(\alpha,\beta)} c_{(\alpha,\beta)} \varphi_\alpha(x) \psi_\beta(y).$$

- Computation by means of **deterministic numerical integration**:

$$s_X \approx Q_X\left(|Q_Y g - Q_X Q_Y g|^2\right),$$
$$s_Y \approx Q_Y\left(|Q_X g - Q_X Q_Y g|^2\right).$$

- Computation by means of **Monte Carlo integration**:

$$s_X \approx \frac{1}{\nu} \sum_{\ell=1}^{\nu} \left(g(x_\ell, y_\ell) - \frac{1}{\nu} \sum_{k=1}^{\nu} g(x_k, y_k) \right) \left(g(x_\ell, \tilde{y}_\ell) - \frac{1}{\nu} \sum_{k=1}^{\nu} g(x_k, \tilde{y}_k) \right),$$
$$s_Y \approx \frac{1}{\nu} \sum_{\ell=1}^{\nu} \left(g(x_\ell, y_\ell) - \frac{1}{\nu} \sum_{k=1}^{\nu} g(x_k, y_k) \right) \left(g(\tilde{x}_\ell, y_\ell) - \frac{1}{\nu} \sum_{k=1}^{\nu} g(\tilde{x}_k, y_k) \right).$$

- References: [B. Sudret. Reliab. Eng. Syst. Safe., 2008], [Crestaux et al. Reliab. Eng. Syst. Safe., 2009], [I. Sobol. Math. Comput. Simulat., 2001], and [A. Owen. ACM T. Model. Comput. S., 2013].

Example: Metal forming

Example: Metal forming

Manufacturing tolerances in metal forming

Raw materials variability:
- Material properties.

. . .

Process variability:
- Blank holder force.
- Initial dimensions.
- Friction.

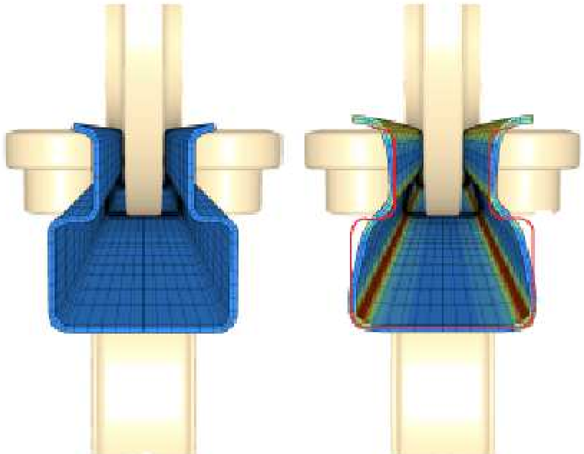
. . .

Modeling limitations:
- Constitutive model.
- FE discretization.

. . .

Input variables.

→



→

Product variability:
- Final dimensions.
- Springback.

. . .

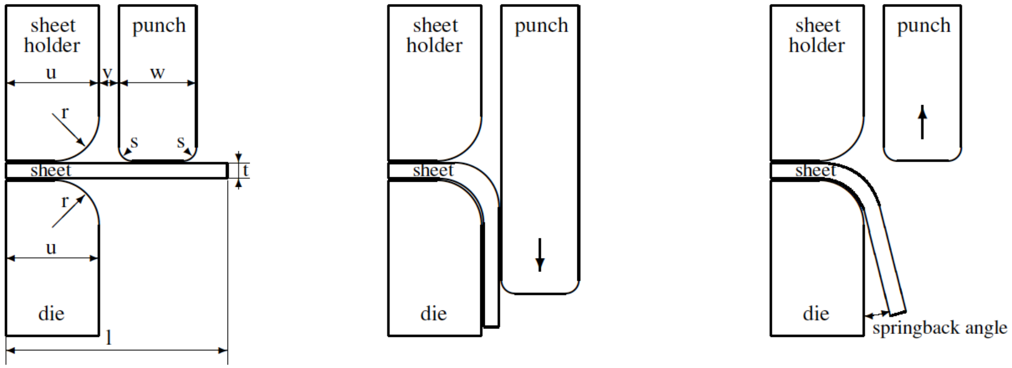
Prediction limitations:
- Numerical noise.

. . .

Output variable.

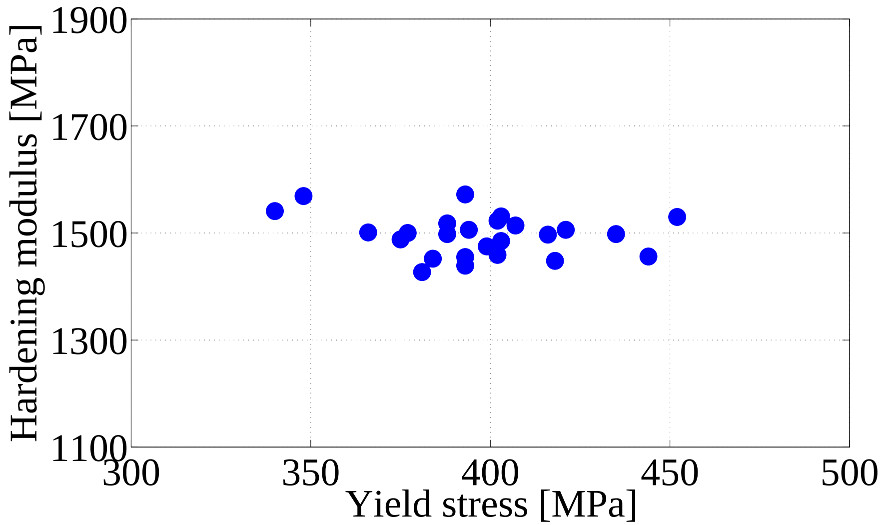
Manufacturing tolerances in metal forming (continued)

- Material properties (h, s) → Model $y = g(h, s)$ → Springback angle y



- Observed samples $(h_1^{\text{obs}}, s_1^{\text{obs}}), (h_2^{\text{obs}}, s_2^{\text{obs}}), \ldots, (h_n^{\text{obs}}, s_n^{\text{obs}})$.

h [MPa]	s [MPa]
1488	375
1485	403
1514	407
1500	377
...	...



- Mechanics and physics impose that h and s be positive.

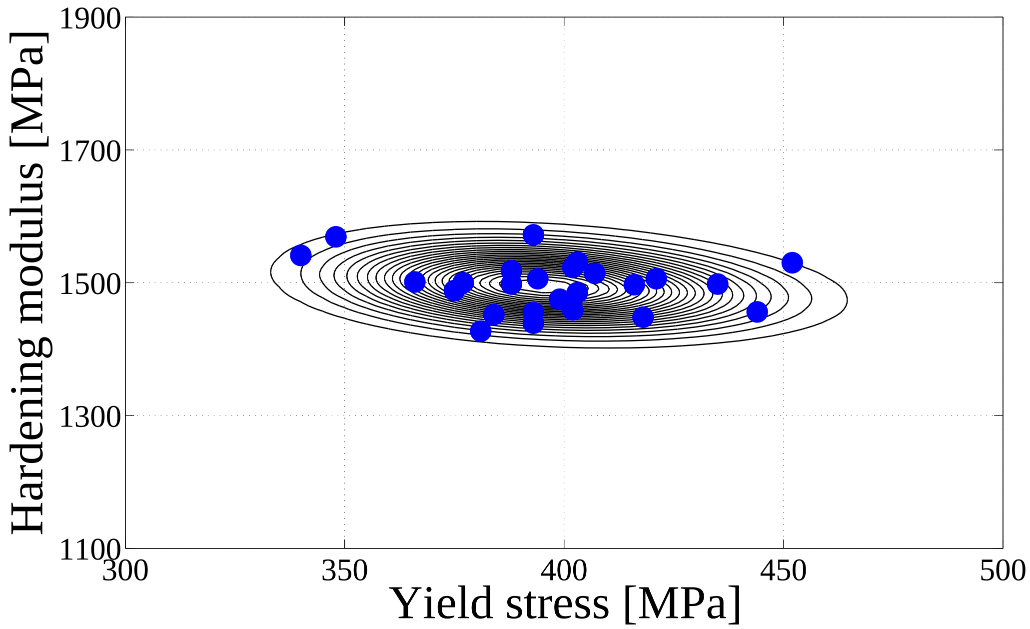
Manufacturing tolerances in metal forming (continued)

- We estimate adequate values for the parameters of the bivariate gamma probability distribution by using the method of maximum likelihood as follows:

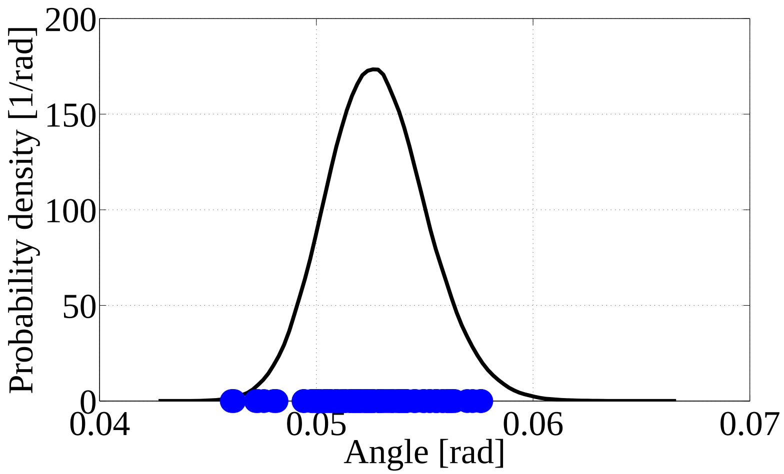
$$(\hat{\overline{h}}, \hat{\sigma}_H^2, \hat{\overline{s}}, \hat{\sigma}_S^2, \hat{\rho}) = \text{solution of} \max_{(\overline{h}, \sigma_H^2, \overline{s}, \sigma_S^2, \rho)} l(\overline{h}, \sigma_H^2, \overline{s}, \sigma_S^2, \rho),$$

where the likelihood of the parameters $\overline{h}$, σ_H^2, $\overline{s}$, σ_S^2, and ρ is given by

$$l(\overline{h}, \sigma_H^2, \overline{s}, \sigma_S^2, \rho) = \prod_{\ell=1}^{n} \rho_{(H,S)}(h_\ell^{\text{obs}}, s_\ell^{\text{obs}}; \overline{h}, \sigma_H^2, \overline{s}, \sigma_S^2, \rho).$$



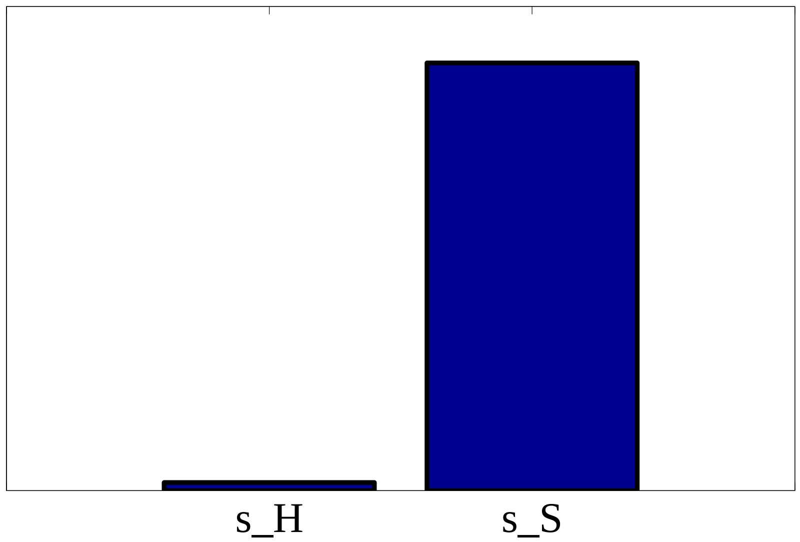
Manufacturing tolerances in metal forming (continued)



• solid: PDF of output.

$[0.046\,\text{rad},\ 0.059\,\text{rad}]$.
95%-confidence interval.

Propagation.



s_H, s_S represent significance of inputs in inducing uncertainties in output.

Sensitivity analysis.

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