Negative parity baryons in the $1/N_c$ expansion: the quark excitation versus the meson-nucleon resonance picture

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Abstract

In order to better understand the fundamental issue regarding the compatibility between the quark-shell picture and that of resonances in meson-nucleon scattering in large N_c QCD we extend the work of Cohen and Lebed on mixed symmetric $\ell = 1$ baryons to analyze excited states with $\ell = 3$. We give an explicit proof on the degeneracy of mass eigenvalues of a simple Hamiltonian including operators up to order $\mathcal{O}(N_c^0)$ *i.e.* neglecting $1/N_c$ corrections in the quark-shell picture in the large N_c limit. We obtain three sets of degenerate states with $\ell = 3$, as in the case of $\ell = 1$ baryons. The compatibility between this picture and that of resonances in meson-nucleon scattering is discussed in the light of the present results.

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I. INTRODUCTION

The usefulness of the $1/N_c$ expansion method [1–4] to describe ground state baryons has been clearly demonstrated [5–7]. It is based on the contracted spin-flavor symmetry SU($2N_f$) that emerges in large N_c QCD. For excited baryons the problem is more involved and needs more investigations.

There are two complementary pictures of large N_c for the baryon resonances. Using the terminology of Refs. [8–10] these are:

(i) the SU($2N_f$) $\times$ O(3) called *the quark-shell picture* where the role of O(3) is essentially to include orbital excitations. This picture allows to classify baryons in excitation bands N, like in the quark model [11, 12]. For $N_c = 3$ each band contains a number of SU(6) $\times$ O(3) multiplets. A practical and commonly used procedure is to consider an excited state as a single quark excitation about a spin-flavor symmetric core [13, 14]. The N_c counting is implemented by introducing operators that break the SU($2N_f$) symmetry in powers of $1/N_c$. The coefficients of these operators encode the quark dynamics and are fitted from experiment.

(ii) the *resonance or scattering picture* derived from symmetry features shared by various chiral soliton models. The role of large N_c QCD is to relate the scattering amplitudes in various channels with K-amplitudes, where K is the grand spin $\vec{K} = \vec{I} + \vec{J}$. These are linear relations in the meson-nucleon scattering amplitudes from which one can infer some patterns of degeneracy among resonances.

For the nonstrange lowest negative parity baryons belonging to the $[70, 1^-]$ multiplet (the $N = 1$ band in quark model terms) Cohen and Lebed have shown [8] that the two pictures share the same pattern of degeneracy from which they concluded that the two pictures are generically compatible. Simultaneously with Cohen and Lebed, Pirjol and Schat [9] found the same sets of degenerate states, corresponding to irreducible representations of the contracted SU$(4)_c$ symmetry and the three degenerate multiplets obtained by them were called three towers of states. Moreover, to the three leading-order operators in the mass formula they added $1/N_c$ corrections and reanalyzed the mass spectrum of the lowest negative parity nonstrange baryons. They found ambiguities in the identification of physical states with $N_c = 3$ with the degenerate large N_c tower states.

One should also mention that in the SU(4) case, prior to Refs. [8, 9], the degeneracy of multiplets corresponding to irreducible representations of the contracted $\mathrm{SU}(4)_c$ symmetry was first discussed by Pirjol and Yan in Ref. [15].

Later on, the compatibility between the two pictures was discussed on a general basis again by Cohen and Lebed [10]. By full compatibility it was understood that any complete spin-flavor multiplet within one picture fills the quantum numbers in the other picture. The analysis involved group theoretical arguments and the nature of quark excitations in a hedgehog picture. The compatibility was generally claimed for completely symmetric, mixed symmetric and completely antisymmetric states of N_c quarks having angular momentum up to $\ell = 3$. However an explicit proof regarding the degeneracy of mass eigenvalues in the quark-shell picture is known only for $\ell = 1$ [8, 9]. For symmetric states with $\ell = 0$ and 2 it is inferred from previous studies [5–7] and [16] respectively. Similar arguments hold for symmetric $\ell = 4$ states as well [17].

The aim of the present work is to give an explicit analytical proof of the degeneracy of mass eigenvalues in the quark-shell picture for $\ell = 3$ and present its pattern of degeneracy as compared to that of the meson-nucleon scattering picture. For this purpose we use the Hamiltonian of Ref. [8] and calculate all the possible eigenvalues. We find a pattern of degeneracy which is compared to that given in Ref. [10] from general arguments and discuss the compatibility of the two pictures explicitly.

II. THE QUARK-SHELL PICTURE

In the quark-shell picture the authors of Ref. [8] start from the leading-order Hamiltonian including operators up to order $\mathcal{O}(N_c^0)$ which has the following form [14]

$$H = c_1 \mathbb{1} + c_2 \ell \cdot s + c_3 \frac{1}{N_c} \ell^{(2)} \cdot g \cdot G_c \tag{1}$$

This operator is defined in the spirit of a Hartree picture (mean field) [2] where the matrix elements of the first term are N_c on all baryons, and the spin-orbit term $\ell \cdot s$ which is a one-body operator and the third term - a two-body operator containing the tensor $\ell^{(2)ij}$ of O(3) - have matrix elements of order $\mathcal{O}(N_c^0)$. The neglect of $1/N_c$ corrections in the $1/N_c$ expansion makes sense for the comparison with the scattering picture in the large N_c limit, as described in the following section.

We remind that the SU(4) generators are S^i, T^a and G^{ia} and ℓ^i are the O(3) generators which form the tensor operator $\ell^{(2)ij} = 1/2\ \{\ell^i, \ell^j\} - 1/3\ \ell^2\ \delta_{i,-j}$. In the manner of Ref. [14] they are decomposed into two parts, one acting on the excited quark and the other on the ground state core. Thus ℓ^i, s^i, t^a and g^{ia} act on the excited quark and S_c^i, T_c^a and G_c^{ia} act on the ground state core. Starting from the Hamiltonian (1) the authors of Ref. [8] show that the masses of the $[70, 1^-]$ multiplet described by this model to leading order in the $1/N_c$ expansion fall into three degenerate multiplets given by three distinct masses denoted by m_0, m_1 and m_2, which are linear in the parameters c_1, c_2 and c_3 . Their expression are

$$m_0 = c_1 N_c - (c_2 + \frac{5}{24} c_3), \quad m_1 = c_1 N_c - \frac{1}{2}(c_2 - \frac{5}{24} c_3), \quad m_2 = c_1 N_c + \frac{1}{2}(c_2 - \frac{1}{24} c_3). \quad (2)$$

From its form one can see that the Hamiltonian (1) incorporates the property that the characteristic N_c scaling for the excitation energy of baryons is N_c^0 [2].

The spectrum obtained from the Hamiltonian (1) is identical to that derived by Pirjol and Schat [9] for $\ell = 1$. Note that the third operator of Ref. [9] contains an extra factor of 3, which should be taken into account when comparing the eigenvalues.

Below we give the mass matrices obtained from the Hamiltonian (1) for $\ell = 3$. As we shall see, this Hamiltonian has the remarkable property that for $\ell = 3$ as well, its eigenvalues are simple linear expressions in the coefficients c_i, which makes the discussion very convenient.

A. The nucleon case

We have the following $[N_c - 1, 1]$ spin-flavor (SF) states which form a symmetric state with the orbital $\ell = 3$ state of partition $[N_c - 1, 1]$ as well

1. $[N_c - 1, 1]_{SF} = \left[\frac{N_c+1}{2}, \frac{N_c-1}{2}\right]_S \times \left[\frac{N_c+1}{2}, \frac{N_c-1}{2}\right]_F$, $N_c \geq 3$
   with $S = 1/2$ and $J = 5/2, 7/2$
2. $[N_c - 1, 1]_{SF} = \left[\frac{N_c+3}{2}, \frac{N_c-3}{2}\right]_S \times \left[\frac{N_c+1}{2}, \frac{N_c-1}{2}\right]_F$, $N_c \geq 3$
   with $S = 3/2$ and $J = 3/2, 5/2, 7/2, 9/2$.

They give rise to matrices of a given J either 2×2 or 1×1. States of symmetry $[N_c - 1, 1]_{SF}$ with $S = 5/2$, like for Δ (see below), which together with $\ell = 3$ could give rise to $J = 11/2$, are not allowed for N, by inner products of the permutation group [18]. Therefore the resonance $N_{11/2}$ should belong to the $N = 5$ band ($\ell = 5$), as suggested in the before last section. For $N_c = 3$ the above states belong to the ^{28} and ^{48} multiplets of SU(2) $\times$ SU(3) respectively.

We calculate the matrix elements using the formulas from Appendix A. The expectation value for the $\ell = 3$ $N_{3/2}$ state is

$$m^{(1)}_{N_{3/2}} = c_1 N_c - 2c_2 - \frac{3}{4}c_3 \tag{3}$$

The matrix for $N_{5/2}$ is

$$M^{\ell=3}_{N_{5/2}} = \begin{pmatrix} c_1 N_c - \frac{4}{3}c_2 & -\frac{\sqrt{5}}{3}c_2 - \frac{3\sqrt{5}}{8}c_3 \\ -\frac{\sqrt{5}}{3}c_2 - \frac{3\sqrt{5}}{8}c_3 & c_1 N_c - \frac{7}{6}c_2 + \frac{3}{16}c_3 \end{pmatrix} \tag{4}$$

Its eigenvalues are

$$m^{(1)}_{N_{5/2}} = c_1 N_c - 2c_2 - \frac{3}{4}c_3 \tag{5}$$

$$m^{(2)}_{N_{5/2}} = c_1 N_c - \frac{1}{2}c_2 + \frac{15}{16}c_3 \tag{6}$$

The matrix for $N_{7/2}$ is

$$M^{\ell=3}_{N_{7/2}} = \begin{pmatrix} c_1 N_c + c_2 & -\frac{\sqrt{3}}{2}c_2 + \frac{5\sqrt{3}}{16}c_3 \\ -\frac{\sqrt{3}}{2}c_2 + \frac{5\sqrt{3}}{16}c_3 & c_1 N_c + \frac{5}{8}c_3 \end{pmatrix} \tag{7}$$

and its eigenvalues are

$$m^{(1)}_{N_{7/2}} = c_1 N_c - \frac{1}{2}c_2 + \frac{15}{16}c_3 \tag{8}$$

$$m^{(2)}_{N_{7/2}} = c_1 N_c + \frac{3}{2}c_2 - \frac{5}{16}c_3 \tag{9}$$

The expectation value of the $N_{9/2}$ is

$$m^{(1)}_{N_{9/2}} = c_1 N_c + \frac{3}{2}c_2 - \frac{5}{16}c_3 \tag{10}$$

B. The Δ case

We have the following basis states in the spin-flavor space compatible with the orbital state $[N_c - 1, 1]$ with $\ell = 3$

1. $[N_c - 1, 1]_{SF} = \left[\frac{N_c+1}{2}, \frac{N_c-1}{2}\right]_S \times \left[\frac{N_c+3}{2}, \frac{N_c-3}{2}\right]_F$, $N_c \geq 3$
   with $S = 1/2$ and $J = 5/2, 7/2$,

2. $[N_c-1,1]_{SF} = \left[\frac{N_c+3}{2},\frac{N_c-3}{2}\right]_S \times \left[\frac{N_c+3}{2},\frac{N_c-3}{2}\right]_F$, $N_c \geq 5$
   with $S=3/2$ and $J=3/2,5/2,7/2,9/2$,

3. $[N_c-1,1]_{SF} = \left[\frac{N_c+5}{2},\frac{N_c-5}{2}\right]_S \times \left[\frac{N_c+3}{2},\frac{N_c-3}{2}\right]_F$, $N_c \geq 7$
   with $S=5/2$ and $J=1/2,3/2,5/2,7/2,9/2,11/2$.

As above, they indicate the size of a matrix of fixed J. For $N_c=3$ the first state belongs to the ${}^2 10$ multiplet. The other two types of states do not appear in the real world with $N_c=3$. Note that both for N_J and Δ_J states the size of a given matrix equals the multiplicity of the corresponding state indicated in Table 1 of Ref. [10] for $\ell=3$.

The expectation value for the $\Delta_{1/2}$ state is

$$m^{(1)}_{\Delta_{1/2}} = c_1 N_c - 2c_2 - \frac{3}{4}c_3 \tag{11}$$

The matrix for $\Delta_{3/2}$ is

$$M^{\ell=3}_{\Delta_{3/2}} = \begin{pmatrix} c_1 N_c - \frac{4}{5}c_2 + \frac{3}{5}c_3 & -\frac{3}{5}c_2 - \frac{27}{40}c_3 \\ -\frac{3}{5}c_2 - \frac{27}{40}c_3 & c_1 N_c - \frac{17}{10}c_2 - \frac{33}{80}c_3 \end{pmatrix} \tag{12}$$

The eigenvalues of this matrix are

$$m^{(1)}_{\Delta_{3/2}} = c_1 N_c - 2c_2 - \frac{3}{4}c_3 \tag{13}$$

$$m^{(2)}_{\Delta_{3/2}} = c_1 N_c - \frac{1}{2}c_2 + \frac{15}{16}c_3 \tag{14}$$

For $\Delta_{5/2}$ we have

$$M^{\ell=3}_{\Delta_{5/2}} = \begin{pmatrix} c_1 N_c + \frac{2}{3}c_2 & \sqrt{\frac{1}{2}}\,(\frac{5}{3}c_2 - \frac{3}{8}c_3) & \frac{3}{4}\sqrt{\frac{1}{2}}c_3 \\ \sqrt{\frac{15}{2}}(\frac{1}{2}c_2 + \frac{1}{16}c_3) & c_1 N_c - \frac{7}{15}c_2 - \frac{3}{20}c_3 & -\frac{3}{2\sqrt{2}}c_2 - \frac{3}{16\sqrt{2}}c_3 \\ \frac{3}{4}\sqrt{\frac{1}{2}}c_3 & -\frac{3}{2\sqrt{2}}c_2 - \frac{3}{16\sqrt{2}}c_3 & c_1 N_c - \frac{6}{5}c_2 + \frac{1}{40}c_3 \end{pmatrix} \tag{15}$$

The eigenvalues of this matrix are

$$m^{(1)}_{\Delta_{5/2}} = c_1 N_c - 2c_2 - \frac{3}{4}c_3 \tag{16}$$

$$m^{(2)}_{\Delta_{5/2}} = c_1 N_c - \frac{1}{2}c_2 + \frac{15}{16}c_3 \tag{17}$$

$$m^{(3)}_{\Delta_{5/2}} = c_1 N_c + \frac{3}{2} c_2 - \frac{5}{16} c_3 \tag{18}$$

For $\Delta_{7/2}$ we obtain

$$M^{\ell=3}_{\Delta_{7/2}} = \begin{pmatrix} c_1 N_c - \frac{1}{2} c_2 & \sqrt{\frac{15}{2}}(\frac{1}{2} c_2 + \frac{1}{16} c_3) & \frac{3\sqrt{15}}{16} c_3 \\ \sqrt{\frac{15}{2}}(\frac{1}{2} c_2 + \frac{1}{16} c_3) & c_1 N_c - \frac{1}{2} c_3 & -\frac{3}{2\sqrt{2}} c_2 - \frac{3}{16\sqrt{2}} c_3 \\ \frac{3\sqrt{15}}{16} c_3 & -\frac{3}{2\sqrt{2}} c_2 - \frac{3}{16\sqrt{2}} c_3 & c_1 N_c - \frac{1}{2} c_2 + \frac{3}{8} c_3 \end{pmatrix} \tag{19}$$

and the eigenvalues of this matrix are

$$m^{(1)}_{\Delta_{7/2}} = c_1 N_c - 2 c_2 - \frac{3}{4} c_3 \tag{20}$$

$$m^{(2)}_{\Delta_{7/2}} = c_1 N_c - \frac{1}{2} c_2 + \frac{15}{16} c_3 \tag{21}$$

$$m^{(3)}_{\Delta_{7/2}} = c_1 N_c + \frac{3}{2} c_2 - \frac{5}{16} c_3 \tag{22}$$

For $\Delta_{9/2}$ we obtain

$$M^{\ell=3}_{\Delta_{9/2}} = \begin{pmatrix} c_1 N_c + \frac{3}{5} c_2 + \frac{1}{4} c_3 & -\frac{3\sqrt{11}}{10} c_2 + \frac{3\sqrt{11}}{16} c_3 \\ -\frac{3\sqrt{11}}{10} c_2 + \frac{3\sqrt{11}}{16} c_3 & c_1 N_c - \frac{1}{2} c_2 - \frac{15}{16} c_3 \end{pmatrix} \tag{23}$$

with eigenvalues

$$m^{(1)}_{\Delta_{9/2}} = c_1 N_c - \frac{1}{2} c_2 + \frac{15}{16} c_3 \tag{24}$$

$$m^{(2)}_{\Delta_{9/2}} = c_1 N_c + \frac{3}{2} c_2 - \frac{5}{16} c_3 \tag{25}$$

Finally, the expectation value of the $\Delta_{11/2}$ one component state is

$$m^{(1)}_{\Delta_{11/2}} = c_1 N_c + \frac{3}{2} c_2 - \frac{5}{16} c_3 \tag{26}$$

It is remarkable that the 18 available eigenstates with $\ell = 3$ fall into three degenerate multiplets, like for $\ell = 1$. If the degenerate masses are denoted by m'_2, m_3 and m_4 we have

$$m'_2 = m^{(1)}_{\Delta_{1/2}} = m^{(1)}_{N_{3/2}} = m^{(1)}_{\Delta_{3/2}} = m^{(1)}_{N_{5/2}} = m^{(1)}_{\Delta_{5/2}} = m^{(1)}_{\Delta_{7/2}}, \tag{27}$$

$$m_3 = m^{(2)}_{\Delta_{3/2}} = m^{(2)}_{N_{5/2}} = m^{(2)}_{\Delta_{5/2}} = m^{(1)}_{N_{7/2}} = m^{(2)}_{\Delta_{7/2}} = m^{(1)}_{\Delta_{9/2}}, \tag{28}$$

$$m_4 = m^{(3)}_{\Delta_{5/2}} = m^{(2)}_{N_{7/2}} = m^{(3)}_{\Delta_{7/2}} = m^{(1)}_{N_{9/2}} = m^{(2)}_{\Delta_{9/2}} = m^{(1)}_{\Delta_{11/2}}. \tag{29}$$

The masses (27)-(29) are indicated in Column 2 of Tables I-III for comparison with results obtained below from the resonance picture amplitudes. Here, the notation m_K (or m'_K) is used for the calculated masses while in Ref. [10] the m_K associated with $\ell = 3$ are generic names related to poles in the reduced amplitudes. One can notice that m_2 found in Ref. [8] for $\ell = 1$, as reproduced in Eq. (2), is different from m'_2 obtained here for $\ell = 3$. In addition to distinct analytical forms for m_2 and m'_2 the coefficients c_i entering these expressions are expected to depend on the band [11]. A more extensive discussion is given at the end of the next section.

III. THE MESON-NUCLEON SCATTERING PICTURE

Here we are concerned with the SU(4) case, as above, and we look for the degeneracy pattern in the resonance picture. Following Refs. [8, 10] the starting point in this analysis are the linear relations of the S matrices $S^{\pi}_{LL'RR'IJ}$ and S^{η}_{LRJ} of π and η scattering off a ground state baryon in terms of K-amplitudes. They are given by the following equations

$$S^{\pi}_{LL'RR'IJ} = \sum_K (-1)^{R'-R} \sqrt{(2R+1)(2R'+1)}(2K+1) \begin{Bmatrix} K & I & J \\ R' & L' & 1 \end{Bmatrix} \begin{Bmatrix} K & I & J \\ R & L & 1 \end{Bmatrix} s^{\pi}_{KLL'} \quad (30)$$

and

$$S^{\eta}_{LRJ} = \sum_K \delta_{KL} \delta(LRJ) s^{\eta}_K \quad (31)$$

in terms of the reduced amplitudes $s^{\pi}_{KL'L}$ and s^{η}_K respectively.

The notation is as follows. For π scattering R and R' are the spin of the incoming and outgoing baryons respectively (R =1/2 for N and $R = 3/2$ for Δ), L and L' are the partial wave angular momentum of the incident and final π respectively (the orbital angular momentum L of η remains unchanged), I and J represent the total isospin and total angular momentum associated with a given resonance (see column 1 of Tables I-III) and K is the magnitude of *grand spin* $\vec{K} = \vec{I} + \vec{J}$. The $6j$ coefficients imply four triangle rules $\delta(LRJ)$, $\delta(R1I)$, $\delta(L1K)$ and $\delta(IJK)$.

These equations were first derived in the context of the chiral soliton model [19–22] where the mean-field breaks the rotational and isospin symmetries, so that J and I are not conserved but the *grand spin* K is conserved and excitations can be labeled by K. These relations are exact in large N_c QCD and are independent of any model assumption.

The meaning of Eq. (30) is that there are more amplitudes $S^{\pi}_{LL'RR'IJ}$ than there are $s^{\pi}_{KLL'}$ amplitudes. The reason is that the IJ as well as the RR' dependence is contained only in the geometrical factor containing the two $6j$ coefficients. Then, for example, in the πN scattering, in order for a resonance to occur in one channel there must be a resonance in at least one of the contributing amplitudes $s^{\pi}_{KLL'}$. But as $s^{\pi}_{KLL'}$ contributes in more than one channel, all these channels resonate at the same energy and this implies degeneracy in the excited spectrum. From the chiral soliton model there is no reason to suspect degeneracy between different K sectors.

From the meson-baryon scattering relations (30) and (31) the following degenerate negative parity multiplets have been found for $\ell = 1$ orbital excitations [8]

$$N_{1/2},\ \Delta_{1/2}, \quad (s^{\eta}_0) \tag{32}$$

$$N_{1/2},\ \Delta_{1/2},\ N_{3/2},\ \Delta_{3/2},\ \Delta_{5/2}, \quad (s^{\pi}_{100}, s^{\pi}_{122}) \tag{33}$$

$$\Delta_{1/2},\ N_{3/2},\ \Delta_{3/2},\ N_{5/2},\ \Delta_{5/2},\ \Delta_{7/2}, \quad (s^{\pi}_{222}, s^{\eta}_2). \tag{34}$$

One can see a clear correspondence between the first three degenerate multiplets of Eqs. (32), (33) and (34) and the three towers of states [8, 9] of the excited quark picture provided by the symmetric core + excited quark scheme [14]. They correspond to $K = 0, 1$ and 2 in the resonance picture. But the resonance picture also provides a $K = 3$ due to the amplitude s^{π}_{322}. As this is different from the other $s^{\pi}_{KL'L}$, in Ref. [8] it was interpreted as belonging to the $N = 3$ band.

Here we extend the work of Ref. [8, 9] to $\ell = 3$ excited states which belong to the $N = 3$ band. In Tables I-III we list the partial wave amplitudes of interest and their expansion in terms of K-amplitudes from Eqs. (30) and (31). They correspond to $L = L' = 2$, $L = L' = 4$ and $L = L' = 6$ respectively. Note that the squared sum of the coefficients of every elastic amplitudes πNN or $\pi\Delta\Delta$ is equal to one. This is due to the sum rule of $6j$ coefficients

$$\sum_K (2R+1)(2K+1) \begin{Bmatrix} K & I & J \\ R' & L & 1 \end{Bmatrix} \begin{Bmatrix} K & I & J \\ R & L & 1 \end{Bmatrix} = \delta(R'R), \tag{35}$$

which can be used for a check. The same relation can be used to check that the coefficients of the $\pi N\Delta$ amplitudes sum up to zero.

From the last column of Tables I-III one can infer the following degenerate towers of states with their contributing amplitudes

$$\Delta_{1/2}, \quad N_{3/2}, \quad \Delta_{3/2}, \quad N_{5/2}, \quad \Delta_{5/2}, \quad \Delta_{7/2}, \qquad (s^{\pi}_{222}, s^{\eta}_2), \tag{36}$$

TABLE I: Partial wave amplitudes and their expansions in terms of K-amplitudes from Eqs. (30) and (31). The superscripts πNN, $\pi N\Delta$, $\pi\Delta\Delta$, ηNN, and $\eta\Delta\Delta$ refer to the scattered meson and the initial and final baryons, respectively. We list amplitudes consistent with a single quark excited to $\ell = 3$ and partial waves having $L = L' = 2$.

State	Quark-shell mass	Partial wave and K-amplitudes
$\Delta_{1/2}$	m_2'	$D_{31}^{\pi\Delta\Delta} = \frac{1}{10}\left(s_{122}^{\pi} + 9s_{222}^{\pi}\right)$
		$D_{31}^{\eta\Delta\Delta} = s_2^{\eta}$
$N_{3/2}$	m_2'	$D_{13}^{\pi NN} = \frac{1}{2}\left(s_{122}^{\pi} + s_{222}^{\pi}\right)$
		$D_{13}^{\eta NN} = s_2^{\eta}$
		$D_{13}^{\pi\Delta\Delta} = \frac{1}{2}\left(s_{122}^{\pi} + s_{222}^{\pi}\right)$
		$D_{13}^{\pi N\Delta} = \frac{1}{2}\left(s_{122}^{\pi} - s_{222}^{\pi}\right)$
$\Delta_{3/2}$	$m_2',\ m_3$	$D_{33}^{\pi NN} = \frac{1}{20}\left(s_{122}^{\pi} + 5s_{222}^{\pi} + 14s_{322}^{\pi}\right)$
		$D_{33}^{\pi\Delta\Delta} = \frac{1}{25}\left(8s_{122}^{\pi} + 10s_{222}^{\pi} + 7s_{322}^{\pi}\right)$
		$D_{33}^{\eta\Delta\Delta} = s_2^{\eta}$
		$D_{33}^{\pi N\Delta} = \frac{1}{5\sqrt{10}}\left(2s_{122}^{\pi} + 5s_{222}^{\pi} - 7s_{322}^{\pi}\right)$
$N_{5/2}$	$m_2',\ m_3$	$D_{15}^{\pi NN} = \frac{1}{9}\left(2s_{222}^{\pi} + 7s_{322}^{\pi}\right)$
		$D_{15}^{\eta NN} = s_2^{\eta}$
		$D_{15}^{\pi\Delta\Delta} = \frac{1}{9}\left(7s_{222}^{\pi} + 2s_{322}^{\pi}\right)$
		$D_{15}^{\pi N\Delta} = \frac{\sqrt{14}}{9}\left(s_{222}^{\pi} - s_{322}^{\pi}\right)$
$\Delta_{5/2}$	$m_2',\ m_3$	$D_{35}^{\pi NN} = \frac{1}{90}\left(27s_{122}^{\pi} + 35s_{222}^{\pi} + 28s_{322}^{\pi}\right)$
		$D_{35}^{\pi\Delta\Delta} = \frac{1}{450}\left(189s_{122}^{\pi} + 5s_{222}^{\pi} + 256s_{322}^{\pi}\right)$
		$D_{35}^{\eta\Delta\Delta} = s_2^{\eta}$
		$D_{35}^{\pi N\Delta} = \frac{1}{90}\sqrt{\frac{7}{5}}\left(27s_{122}^{\pi} + 5s_{222}^{\pi} - 32s_{322}^{\pi}\right)$
$\Delta_{7/2}$	$m_2',\ m_3$	$D_{37}^{\pi\Delta\Delta} = \frac{1}{5}\left(2s_{222}^{\pi} + 3s_{322}^{\pi}\right)$
		$D_{37}^{\eta\Delta\Delta} = s_2^{\eta}$

TABLE II: Same as Table I but for partial waves $L = L' = 4$.

State	Quark-shell mass	Partial wave and K-amplitudes
$N_{5/2}$	m_3	$G_{15}^{\pi\Delta\Delta} = s_{344}^{\pi}$
$\Delta_{5/2}$	$m_3,\ m_4$	$G_{35}^{\pi\Delta\Delta} = \frac{1}{4}\,(s_{344}^{\pi} + 3s_{444}^{\pi})$
		$G_{35}^{\eta\Delta\Delta} = s_{4}^{\eta}$
$N_{7/2}$	$m_3,\ m_4$	$G_{17}^{\pi NN} = \frac{1}{12}\,(7s_{344}^{\pi} + 5s_{444}^{\pi})$
		$G_{17}^{\eta NN} = s_{4}^{\eta}$
		$G_{17}^{\pi\Delta\Delta} = \frac{1}{12}\,(5s_{344}^{\pi} + 7s_{444}^{\pi})$
		$G_{17}^{\pi N\Delta} = \frac{\sqrt{35}}{12}\,(s_{344}^{\pi} - s_{444}^{\pi})$
$\Delta_{7/2}$	$m_3,\ m_4$	$G_{37}^{\pi NN} = \frac{1}{72}\,(7s_{344}^{\pi} + 21s_{444}^{\pi} + 44s_{544}^{\pi})$
		$G_{37}^{\pi\Delta\Delta} = \frac{1}{225}\,(100s_{344}^{\pi} + 48s_{444}^{\pi} + 77s_{544}^{\pi})$
		$G_{37}^{\eta\Delta\Delta} = s_{4}^{\eta}$
		$G_{37}^{\pi N\Delta} = \frac{\sqrt{14}}{90}\,(5s_{344}^{\pi} + 6s_{444}^{\pi} - 11s_{544}^{\pi})$
$N_{9/2}$	m_4	$G_{19}^{\pi NN} = \frac{1}{15}\,(4s_{444}^{\pi} + 11s_{544}^{\pi})$
		$G_{19}^{\eta NN} = s_{4}^{\eta}$
		$G_{19}^{\pi\Delta\Delta} = \frac{1}{15}\,(11s_{444}^{\pi} + 4s_{544}^{\pi})$
		$G_{19}^{\pi N\Delta} = \frac{2\sqrt{11}}{15}\,(s_{444}^{\pi} - s_{544}^{\pi})$
$\Delta_{9/2}$	$m_3,\ m_4$	$G_{39}^{\pi NN} = \frac{1}{90}\,(35s_{344}^{\pi} + 33s_{444}^{\pi} + 22s_{544}^{\pi})$
		$G_{39}^{\pi\Delta\Delta} = \frac{1}{900}\,(385s_{344}^{\pi} + 3s_{444}^{\pi} + 512s_{544}^{\pi})$
		$G_{39}^{\eta\Delta\Delta} = s_{4}^{\eta}$
		$G_{39}^{\pi N\Delta} = \frac{1}{90}\sqrt{\frac{11}{10}}\,(35s_{344}^{\pi} - 3s_{444}^{\pi} - 32s_{544}^{\pi})$
$\Delta_{11/2}$	m_4	$G_{3,11}^{\pi\Delta\Delta} = \frac{1}{25}\,(12s_{444}^{\pi} + 13s_{544}^{\pi})$
		$G_{3,11}^{\eta\Delta\Delta} = s_{4}^{\eta}$

TABLE III: Same as Table I but for partial waves $L = L' = 6$.

State	Quark-shell mass	Partial wave and K-amplitudes
$N_{9/2}$	m_4	$I_{19}^{\pi\Delta\Delta} = s_{566}^{\pi}$
$\Delta_{9/2}$	$m_3,\ m_4$	$I_{39}^{\pi\Delta\Delta} = \frac{1}{10}(3s_{566}^{\pi} + 7s_{666}^{\pi})$
		$I_{39}^{\eta\Delta\Delta} = s_6^{\eta}$
$\Delta_{11/2}$	m_4	$I_{1,11}^{\pi NN} = \frac{1}{468}(55s_{566}^{\pi} + 143s_{666}^{\pi} + 270s_{766}^{\pi})$
		$I_{1,11}^{\pi\Delta\Delta} = \frac{1}{819}(392s_{566}^{\pi} + 130s_{666}^{\pi} + 297s_{766}^{\pi})$
		$I_{1,11}^{\eta\Delta\Delta} = s_6^{\eta}$
		$I_{1,11}^{\pi N\Delta} = \frac{\sqrt{55}}{117\sqrt{14}}(14s_{566}^{\pi} + 13s_{666}^{\pi} - 27s_{766}^{\pi})$

$$\Delta_{3/2}, \quad N_{5/2}, \quad \Delta_{5/2}, \quad N_{7/2}, \quad \Delta_{7/2}, \quad \Delta_{9/2}, \quad (s_{322}^{\pi}, s_{344}^{\pi}), \tag{37}$$

$$\Delta_{5/2}, \quad N_{7/2}, \quad \Delta_{7/2}, \quad N_{9/2}, \quad \Delta_{9/2}, \quad \Delta_{11/2}, \quad (s_{444}^{\pi}, s_4^{\eta}), \tag{38}$$

$$\Delta_{7/2}, \quad N_{9/2}, \quad \Delta_{9/2}, \quad \Delta_{11/2}, \quad (s_{544}^{\pi}, s_{566}^{\pi}), \tag{39}$$

$$\Delta_{9/2}, \quad \Delta_{11/2}, \quad (s_{666}^{\pi}, s_6^{\eta}) \tag{40}$$

associated with $K = 2, 3, 4, 5$ and 6 respectively. Here one can recognize patterns of degeneracy similar to those observed in Table II of Ref. [10]. Note that m_K of column 2 of that table represents the name associated with the position of a possible pole in an amplitude with K-spin.

We can now compare the towers (36)-(40) with the quark-shell model results of (27)-(29). The first observation is that the agreement of (36) ($K = 2$) with (27), of (37) ($K = 3$) with (28) and of (38) ($K = 4$) with (29) is perfect regarding the quantum numbers. Second, we note that the resonance picture can have poles with $K = 5, 6$ which imply the towers (39) and (40). They have no counterpart in the quark-shell picture for $\ell = 3$. But there is no problem because the poles with $K = 5, 6$ can belong to a higher band, namely $N = 5$ ($\ell = 5$) without spoiling the compatibility.

A discussion is necessary for the tower (27) of the quark-shell picture associated with the degenerate mass m_2'. The expression of m_2' is entirely different from that of m_2 obtained for $\ell = 1$. This is quite natural from the algebra described in the Appendix. Moreover, in practice, the $\ell = 3$ states should lie higher than the $\ell = 1$ states (as they include more orbital excitation). In fact the analysis of Ref. [11] suggests that the coefficients c_i are expected to depend on the band. If so, some constraints could be imposed on the values of these coefficients to be found phenomenologically, after including $1/N_c$ corrections. For example, the first panel in Fig. 1 of Ref. [11] indicates a linear behavior of the coefficient c_1 as a function of the band N. From that figure one can extract the value of c_1 associated with the band N = 3. This gives

$$c_1 \approx 640 \text{ MeV} \tag{41}$$

Such a value can safely be used in a first step analysis of the N = 3 band, which would mean that there is one less parameter to fit.

Thus one can associate a common $K = 2$ to $\ell = 1$ and $\ell = 3$. For this value of K the triangular rule $\delta(K\ell 1)$ proposed in Ref [10] is satisfied. The quark-shell picture brings however more information than the resonance picture because it implies an energy dependence via the ℓ dependence which measures the orbital excitation. As m_2' is different from m_2 and in the resonance picture they stem from the same amplitude s^{π}_{222} one should expect that this amplitude possesses two poles at two distinct energies, in order to have compatibility. Thus the number of poles of the reduced amplitudes s^{π}_{KLL} remains an open question.

We anticipate that a similar situation will appear for every value of K associated with two distinct values of ℓ, satisfying the $\delta(K\ell 1)$ rule, for example, for $K = 4$ which is common to $\ell = 3$ and $\ell = 5$.

IV. THE EXPERIMENTAL SITUATION FOR RESONANCES WITH $\ell \geq 3$

Here we are essentially concerned with resonances which can be explained as orbital excitations with $\ell = 3$. Examples of experimentally known negative parity resonances of this category [23] are indicated in Table IV. They are located at about 2.2 GeV and in quark model terms they belong to the N = 3 band. For completeness we have also indicated two resonances $N(2600)I_{1,11}$ and $\Delta(2750)I_{3,13}$ which should belong to the N = 5 band, as their total angular momentum require an orbital excitation with $\ell = 5$.

The SU(6) $\times$ O(3) multiplet content of the N = 3 band is [24, 25] : $[70',1^-], [70'',1^-], [56,3^-], [20,3^-], [70,3^-], [70,2^-], [56,1^-]$ and $[20,1^-]$ where $70'$ and $70''$

TABLE IV: Examples of nonstrange negative parity resonances from Particle Data Group [23] and their possible main component expressed in terms of SU(6) multiplets.

Resonance	Multiplet	Status
$N(2190)G_{17}$	$^2N[70,3^-]$	$****$
$N(2250)G_{19}$	$^4N[70,3^-]$	$****$
$N(2600)I_{1,11}$	$^2N[70,5^-]$	$***$
$\Delta(2220)G_{37}$	$^2N[70,3^-]$	$*$
$\Delta(2400)G_{39}$	$^4\Delta[56,3^-]$	$**$
$\Delta(2750)I_{3,13}$	$^4\Delta[56,5^-]$	$**$

represent radial excitations. In column 2 we have indicated the multiplet to which the listed resonances can belong, using the notations of Ref. [25]. Both references [24] and [25] were independently concerned with the resonance D_{35} observed by Cutkosky et al. [26]. In Ref. [24] a sum rule was derived in a harmonic oscillator basis to calculate the mass of D_{35} as a pure $[56,1^-]$ state by neglecting the tensor force. In Ref. [25] the masses of negative parity nonstrange resonances were obtained in a quark model with a linear confinement and a chromomagnetic interaction (spin-spin and tensor forces) including interband mixing. The D_{35} resonance was found to be mainly a $[56,1^-]$ state.

For the resonances expected to belong to the N = 5 band the multiplet is only suggested. To our knowledge, the SU(6) × O(3) multiplet decomposition of this band is not known.

V. CONCLUSIONS

The compatibility between the quark-excitation and the meson-nucleon resonance pictures of negative parity baryons with $\ell = 3$ has been analyzed in the spirit of Refs. [8–10]. We have found patterns of degeneracy with a common resonance content in both pictures. This supports the idea of full compatibility of Ref. [10] in the sense that any complete spin-flavor multiplet within one picture fills the quantum numbers of the other picture. However the quark-shell picture is richer in information, by making a clear distinction between degenerate sets of states of different values of the angular momentum but associated with the same grand spin K.

The low-energy baryons of the $[70, 1^-]$ multiplet have been very extensively studied in large N_c QCD but the highly excited $\ell = 3$ baryons have been nearly entirely neglected so far. To our knowledge, there is only one general work including $\ell = 3$ baryons [27]. The experimental situation described in Sec. 4 encourages such analysis. The $1/N_c$ expansion method could, in principle, predict many more resonances to guide the experimentalists. This work supplies an incentive for the study of highly excited negative parity baryon in the $1/N_c$ expansion method. Including $1/N_c$ corrections in the mass formula means that, besides c_1, c_2 and c_3, more parameters are involved in the fit. As the data is presently scarce, in a first attempt, the number of parameters must remain small. A strategy would be to fix the value of c_1 in agreement with Eq. (41) and restrict the number of operators in the mass formula to the most dominant ones, as for example, the spin and isospin operators described in Ref. [28] for $\ell = 1$. This would involve at most four parameters to fit.

Appendix A

In our notation [28] the total wave function of the symmetric core+excited quark procedure [14] takes the following form

$$|\ell S J J_3; I I_3\rangle_{p=2} = \sum_{m_\ell, S_3} \left(\begin{array}{cc|c} \ell & S & J \\ m_\ell & S_3 & J_3 \end{array} \right) |\ell m\rangle |[N_c-1,1]p=2; S S_3; I I_3\rangle. \qquad \text{(A1)}$$

It contains a Clebsch-Gordan coefficient, an orbital part $|\ell m\rangle$ and a spin-flavor part $|[N_c - 1, 1]p = 2; SS_3; II_3\rangle$ depending on an index p which takes the value 2, which signifies that the excited quark is in the second row of the Young diagram of partition $[N_c - 1, 1]$ in the flavor-spin space. The expression of the spin-flavor part is

$$|[N_c-1,1]p=2; SS_3; II_3\rangle = \sum_{p_1 p_2} K([f_1]p_1[f_2]p_2|[N_c-1,1]p=2)|SS_3;p_1\rangle|II_3;p_2\rangle, \quad \text{(A2)}$$

in terms of the spin part

$$|SS_3; p_1\rangle = \sum_{m_1, m_2} \left(\begin{array}{cc|c} S_c & \frac{1}{2} & S \\ m_1 & m_2 & S_3 \end{array} \right) |S_c m_1\rangle |1/2 m_2\rangle, \qquad \text{(A3)}$$

with $S_c = S - 1/2$ if $p_1 = 1$ and $S_c = S + 1/2$ if $p_1 = 2$ and the isospin part

$$|II_3; p_2\rangle = \sum_{i_1,i_2} \left(\begin{array}{cc|c} I_c & \frac{1}{2} & I \\ i_1 & i_2 & I_3 \end{array} \right) |I_c i_1\rangle |1/2 i_2\rangle, \tag{A4}$$

with $I_c = I - 1/2$ if $p_2 = 1$ and $I_c = I + 1/2$ if $p_2 = 2$. Here S_c and I_c are the spin and isospin of the core and p_1 and p_2 represent the position of the N_c-th quark in the spin and isospin parts of the wave function respectively, both consistent with $p = 2$ and the inner product rules generating the wave function in the flavor-spin space. The coefficients $K([f_1]p_1[f_2]p_2|[N_c - 1, 1]p = 2)$ are isoscalar factors of the permutation group S_{N_c}. At p fixed, one can use an alternative notation

$$K([f_1]p_1[f_2]p_2|[N_c - 1, 1]p = 2) = c^{[N_c-1,1]}_{p_1 p_2}(S) \tag{A5}$$

For the representation $[N_c - 1, 1]$ the only non vanishing expressions are

$$c^{[N_c-1,1]}_{11}(S) = -\sqrt{\frac{(S+1)(N_c - 2S)}{N_c(2S+1)}}, \tag{A6}$$

$$c^{[N_c-1,1]}_{22}(S) = \sqrt{\frac{S[(N_c + 2(S+1)]}{N_c(2S+1)}}, \tag{A7}$$

$$c^{[N_c-1,1]}_{12}(S) = c^{[N_c-1,1]}_{21}(S) = 1. \tag{A8}$$

Actually we need the above coefficients in the limit $N_c \to \infty$. Therefore for N resonances where $S = 1/2$ we have to take

$$c^{[N_c-1,1]}_{11}(1/2) \to -\sqrt{\frac{3}{4}}; \qquad c^{[N_c-1,1]}_{22}(1/2) \to \sqrt{\frac{1}{4}}, \tag{A9}$$

and for Δ resonances

$$c^{[N_c-1,1]}_{11}(3/2) \to -\sqrt{\frac{5}{8}}; \qquad c^{[N_c-1,1]}_{22}(3/2) \to \sqrt{\frac{3}{8}}. \tag{A10}$$

With the above notations, the matrix elements with a given J, between states with S' and S take the following form

$$\begin{aligned} &\langle \ell' S' J' J'_3; I' I'_3 | \ell \cdot s | \ell S J J_3; I I_3 \rangle_{p=2} = \\ &(-1)^{J+\ell+1/2} \delta_{J'J} \delta_{J'_3 J_3} \delta_{\ell'\ell} \delta_{I'I} \delta_{I'_3 I_3} \sqrt{\frac{3}{2}(2S+1)(2S'+1)\ell(\ell+1)(2\ell+1)} \left\{ \begin{array}{ccc} \ell & \ell & 1 \\ S & S' & J \end{array} \right\} \\ &\times \sum_{p_1,p_2} (-1)^{-S_c} c^{[N_c-1,1]}_{p_1 p_2}(S') c^{[N_c-1,1]}_{p_1 p_2}(S) \left\{ \begin{array}{ccc} 1 & \frac{1}{2} & \frac{1}{2} \\ S_c & S & S' \end{array} \right\} \end{aligned} \tag{A11}$$

This expression is equivalent to Eq. (A7) of Ref. [14]. The correspondence in the isoscalar factors denoted there by $c_{\rho\eta}$ is

$$c_{11}^{[N_c-1,1]}(S) \to c_{0-}; \quad c_{22}^{[N_c-1,1]}(S) \to c_{0+}; \quad c_{12}^{[N_c-1,1]}(S) \to c_{++}; \quad c_{21}^{[N_c-1,1]}(S) \to c_{--}. \tag{A12}$$

The expectation value of the operator containing the tensor term is

$$\begin{aligned}
&\langle \ell' S' J' J_3'; I' I_3' | \ell^{(2)} \cdot g \cdot G_c | \ell S J J_3; I I_3 \rangle_{p=2} = (-)^{J+I+\ell+S+1/2} \\
&\times \delta_{J'J}\delta_{J_3'J_3}\delta_{\ell'\ell}\delta_{I'I}\delta_{I_3'I_3}\frac{1}{8}\sqrt{\frac{15}{2}\ell(\ell+1)(2\ell-1)(2\ell+1)(2\ell+3)(2S'+1)(2S+1)}\left\{\begin{array}{ccc} 2 & \ell & \ell \\ J & S & S' \end{array}\right\} \\
&\times \sum_{p_1',p_2',p_1,p_2} (-1)^{S_c'} c_{p_1'p_2'}^{[N_c-1,1]}(S') c_{p_1p_2}^{[N_c-1,1]}(S) \sqrt{(2I_c'+1)(2I_c+1)} \\
&\times \sqrt{(N_c+1)^2 - (S_c'-S_c)^2(2I+1)^2}\left\{\begin{array}{ccc} \frac{1}{2} & 1 & \frac{1}{2} \\ I_c' & I & I_c \end{array}\right\}\left\{\begin{array}{ccc} I_c' & I_c & 1 \\ S' & S & 2 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{array}\right\}
\end{aligned} \tag{A13}$$

One can recover Eq. (A9) of Ref. [14] using the correspondence (A12). In the large N_c limit considered here the term $(S_c'-S_c)^2(2I+1)^2$ under the squared root should be ignored.

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