Constant 2-labelling of weighted cycles

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Extended Abstract

We introduce the concept of constant 2-labelling of a weighted graph. Roughly speaking, a *constant 2-labelling* of a weighted graph is a 2-coloring $\{\bullet, \circ\}$ of its vertex set which preserves the sum of the weight of black vertices under some automorphisms.

The motivation about introducing such labellings comes from covering problems in graphs. These lattest are coverings with balls of constant radius satisfying special multiplicity condition. Let $G = (V, E)$ be a graph and r, a, b be positive integers. A set $S \subseteq V$ of vertices is an (r, a, b)-*code* if every element of S belongs to exactly a balls of radius r centered at elements of S and every element of $V \setminus S$ belongs to exactly b balls of radius r centered at elements of S. Such codes are also known as (r, a, b)-covering codes [1], (r, a, b)-isotropic colorings [1] or as perfect colorings [5]. For (r, a, b)-codes of the infinite grid with $|a - b| > 4$ and $r \geq 2$, constant 2-labellings help us to give all possible values of a and b.

The notion of (r, a, b)-codes generalizes the notion of domination and perfect codes in graphs. An r-perfect code in a graph is nothing less than an $(r, 1, 1)$-code. Perfect codes were introduced in terms of graphs by Biggs in [2]. It was shown by Kratochvil [4] that the problem of finding an r-perfect code in graphs (*i.e.*, an $(r, 1, 1)$-code) is NP-complete. Moreover, this problem is even NP-complete in the case of bipartite graphs with maximum degree three. For more information about perfect codes, see [3, Chapter 11].

Constant 2-labellings. Given a graph $G = (V, E)$, a vertex v of G, a map $w : V \to \mathbb{R}$ and a subset A of the set $Aut(G)$ of all automorphisms of G, a *constant 2-labelling* of G is a mapping $\varphi : V \to \{\bullet, \circ\}$ such that

$$\sum_{\{u \in V \mid \varphi \circ \xi(u) = \bullet\}} w(u) \text{ is constant for all } \xi \in A_\bullet \text{ (respectively } \xi \in A_\circ)$$

where $A_\bullet = \{\xi \in A \mid \varphi \circ \xi(v) = \bullet\}$ (resp. $A_\circ = \{\xi \in A \mid \varphi \circ \xi(v) = \circ\}$). Observe that any coloring using only one color is a constant 2-labelling. Such constant 2-labellings are called *trivial.* Moreover, if φ is a constant 2-labelling of a graph G, then the coloring obtained by exchanging colors is also a constant 2-labelling of G.

We look at weighted cycles with p vertices denoted by $\mathcal{C}_p$. These vertices $0, \ldots, p-1$ have respectively weights $w(0), \ldots, w(p-1)$. We will represent such a cycle by the word $w(0) \ldots w(p-1)$. Let $\mathcal{R}_k$ denote a k-rotation of $\mathcal{C}_p$, i.e.,

$$\mathcal{R}_k : \{0, \ldots, p-1\} \to \{0, \ldots, p-1\} : i \mapsto i + k \bmod p.$$

We set $A = \{\mathcal{R}_k \mid k \in \mathbb{Z}\}$ and $v = 0$. A coloring $\varphi : \{0, \ldots, p-1\} \to \{\bullet, \circ\}$ of a cycle $\mathcal{C}_p$ is a constant 2-labelling if, for every k-rotation of the coloring, the weighted sum of black vertices is a constant a (resp. b) whenever the vertex 0 is black (resp. white).

We consider eight particular weighted cycles $\mathcal{C}_p$ with at most 4 different weights, namely z, x, y and t. The following words represent respectively cycles of Type 1–8 (see Figure 1) :

$$zx^{p-1},\ zx^{\frac{p-2}{2}}tx^{\frac{p-2}{2}},\ z(xy)^{\frac{p-1}{2}},\ z(xy)^{\frac{p-2}{2}}x,\ z(xy)^{\frac{p-1}{4}}(yx)^{\frac{p-1}{4}},$$
$$z(xy)^{\frac{p-3}{4}}xx(yx)^{\frac{p-3}{4}},\ z(xy)^{\frac{p-2}{4}}t(yx)^{\frac{p-2}{4}},\ z(xy)^{\frac{p-4}{4}}xtx(yx)^{\frac{p-4}{4}}$$

with $x \neq y$ and $p \geq 2$. Note that the exponents appearing in the representation of cycles must be integers. This implies extra conditions on p depending on the type of $\mathcal{C}_p$. We can give a characterization of all constant 2-labellings of these weighted cycles. For example, if we set $a = \sum_{\{u \in V \mid \varphi \circ \xi(u) = \bullet\}} w(u)$ and $b = \sum_{\{u \in V \mid \varphi \circ \xi'(u) = \bullet\}} w(u)$ for $\xi \in A_\bullet, \xi' \in A_\circ$, then we have the following result for Type 7 cycles.

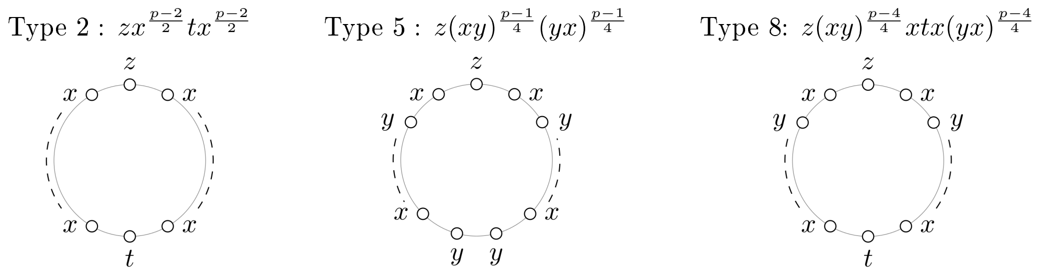

Figure 1: Types of weighted cycles $\mathcal{C}_p$.

Lemma 1 *For cycles $\mathcal{C}_p$ of Type 7, i.e., $z(xy)^{\frac{p-2}{4}}t(yx)^{\frac{p-2}{4}}$ with $t \neq x \neq y$ and $2 < p \in \mathbb{N}$, if φ is a non trivial constant 2-labelling, then φ is either alternate with $a = (\frac{p}{2}-1)y + z$ and $b = (\frac{p}{2}-1)x + t$ or $\frac{p}{2}$-periodic with $a = \alpha(x+y) + t + z$ and $b = (\alpha+1)(x+y)$ for $\alpha \in \{0, \dots, \frac{p}{2}-1\}$.*

Application to (r, a, b)-codes. We will focus on the graph of the infinite grid $\mathbb{Z}^2$. we consider balls defined relative to the Manhattan metric. We can view an (r, a, b)-code of $\mathbb{Z}^2$ as a particular coloring φ with two colors black and white where the black vertices are the elements of the code. In other words, the coloring φ is such that a ball of radius r centered on a black (respectively white) vertex contains exactly a (resp. b) black vertices.

For 2-colorings of the infinite grid satisfying specific periodicity properties, we will present a *projection and folding method* that associates a weighted cycle to a ball of radius r in $\mathbb{Z}^2$. For $r \geq 2$, Puzynina [5] showed that every (r, a, b)-codes of $\mathbb{Z}^2$ are periodic. Moreover Axenovich [1] gave a characterization of all (r, a, b)-codes of $\mathbb{Z}^2$ with $r \geq 2$ and $|a-b| > 4$. Using these results, we can apply our method to any (r, a, b)-codes with $r \geq 2$ and $|a-b| > 4$. The weighted cycles obtained by this procedure are of Type 1–8.

So, for $r \geq 2$ and $|a-b| > 4$, there exists an (r, a, b)-code of $\mathbb{Z}^2$ if and only if there exists a constant 2-labelling of some cycle $\mathcal{C}_p$, with $v = 0$, $A = \{\mathcal{R}_k \mid k \in \mathbb{Z}\}$ and a mapping w defined as before, such that

$$a = \sum_{\{u \in V \mid \varphi\circ\xi(u) = \bullet\}} w(u) \text{ and } b = \sum_{\{u \in V \mid \varphi\circ\xi' = \bullet\}} w(u) \quad \forall \xi \in A_\bullet,\ \xi' \in A_\circ.$$

Hence, the concept of constant 2-labellings allows us to obtain a new characterization of (r, a, b)-codes of $\mathbb{Z}^2$ with $r \geq 2$ and $|a-b| > 4$. Moreover, we can give all the possible values of constants a and b.

References

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